A new coupled ARMA-FGM model and its application in the internet third-party payment forecasting in China

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Abstract

Purpose – With the rapid development of e-commerce in China, the third-party payment system greatly improved the efficiency and volume of the entire trading market. The purpose of this paper is to put forward a suitable prediction model to analyse its development trend.

Design/methodology/approach – The authors analyse internet third-party payments in China, taking into account online payment transaction values coupled with an ARMA model and the fractional grey model (FGM). First, the rolling FGM model is applied in order to characterise the trends of the transaction volume. The influence of the initial value change on the FGM model is analysed. The optimisation mean absolute percentage error (MAPE) model is constructed to determine the optimal translational values, the corresponding optimal accumulation order and optimal inverse accumulation order.

Findings – This paper uses China’s recent third-party online payment data to quantify its development trend. The authors find the coupling model suitable for the development trend of third-party online payment transaction. The results show that the model is suitable to quantify its development trend of China’s recent third-party online payment.

Originality/value – Considering the complex influence factors that lead to the third-party online payment volume data of time-varying grey feature, this paper combines the FGM with ARMA model to describe the development of third-party payment mode.

Keywords Time series, Coupled forecasting model, Fractional grey model, The third-party internet online payment

Paper type Research paper

1. Introduction

Based on the internet and mobile communications, third-party payment is rapidly developing. This form of payment is changing the inherent modes of production and payment habits and becoming the new growth point for the world economy. The total transaction volume of China’s third-party internet payments in 2015 increased by 874.41 per cent, from 16 billion in 2006 to 14,006.57 billion yuan (Figure 1). In the fourth quarter of the transaction composition, the top three components were internet shopping, funds and aviation, accounting for 51.7 per cent of the internet pay market share. The remainder is communication recharge, e-commerce B2B and game recharge (iResearch, 2016). Third-party payments have penetrated into all aspects of life and become an indispensable payment method. We aim to study the development trend of third-party payments under comprehensive factors. Understanding the achievements of this industry will have a positive effect on future innovation and development.

In China, the current online payment businesses are commercial bank business and third-party payment business. With the online payment of the bank and the third-party payment platform, there is a time-varying complex relationship. In the initial payment of a third-party payment, small businesses comprise mostly small transactions, while banks account for larger transactions.

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Due to the quick rise of third-party payments, there exists little quantitative analysis on this topic. However, in the current econometric model system, the method of forecasting the economic variables with time is becoming increasingly rich and mature. Commonly used methods are as follows:

1. A linear regression model with the simple algorithm but no reflection of economic variables with uncertainties (Feng and Cao, 2011);
2. ARMA system suitable for discriminating periodic changes (Cao and Lou, 2008);
3. Kalman filter model with stronger abilities, larger calculation sizes and flexible choice for predictive factors (Liu et al., 2014);
4. Traditional grey prediction model for less data volume and poor information data, but it cannot reflect the new information priority (Zhang et al., 2010);
5. A non-parametric regression model with a simpler algorithm, looser data requirements and is more adaptable but has larger data volume (Liu et al., 2011);
6. Neural network method with data self-organising adaptive feature, high precision and robustness (Yan et al., 2013); and
7. Support vector machine for less data, non-linearity and dimensionality data (Tong and Lim, 1980).

Theories and applications of these methods enrich and improve the econometric model. However, internet online payment data have their own unique characteristics. This payment method relies on the shopping festival (Double 11), tourism and other seasonal consumption that will show a seasonal cyclic phenomenon and grey nature (its history data are not sufficient, and the evolutionary trend is uncertain).

Time series analysis assumes that the past state of things continues into the future. The use of time series before and after the autocorrelation according to the previous state trend prediction of future development can reflect the variable time before and after the dynamic impact relationship. Since the British statistician G.U. Yule used the autoregressive model (AR (2)) in 1927 to predict the law of market changes, domestic and foreign scholars have conducted numerous studies on such methods (Zhu and Liu, 2004). This model includes the proposed threshold autoregressive (TAR) and the generalised autoregressive conditional heteroskedasticity model (GARCH) (Engle, 1982; Bollerslev, 1986). Many scholars have...
improved the model system in terms of parameter estimation (Granger and Joyeux, 1980),
order determination (Granger, 1980) and modelling improvement (Aue et al., 2017). They have
applied these to the real relevant field (Yan et al., 2015; Bao, 2013). The theory of time series
modelling is becoming more mature.

Grey theory, proposed by Professor Deng Julong (Deng, 1982, 1989), is primarily used to
solve the problem of forecasting with uncertain information systems. This theory has been
widely used in the prediction of complex systems, such as Lorenz chaotic (Zhang et al., 2009),
carbon dioxide emissions (Lin et al., 2011) and typhoon trajectories (Chen and Huang, 2013).
The classical grey model (GM (1, 1)) has a good fitting effect on slow growing or decreasing
data, but it does not solve the problem of greater value for new information. In the internet
era where information updates are quick, new information is more meaningful than old.
Wu et al. (2013) used the least squares method of the perturbation theory to demonstrate that
the traditional cumulative generation operator violates the principle that the new information
has a greater value. The example demonstrates that the fractional cumulative generation
model makes up for the defect of GM (1, 1). Wu et al. (2014) proposed a fractional-order
cumulative discrete model (FAGM (1, 1)) that proved that the solution is relatively stable with
respect to the first-order additive discrete grey model. Mao et al. (2014, 2016) proved that
FAGM (1, 1) can break the GM (1, 1) modelling level boundary limit, extended the integer
order of the derivative to the fractional order, and proposed a fractional derivative grey model
(FGM (Q, 1)). It is considered that the fractional-order cumulative generation can replace
the traditional one-time cumulative generation to improve the exponential law of the data,
and the heredity and memory of the fractional model can describe the grey characteristic of
the third-party payment data more ideally.

Time series forecasting methods and grey prediction models not only have their own
irreplaceable advantages but also have their own inevitable problems. Considering the
possible shortcomings of the single prediction model, Bates and Granger (1969) proposed
using the different methods to provide information to achieve the purpose of optimising the
model. Currently, combinatorial thinking (also called coupling) is introduced into economic,
financial and other fields, and the combination model is better than the single model
(Shi et al., 1996; Lemke and Gabrys, 2010; Zhang and Ren, 2010). The researchers combined
the grey model, the neural network model, the chaos theory, the wavelet theory and the
ARMA system models in combination with the short-term output, short-term traffic volume
and tourism demand forecasts (Wang and Zhang, 2012; Liang et al., 2015; Zheng and Shi,
2005). As a special form of combinatorial prediction, the time series combined with the grey
model (Shan et al., 2012) has the advantage of being underestimated in residual correction.

As a special form of the measurement model, the time series combined with the grey
model considers the characteristics of transaction data and has important advantages in
residual correction. In this paper, the rolling fractional grey model (FGM) is established to
calculate the overall development trend of the transaction volume, and the time series
model is used to correct the error term to establish the coupling model. The influence of the
initial value change on the coupled model is analysed, and the optimal model for parameter
estimation is constructed. Finally, this paper uses China’s recent third-party online payment
data as an example to quantify its development trend.

2. ARIMA-FGM coupling model of internet third-party online payment
transaction
2.1 Fractional-order accumulation generation

Definition 1. Take the raw non-negative sequence \( x^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))^T \), where
\( x^{(0)} \) is the \( r \)-order accumulated sequence of \( x^{(0)} \). \( r \) is a rational number,
\( x^{(r)}(k) = \sum_{i=1}^{k} x^{(r-1)}(i), \quad k = 1, 2 \cdots n \), \( x^{(r)} = \Lambda^r x^{(0)} \).
\[ A^r \] is \( r \)-order accumulation generating matrix, note \( A^r = (a_{ij}^r)_{n \times n} \), and 
\[
\begin{bmatrix}
  r \\
  i-j
\end{bmatrix}
\]
for generalized factorial function, then:

\[
\begin{pmatrix}
  a_{ij}^r \\
  n
\end{pmatrix} = \begin{cases}
  1, & i = j \\
  \frac{r(r+1)\cdots(r+i-j-1)}{(i-j)!}, & i > j \\
  0, & i < j
\end{cases}
\]

for \( x^{(r-1)}(k) = \sum_{m=1}^{k} x^{(r-1)}(m) - \sum_{m=1}^{k-1} x^{(r-1)}(m) = x^{(r)}(k) - x^{(r)}(k-1) \)

for \( x^{(0)} = A^{-1}x^{(1)} = A^{-2}x^{(2)} = \ldots = A^{-r}x^{(r)} \), note \( A^{-r} \) as \( r \)-order reduce generation matrix, which satisfies \( (A^r)^{-1} = A^{-r} \), i.e., it is the inverse operation of \( r \)-order accumulation matrix.

Because the actual system is usually of fractional order, and the grey model with integer order derivative concerned in present study is inappropriate to describe the irregular phenomenon, introducing the fractional derivative into the grey model can better reveal the characteristics of the phenomenon. Fractional differential equations, moreover, are simple in modelling complex systems, accurate in description and widely used in practice, so we use the fractional-order differential equation rather than traditional integer order. This change can better break the grey exponential rate limit of the model, expand the scope of the model and improve the practicability of the model.

2.2 FGM model

Let the raw non-negative sequence \( x^{(0)} = (x^{(0)}(1), x^{(0)}(2), \ldots, x^{(0)}(n))^T \), \( x^{(r)} = A^r x^{(0)}, z^{(r)}(k) = ax^{(r)}(k-1) + (1-\alpha)x^{(r)}(k), \alpha \in (0, 1) \), i.e., \( z^{(r)} = (z^{(r)}(2), z^{(r)}(3), \ldots, z^{(r)}(n))^T \) is the \( \alpha \) mean generating sequence, and \( \alpha \) is the background value coefficient:

\[ \text{Definition 2.} \text{ The differential equation:} \]

\[
\frac{d^q x^{(r)}}{dt^q} + ax^{(r)} = b
\]

is the whitening form of (FGM (q, 1)) model.
For $x^{(r-q)}$, the $q$-order derivatives of $n$-order differentiable function $x(t)$ satisfy 

$$x_k^{(r-q)} = \frac{1}{h^q} \sum_{i=1}^{k} \left( -q \right) \frac{x_{k-i}}{k-i}$$

when $h \to 0$; thus, for $h = 1$, consider the $q$-order difference to estimate the $q$-order derivative of $x^{(r)}$, then:

**Definition 3.** Note:

$$x^{(r-q)}(k) + az^{(r)}(k) = b$$

(3)

where $x^{(r-q)} = A^{-q}x^{(r)}$, which is the definition form of FGM $(q, 1)$ model.

Let the non-negative sequence $x^{(r)} = (x^{(r)}(1), x^{(r)}(2), \ldots, x^{(r)}(n))^T$, $z^{(r)}$ is the $\alpha$ mean generating sequence, and $x^{(r-q)} = (x^{(r-q)}(1), x^{(r-q)}(2), \ldots, x^{(r-q)}(n))^T$ is the $q$-order reduction sequence, where $x^{(r-q)} = A^{-q}x^{(r)}$.

Let $p = (a, b)^T$ be parameters and:

$$X^* = \begin{bmatrix}
  x^{(r-q)}(2) \\
x^{(r-q)}(3) \\
  \vdots \\
x^{(r-q)}(n)
\end{bmatrix}$$

$$B = \begin{bmatrix}
  -z^{(r)}(2) & 1 \\
  -z^{(r)}(3) & 1 \\
  \vdots & \vdots \\
  -z^{(r)}(n) & 1
\end{bmatrix}$$

If $X^*$ is an difference matrix of accumulative data, then the least squares estimated parameters of $x^{(r-q)}(k) = -az^{(r)}(k) + b$ will satisfy:

$$\hat{p} = (B^TB)^{-1}B^TY$$

(4)

Data matrix $B$ and cumulative data difference matrix $X^*$ can be decomposed as:

$$B = \begin{bmatrix}
  -z & -(1-z) & 0 & \cdots & 0 \\
  0 & z & -(1-z) & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & -(1-z)
\end{bmatrix}$$

$$\begin{bmatrix}
  r & 0 & 0 & \cdots & 0 \\
  0 & r & 0 & \cdots & 0 \\
  1 & 0 & r & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  n-1 & n-2 & n-3 & \cdots & r
\end{bmatrix} \begin{bmatrix}
  y^{(0)}(1)d & -1 & 0 \\
  y^{(0)}(2)d & -1 & 0 \\
  \vdots & \vdots & \ddots \\
  y^{(0)}(n)d & -1 & 0
\end{bmatrix} = B_1A'M$$
\[ \mathbf{X}^* = \begin{pmatrix} -q & -q & 0 & \ldots & 0 \\ 1 & 0 & \ldots & \ldots & \ldots \\ -q & -q & -q & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -q & -q & -q & \ldots & -q \\ n-1 & n-2 & n-3 & \ldots & 0 \end{pmatrix} \begin{pmatrix} -q \\ 0 \\ -q \\ \vdots \\ n-1 \end{pmatrix} \begin{pmatrix} x^{(0)}(1) \\ x^{(0)}(2) \\ \vdots \\ x^{(0)}(n) \end{pmatrix} \]

\[ = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -q \\ 0 \\ -q \\ \vdots \\ n-1 \end{pmatrix} \begin{pmatrix} -q \\ 1 \\ -q \\ \vdots \\ n-2 \end{pmatrix} \begin{pmatrix} -q \\ 0 \\ -q \\ \vdots \\ n-3 \end{pmatrix} \begin{pmatrix} -q \\ \vdots \\ 0 \end{pmatrix} \begin{pmatrix} x^{(0)}(1) \\ x^{(0)}(2) \\ \vdots \\ x^{(0)}(n) \end{pmatrix} \]

\[ = \mathbf{Q} \mathbf{A}' \mathbf{X}_N = \mathbf{E}^* \mathbf{A}^{-q} \mathbf{A}' \mathbf{X}_N = \mathbf{E}^* \mathbf{A}^{r-q} \mathbf{X}_N = \mathbf{E}^* \mathbf{A}^{r-q} \mathbf{Y}_N \mathbf{d} \]

where:

\[ \mathbf{B}_1 = \begin{pmatrix} -\alpha & -(1-\alpha) & 0 & \ldots & 0 \\ 0 & \alpha & -(1-\alpha) & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -(1-\alpha) \end{pmatrix}_{(n-1) \times n} \]
2.3 Effect of the initial value transformation on the model

The following considers how the model parameters change when the initial point changes to \( t \):

**Theorem 1.** Let the initial point \( x_0(1) \rightarrow x_0(1) = x_0(1) + t \) and \( x_0(k) = x_0(k), k = 2, 3, ..., n \), then \( (a_t, b_t)^T = C^{-1}(B^TB)^{-1}B^T(X+D) \), where \( C^{-1} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \), \( D = \begin{pmatrix} r-q \\ 1 \\ r-q \\ 2 \\ \vdots \\ r-q \\ n-1 \end{pmatrix} \), \( (X+D) = \begin{pmatrix} a_t^* \\ b_t^* \end{pmatrix} \).

By Theorem 1, if the initial point translated \( t \), then the FGM \((q, 1)\) model of the changed data is established; at this time, the parameters of the model can be decomposed into matrices \( C^{-1}, B, D \) and \( X^* \). For the components of each matrix, we can understand that the impact mechanism of \( t \) on the parameters is as follows. Let the \( i \)th element of \( X^* \) changes to \( t[r-q] \) such that we get \( X^* + D \), then consider \( X^* + D \) as a whole to construct \((B^TB)^{-1}B^T(X^*+D)\) as a parameter \((a_t^*, b_t^*)^T\). Next, multiply \( C^{-1} \) on the left-hand side to get \((a_t, b_t)^T = (a_t^*, t\cdot a_t^* + b_t^*)^T\), so we know that the development coefficient \( a_t \) becomes \( a_t^* \) under the influence of \( q \), translation value \( t \) and cumulative order number \( r \), so does \( b_t^* \), and the control coefficient \( b_t \) is the linear combination of the affected \( b_t^* \) and \( t a_t^* \). If \( r = q \), then \( a_t = a_t^* \), \( b_t = b_t^* \), and \( r = q = 1 \), which agrees with other studies (Bao, 2013).

2.4 Time series coupling fractional grey model (ARIMA-FGM)

Internet third-party payment transaction data are a non-stationary time series data set. After \( d \)-order difference, the stationary sequence is obtained, and the ARMA \((p, s)\) model is subsequently established:

**Definition 4.**

\[
X_t = \sum_{j=1}^{p} a_j X_{t-j} + \sum_{j=0}^{s} b_j \tilde{e}_{t-j}, t \in \mathbb{Z} \tag{5}
\]
is called an autoregressive moving average model. If \( b_0 = 1 \), then:

\[
A(z) = 1 - \sum_{j=1}^{p} a_j z^j \neq 0, |z| < 1
\]

\[
B(z) = 1 + \sum_{j=1}^{s} b_j z^j \neq 0, |z| < 1
\]

and \( A(z) \) and \( B(z) \) have no common roots.

In this way, we use the linear model to characterize the periodic, trend and randomness of non-stationary sequences. However, from 2006 to 2016, the transaction amount of data differs on three orders of magnitude, resulting in a large estimation error with linear models, even after the data and differential transformations. Therefore, the rolling FGM is used to fit the original transaction data and determine the development trend. Next, the ARMA system model is used to correct the error. In other words, set a third-party internet online payment the original transaction data and determine the development trend. Next, the ARMA system model is used to correct the error. In other words, set a third-party internet online payment

\[
\text{X}_t = \text{T}_t + \text{R}_t
\]

where \( T_t \) is the trend item, which includes the rising and decreasing trend of the transaction amount, and \( R_t \) is the random error term, which includes the periodic change and the random fluctuation. In this paper, we use the FGM to estimate the trend and the rolling FGM to fully utilise the data. Thus, we can separate the trend item \( T_t \), so that the magnitude of the remaining error decreases and the impact of errors is weakened, then use ARIMA (\( p, d, s \)) to get the corrected residual term \( R_t \).

3. Modelling steps

Specific modelling steps are as follows:

(1) Grouping raw data: grouping \( n \) raw data into \( K \) groups, where \( K = n - m + 1 \), each group has \( m \) data, and the initial points within each set of data are, respectively, transformed into \( t_i \):

\[
\begin{align*}
X_{(0)m,1,t_i} &= (x_{(0)}^{(0)}(1) + t_1, x_{(0)}^{(0)}(2), \ldots, x_{(0)}^{(0)}(m)) \quad (m) \\
X_{(0)m,2,t_i} &= (x_{(0)}^{(0)}(2) + t_2, x_{(0)}^{(0)}(3), \ldots, x_{(0)}^{(0)}(m+1)) \quad (m) \\
&\vdots \\
X_{(0)m,K,t_i} &= (x_{(0)}^{(0)}(K) + t_K, x_{(0)}^{(0)}(K+1), \ldots, x_{(0)}^{(0)}(n)) \quad (m)
\end{align*}
\]

(2) Thus, fractional-order FGM models are established from Equation (2) based on grouping data represented by Equation (6):

\[
\frac{d^{q_{(m,i)}} x_{(m,i,t_i)}^{(r)}}{d t^{p_{(m,i)}}} + a_{(m,i,t_i)} x_{(m,i,t_i)}^{(r)} = b_{(m,i,t_i)}
\]

where \( r_{(m,i)} \) is the cumulative order of the \( i \)th model, \( q_{(m,i)} \) is the derivative order of the \( i \)th model, \( a_{(m,i)} \) is the development coefficient of the \( i \)th model, \( b_{(m,i)} \) is the control coefficient for the \( i \)th model. The model (7) is discretized into:

\[
x_{(m,i,t_i)}^{(r_{(m,i)} - q_{(m,i)})}(k) + a_{(m,i,t_i)} x_{(m,i,t_i)}^{(r_{(m,i)})}(k) = b_{(m,i,t_i)}, i = 1, 2, \ldots, K
\]
where:
\[
\hat{x}_{(m, i, t)}^{(r_{(m, i)})}(k) = \alpha x_{(m, i, t)}^{(r_{(m, i)})}(k) + (1-\alpha) x_{(m, i, t)}^{(r_{(m, i)})}(k)
\]
is the background value for the \(i\)th model.

(3) Combined with the least squares method, we construct a fitting error optimisation mean absolute percentage error model (MAPE model):
\[
\min_{t, r, q} \text{MAPE} = \min_{t-1} \frac{1}{n-1} \sum_{k=1}^{n-1} \left| \frac{x_{(m, i, t)}^{(0)}(k) - x_{(m, i, t)}^{(0)}(k)}{x_{(m, i, t)}^{(0)}(k)} \right| \times 100\%
\]

We can choose the optimal translation value \(t\), cumulative generation order \(r\), reduce generation order \(q\), \(i = 1, 2, ..., k\), and corresponding \(t, r, q\), then:
\[
\hat{x}_{(m, i, t)}^{(r_{(m, i)})} + \hat{a}_{(m, i, t)} x_{(m, i, t)}^{(r_{(m, i)})}(k) = \hat{b}_{(m, i, t)}(m, i, t)
\]

And the numerical solution of each model is obtained as \(\hat{x}_{(m, i, t)}^{(r_{(m, i)})}\), then:
\[
\hat{x}_{(m, i, t)}^{(0)}(m, i, t) = A^{-T} \hat{x}_{(m, i, t)}^{(r_{(m, i)})}
\]
The optimal fitting estimate under the optimal rolling model is obtained by:
\[
\begin{cases}
\hat{x}_{(m, 1, t)}^{(0)} = \left( \hat{x}_{(m, 1)}^{(0)}(1), \hat{x}_{(m, 1)}^{(0)}(2), ..., \hat{x}_{(m, 1)}^{(0)}(m+1) \right)^T \\
\hat{x}_{(m, 2, t)}^{(0)} = \left( \hat{x}_{(m, 2)}^{(0)}(2), \hat{x}_{(m, 1)}^{(0)}(3), ..., \hat{x}_{(m, 1)}^{(0)}(m+1), \hat{x}_{(m, 1)}^{(0)}(m+2) \right)^T \\
\vdots \\
\hat{x}_{(m, K, t)}^{(0)} = \left( \hat{x}_{(m, K)}^{(0)}(K), \hat{x}_{(m, 1)}^{(0)}(K+1), ..., \hat{x}_{(m, 1)}^{(0)}(K+m-1), \hat{x}_{(m, 1)}^{(0)}(K+m) \right)^T
\end{cases}
\]

From (3) and (4), \(\hat{x}_{(m, i, t)}^{(0)}(k)\) are obtained. Because there are intersections values in the estimation of different groups of models, we then calculate the mean value of the intersections to obtain the predicted value \(\hat{T}_t\) of the trend item \(T_t\). It can be seen that different optimal models have crossed predictive estimates. To make full use of the information, improve prediction accuracy and simplify the calculation. The weighted average of cross-estimates is as follows:
\[
\hat{T}_t = \hat{x}_{(m, i, t)}^{(0)}(j) = \begin{cases}
\sum_{i=1}^{j} \hat{x}_{(m, i)}^{(0)}(j), & 1 < j < m, j \in N \\
\sum_{i=j-m+1}^{j} \hat{x}_{(m, i)}^{(0)}(j), & m \leq j < n-m+1, j \in N \\
\sum_{i=j-m+1}^{n-m+1} \hat{x}_{(m, i)}^{(0)}(j), & n-m+1 \leq j < n, j \in N
\end{cases}
\]
(4) Get the error term $R_t = X_t - \hat{T}_t$, then the ARIMA correction error term is established, and finally the coupling model (ARIMA-FGM) is accorded with the trend of the third-party payment transaction. $R_t$ be recorded as $e_t$. The ARIMA $(p, d, s)$ model was established as follows:

$$u_j = \sum_{i=1}^{p} c_i u_{j-i} + \sum_{i=0}^{s} f_i v_{j-i}, f_0 = 1$$

where $u_j = u(j) = \sum_{k=0}^{d} C_d^k (-1)^k e_{j-k}$, it is the residual value after the $d$-order difference.

On applying the non-linear least squares method to solve the parameters of the above model, the updated error $\hat{e}^*(j)$ is obtained and we get:

$$\hat{x}(j) = \hat{x}^{(0)}(j) + e^*(j) \rightarrow \hat{T}_t = \hat{T}_t + \hat{R}_t$$

The estimated values for the coupling model are fitted. The concrete steps are shown in Figure 2.

4. Empirical analysis and modelling comparison
The data regarding third-party payment in China are collected from the Easy View of the think tank and iResearch. We acquired the quarterly transaction data from the first quarter of 2006 till the second quarter of 2016.

4.1 ARIMA model for the third-party internet online payment transactions
Set the original data as $y^{(0)}(0) = (y^{(0)}(1), y^{(0)}(2), ..., y^{(0)}(n))^T$, and the observed data show a similar logistic growth trend, while the growth rate decreases. The visible data are non-stationary; therefore, to establish a suitable model, we create two-order difference smooth sequence after the logarithmic transformation:

$$x_t = (\log (y_t) - \log (y_{t-1})) - (\log (y_{t-1}) - \log (y_{t-2}))$$

$$= \log (y_t) - 2 \log (y_{t-1}) + \log (y_{t-2})$$

The results of the consistency test after the data transformation are shown in Figure 3.

As seen from Figure 3, the autocorrelation coefficient of the processed data shows the first-order trailing property, whereas the partial autocorrelation coefficient exhibits the third-order trailing property. However, after debugging, it is found that the AIC value is smallest when the optimal ARMA (1, 2) model is established for the processed data. Using the model check in Table I, we can see that the DW value of the model is 2.15. After querying the DW statistical table, we found that the value falls in the region without autocorrelation, and the unit is rooted in the unit circle. Therefore, the establishment of the ARMA (1, 2) model is reasonable.

Therefore, the ARMA (1, 2) model is established for the processed data:

$$y_t = -0.82x_t + e_t - 0.61e_{t-2}$$

Since $x_t$ is the sequence after the logarithmic transformation of the original data and the two-order difference, then:

$$\hat{x}_t = (\log (\hat{y}_t) - \log (\hat{y}_{t-1})) - (\log (\hat{y}_{t-1}) - \log (\hat{y}_{t-2}))$$

$$= \log (\hat{y}_t) - 2 \log (\hat{y}_{t-1}) + \log (\hat{y}_{t-2})$$
By the above formula and \( y_1 = y_1, \ y_2 = y_2, \) we can get \( \hat{y}_t. \) The model estimates can be obtained as shown in Figure 4, where the curve is plotted in red. To distinguish the differences between the curves clearly, we take the partial time scale magnification curves, as shown in Figure 5. The easy-to-see time series fit curve has four obvious lags in 2012, 2013, the fourth quarter of 2014 and the first quarter of 2015. We see that the original
transaction volume in the fourth quarter of 2013 increased more obviously and the growth rate of the first quarter of 2014 slowed down. However, when the fitting curve in the first quarter of 2014 showed a surge in growth but slowed in the second quarter, then the original state lagged for a quarter. Since the original transaction volume is a distinct non-stationary
time series, with the change of time, the magnitude of the gap is large. When the third-party transaction volume after the logarithmic transformation is modelled to the later time, then there is greater error and the emergence of the cycle lags.

4.2 Coupled ARMA-FGM model under optimal initial point

By observing the characteristics of the internet transaction data paid by the third party, we divide each of the eight original data into eight groups \((m = 8)\), the initial value of each
group is translated by $t_1, t_2, ..., t_K$ units, the background value coefficient $\alpha = 0.5$, and the rolling FGM is established. According to Equations (3)-(8), the optimal translational values $\hat{t}_1, \hat{t}_2, ..., \hat{t}_K$ of the individual FGM, corresponding optimal accumulation order $\hat{r}_1, \hat{r}_2, ..., \hat{r}_K$ and optimal inverse accumulation order $\hat{q}_1, \hat{q}_2, ..., \hat{q}_K$ are obtained. The results are shown in Figure 4.

In Figure 4, the $x$-axis is a single model in the rolling model, the left $y$-coordinate represents optimal value $\hat{t}$, and the right $y$-coordinate represents the optimal accumulation order $\hat{r}$ and the inverse cumulative order $\hat{q}$. The curve of the hollow four-star is the optimal value $\hat{t}$ corresponding to the single model, and the optimal $t$ expression increases with the increase of the numerical value. The curve of the solid quadrangular star is the optimal accumulation order $\hat{r}$, which has the periodic change similar to the optimal value, but the overall growth trend is not obvious. The curve of the triangle is the optimal inverse cumulative $\hat{q}$ corresponding to the single model, and the change is not obvious, which is approximately 1. It can be observed that the transaction data of the third-party payment in different periods correspond to different FGMs. In addition, the optimal translational amount of the initial point of the FGM increases with the passage of time, and the optimal accumulation order of the individual fractional-order model corresponding to each optimal value is very different. Therefore, it is reasonable to establish the FGM under the optimal translation value for the overall data, respectively, and the method is effective because of the high precision of the FGM.

At present, the first FGM established in the first quarter of 2006 to the fourth quarter of 2007 is taken as an example to explain the relationship between matrix decomposition and translation values and model parameters. On substituting, the optimal translation value of the first fractional model $\hat{t}_1 = 3.33$, the optimal accumulation order $\hat{r}_1 = 0$ and the inverse cumulative order $\hat{q}_1 = 1.59$, we get:

\[
C^{-1} = \begin{pmatrix} 1 & 0 \\ 3.33 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -42.50 & 1 \\ -70.00 & 1 \\ -104.55 & 1 \\ -142.05 & 1 \\ -185.00 & 1 \\ -232.50 & 1 \\ -242.10 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} -1.59 \\ 0.469 \\ 0.064 \\ 0.023 \\ 0.011 \\ 0.006 \\ 0.004 \end{pmatrix}
\]

such that:

\[
(a_1, b_1)^T = C^{-1}(B^T B)^{-1} B^T (X + t_1 F)
\]

The corresponding FGM can be obtained by:

\[
\frac{d^{1.59}x^{(0)}(8.1, t_1)}{dt^{1.59}} + 0.181x^{(0)}(8.1, t_1) = 27.47
\]

Specifically, the initial shift value of the first set is 3.33 units, and the FGM of the 1.59 order derivative is established. The other FGMs are the same.

4.3 Coupling model for forecasting third-party internet online payment transactions

According to Theorem 1, the optimal rolling FGM is established. The fitting value of the original transaction amount is obtained by the mean method, and the residual graph is
obtained. In the graphs shown in Figure 6, the original transaction volume shows a regular increase in the exponential form. In the first quarter of 2009, the original transaction volume increase value is small, and the growth is smooth; therefore, in Figure 5, the black solid line rolling FGM fitting to obtain the residual items is more stable. However, after 2009Q1, with the increase of the order of magnitude, the error generated by the rolling model approximates the cyclical change, and the fitting accuracy is not ideal. However, with the passage of time, in the rolling model of the average, the latter large data estimate the use of the previous smaller values, making the error larger. Conversely, because the evaluation criteria used in this paper are the minimum sum of relative absolute error, the order of magnitude may lead to the increase of error.

To reduce the fitting error, an appropriate ARMA system model is established for the residual items generated by the rolling model. After the stationarity test, the optimal model ARMA (2, 2) is selected according to the minimum criterion of AIC value. The square sum $R^2$ is 0.47 and the DW value is approximately 2.34. In the vicinity of 2, it is considered that the autocorrelation of the residual term is eliminated and the unit root is within the unit circle, so the selected model sequence is stationary.

The established ARMA (2, 2) model is:

$$\hat{e}_t = -0.92e_{t-2} + e_t + 0.57e_{t-2},$$

that is to say:

$$e^*(j) = -0.92e^*(j-2) + e(j) + 0.57e(j-2)$$

We can get the estimate of the error:

$$\hat{x}(j) = \hat{x}^{(0)}(j) + e^*(j)$$

The fitting curve of the third-party payment transaction under different models is plotted in Figure 6 to get the comparison chart.

![Model comparison](image_url)

**Figure 6.** Comparison of third-party payment transactions under different models 2006Q1-2016Q4
In Figures 6 and 7, the black curve marked in the circle is the original data curve, and the triangle is labelled with the original transaction volume to establish the ARMA (1, 2) model. The fitting curve of the four-star is given for the coupling model.

From Figures 6 and 7, the following can be observed:

1. When the time series model estimates a larger error, there is a more obvious lag.
2. The fitting accuracy of the coupled model is significantly higher than that of the single time series model. The reason is that the coupling model first considers the overall increasing trend of the data using the FGM, and then considers the periodic and random fluctuations of the residual random error term from the time series model.
3. Observing Figure 6 shows that the amount of online payments made via third-party internet companies has grown moderately from 2006 to 2011. The rate of growth has been rapid since 2011, although there has been a slowdown in growth until the end of 2015. It is expected that third-party internet online payments will enter a period of steady growth and the transaction volume will continue to grow. However, due to the saturation of the market, the convenience of mobile payment and the diversification of services, consumers rely more on two-dimensional code payment, near-end payment and other mobile payment services; therefore, internet online payment transactions will grow slowly.
4. By the coupling model to predict the third quarter of 2016, the number of online transactions may be 5,229.739 billion, published by the Analysis think tank data source, this value and the actual value of the relative absolute error of 6.6 per cent can be predicted in the fourth quarter for 57,593.34 billion.

![Figure 7. Comparison of third-party payment transactions under different models 2011Q1-2015Q3](image-url)
References


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