Testing derivatives pricing models under higher-order moment swaps

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Abstract

Purpose – This paper aims to test three parametric models in pricing and hedging higher-order moment swaps. Using vanilla option prices from the volatility surface of the Euro Stoxx 50 Index, the paper shows that the pricing accuracy of these models is very satisfactory under four different pricing error functions. The result is that taking a position in a third moment swap considerably improves the performance of the standard hedge of a variance swap based on a static position in the log-contract and a dynamic trading strategy. The position in the third moment swap is taken by running a Monte Carlo simulation.

Design/methodology/approach – This paper undertook empirical tests of three parametric models. The aim of the paper is twofold: assess the pricing accuracy of these models and show how the classical hedge of the variance swap in terms of a position in a log-contract and a dynamic trading strategy can be significantly enhanced by using third-order moment swaps. The pricing accuracy was measured under four different pricing error functions. A Monte Carlo simulation was run to take a position in the third moment swap.

Findings – The results of the paper are twofold: the pricing accuracy of the Heston (1993) model and that of two Levy models with stochastic time and stochastic volatility are satisfactory; taking a position in third-order moment swaps can significantly improve the performance of the standard hedge of a variance swap.

Research limitations/implications – The limitation is that these empirical tests are conducted on existing three parametric models. Maybe more critical insights could have been revealed had these tests been conducted in a brand new derivatives pricing model.

Originality/value – This work is 100 per cent original, and it undertook empirical tests of the pricing and hedging accuracy of existing three parametric models.

Keywords Higher-order moment swaps, Log-contract, Static position, Variance swaps, Volatility surface

Paper type Research paper

1. Introduction

Moment swaps are derivative securities whose payoff depends on the realized higher moments of the underlying asset price or state variable. This payoff is linked to the powers of the log-returns and provides protection against various types of market conditions. Variance swaps which are liquidly traded today are obtained in the case of squared

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log-returns and offer protection against changes in the volatility process. Higher-order moment derivatives can be useful to protect against inaccurately estimated skewness or kurtosis. Rompolis and Tzavalis (2017) and Schoutens (2005) show that moment swaps can be used to significantly improve the standard hedge of the variance swap.

Some recent studies in the finance literature suggest that power-jump assets can be used to complete the market. For instance, an incomplete Levy market where power assets of any order can be traded will yield a complete market. Power assets and realized higher moments are highly linked and they are virtually the same in a discrete time framework (Corcuera *et al.*, 2005).

There has been a proliferation of studies extending the Black–Scholes (BS) option pricing model. However, only a few of these studies addressed hedging contingent claims under more general assumptions about the state variable stochastic process. Well known examples of such studies are the stochastic volatility (SV) model of Heston (1993); the SV and random jumps model (SVJ) of Bates (1996); and the SV, stochastic interest rates and random jumps model of Doffou and Hilliard (2001). Using delta hedging strategies in an incomplete market cannot lead to a perfect hedge against jump risk and volatility risk linked to a position in contingent claims.

Locally risk-minimizing delta hedging strategies that attempt to hedge the option contract using only the state variable and minimizing the variance of the cost process of a non-self-financed hedging position are not adequate. In the presence of jumps, these strategies perform very poorly like the classic delta hedging strategies (Tankov *et al.*, 2007). Other hedging strategies use option contracts to reduce or eliminate volatility risk (Bakshi *et al.*, 1997) or protect against jump risk (Coleman *et al.*, 2006; Cheang *et al.*, 2015). Cross hedging strategies or delta-vega hedging can totally remove an option contract exposure to volatility risk but not its exposure to jump risk. Jump risk can be hedged either by using a risk-minimization strategy (Coleman *et al.*, 2006; Tankov *et al.*, 2007) or by discretizing jump sizes to compute the hedge ratios of the other options (Utzet *et al.*, 2002). However, these two methodologies are limited. First, they do not characterize the maturity and moneyness of the option contracts to be used to efficiently and effectively set up the hedge. Further, these two approaches do not pick up the options' exposure to volatility risk. Finally, empirical evidences show that these two methods do not outperform the delta hedging strategy (Cheang *et al.*, 2015).

This paper tests empirically the Schoutens (2005) along with the Heston (1993) and the time-changed Levy models in their effectiveness in pricing and hedging moment swaps, particularly the variance swaps which are liquidly traded today. This paper is organized as follows. Section 2 defines moment swaps. Section 3 introduces the pricing and hedging model. The model parameters are estimated in Section 4. Section 5 tests the out-of-sample pricing performance of the model. Finally, Section 6 concludes the paper.

2. The moment swaps pricing model

Consider an asset (stock or stock index) with a continuous dividend yield $y \geq 0$. The asset price process is modeled by an Ito semi martingale $P = \{P_t, t \geq 0\}$ such that $P > 0$ and $P\_ >$ 0. In addition to this asset, a bond or money market account with a constant compound interest rate α is available with a price process $\Theta = \{\Theta_t = e^{\alpha t}, t \geq 0\}$. Consider n equally spaced time intervals of length Δt such that $t_j = j\Delta t$, with $j = 0, 1, 2, \ldots, n$ and $\Psi = n\Delta t$ is the expiration date of the derivative contract written on the state variable price P_t. The price of the state variable at each interval j is denoted P_j for simplicity. In practice, the t_j are the daily closing times and P_j is the closing price at day j. It follows that the daily log-returns are given by $\ln(P_j) - \ln(P_{j-1}), j = 1, 2, \ldots n$.

Assume futures contracts written on the state variable price P exist with expiration date Ψ. In the risk-neutral world, the price of the futures contract is given by $F_t = P_t\, exp(\alpha - y)(\Psi - t)$. For simplification, the futures price at the discrete time t_j is denoted F_j. The mth-moment swap on the stock is a contract in which the two counterparties agree to exchange at maturity a nominal amount multiplied by the difference between a fixed level contract price and the realized level of the mth-order non-central sample moment of the log-return over the life of the contract. The payoff function is defined by:

$$MS_P^m = NA\sum_{j=1}^{n}\ln\left(\frac{P_j}{P_{j-1}}\right)^m \tag{1}$$

where NA is the nominal or notional amount and n is the number of segments of length Δt within the time interval $[0, \Psi]$.

For $m = 2$, equation (1) gives the expression of the $2nd$-moment swap or variance swap. Variance swaps are basically forward contracts in which the counterparties agree to exchange a notional amount multiplied by the difference between a fixed variance and the realized variance. The fixed variance is the variance swap rate or the variance forward price. Variance swaps offer protection against volatility shocks. The $3rd$-moment swap is linked to the realized skewness and offers protection against changes in the symmetry of the underlying distribution. Changes in the tail behavior of the underlying distribution created by the occurrences of unexpected large jumps are shielded by the $4th$-moment swap related to the realized kurtosis. If the futures price is the state variable driving the moment derivatives, then the payoff function of the mth-moment swap on the futures is:

$$MS_F^m = NA\sum_{j=1}^{n}\ln\left(\frac{F_j}{F_{j-1}}\right)^m. \tag{2}$$

Using the link between the stock and futures prices, $F_j = P_j\, exp(\alpha - y)(\Psi - t_j)$, and setting the notional amount NA equal to one, the relationship between the futures and the stock moment swaps is derived as follows:

$$\begin{aligned} MS_F^m &= \sum_{j=1}^{n}\left[-(\alpha-y)\Delta t + \ln\left(\frac{P_j}{P_{j-1}}\right)\right]^m = \sum_{j=1}^{n}\sum_{h=0}^{m}\binom{m}{h}\left(-(\alpha-y)\ \Delta t\right)^h \ln\left(\frac{P_j}{P_{j-1}}\right)^{m-h} \\ &= \sum_{h=0}^{m-2}\binom{m}{h}\left(-(\alpha-y)\ \Delta t\right)^h MS_P^{(m-h)} - (\alpha-y)\Psi\left(-(\alpha-y)\ \Delta t\right)^{m-1} \\ &\quad + m\left(-(\alpha-y)\ \Delta t\right)^{m-1}\ \Omega_P \end{aligned}$$

where $\Omega_P = \sum_{j=1}^{n}\ln\left(\frac{P_j}{P_{j-1}}\right) = \ln\left(\frac{P_\Psi}{P_0}\right) = \ln(P_\Psi) - \ln(P_0)$.

The term Ω_P is the *log-contract* on the stock and plays a critical role in the hedging of moment swaps. If Δt raised to the higher order powers is very small and therefore negligible, the above listed expression can be approximated by equation (3) below:

$$MS_F^m \approx MS_P^m - m(\alpha - y)\ \Delta t MS_P^{(m-1)} \tag{3}$$

3. Hedging moment swaps driven by futures contracts

The focus is on hedging the moment swaps driven by the futures price as the state variable or underlying asset. Following Carr and Lewis (2004), as well as Schoutens (2005), a new expression for $\ln\left(\frac{F_j}{F_{j-1}}\right)^m$ can be derived by combining the Taylor series expansion of the mth power of the logarithm function and the power series representation of the exponential function. Most specifically, we have:

$$\ln\lambda^m = m!\left(\lambda - 1 - \ln\lambda - \frac{\ln\lambda^2}{2!} - \frac{\ln\lambda^3}{3!} - \ldots - \frac{\ln\lambda^{m-1}}{(m-1)!} + \xi\left((\lambda - 1)^{m+1}\right)\right) \tag{4}$$

$$\text{and} \exp(\beta) = e^\beta = 1 + \beta + \frac{\beta^2}{2!} + \ldots + \frac{\beta^m}{m!} + \xi\left(\beta^{m+1}\right) \tag{5}$$

Equation (4) is obtained from equation (5) by replacing β by $\ln(\lambda)$, rearranging terms and considering the fact that $\xi(\ln \lambda^{m+1}) = \xi((\lambda - 1)^{m+1})$.

Replacing λ by $\frac{F_j}{F_{j-1}}$ in equation (4) leads to:

$$\ln\left(\frac{F_j}{F_{j-1}}\right)^m = m!\left(\frac{\Delta F_j}{F_{j-1}} - \ln\frac{F_j}{F_{j-1}} - \sum_{h=2}^{m-1}\frac{\ln\left(\frac{F_j}{F_{j-1}}\right)^h}{h!} + \xi\left(\left(\frac{\Delta F_j}{F_{j-1}}\right)^{m+1}\right)\right),$$

where $\Delta F_j = F_j - F_{j-1}$. Hence, with a notional amount of one, the moment swap given in equation (2) takes the following new expression:

$$\mathrm{MS}_F^m = \sum_{j=1}^{n}\ln\left(\frac{F_j}{F_{j-1}}\right)^m$$

$$\mathrm{MS}_F^m = m!\sum_{j=1}^{n}\frac{\Delta F_j}{F_{j-1}} - m!(\ln F_\Psi - \ln F_0) - \sum_{h=2}^{m-1}\frac{m!}{h!}MS_F^h + \xi\left(\sum_{j=1}^{n}\left(\frac{\Delta F_j}{F_{j-1}}\right)^{m+1}\right). \tag{6}$$

Equation (6) shows that the payoff function of the mth-moment swap on the futures can be decomposed into:

- the payoff from $-m!$ number of log-contracts on the futures;
- the payoff from a dynamic trading strategy in futures given by $m!\sum_{j=1}^{n}\frac{\Delta F_j}{F_{j-1}}$; and
- the payoff from a series of moment swaps of order strictly lower than m.

Such decomposition is an approximation because it is derived from the Taylor series expansion of the mth power of the logarithm function and the power series representation of the exponential function which are both approximations.

The log-contract on the futures, Ω_F, is given by $\Omega_F = \ln F_\Psi - \ln F_0 = \Omega_P - (\alpha - y)\Psi$. Carr and Lewis (2004) shows that this log-contract can be hedged by taking a static position in bonds and vanilla options.

Equation (6) leads to the classical hedge of the variance swap in terms of the log-contracts on the futures when $m = 2$, namely:

$$\mathrm{V}S_F = MS_F^2 \approx 2\left(\sum_{j=1}^{n}\frac{\Delta F_j}{F_{j-1}} - \ \Omega_F\right) \tag{7}$$

To hedge the variance swap requires taking a short position in two log-contracts and a dynamic strategy in futures. The dynamic strategy can be executed at a zero cost.

For $m = 3$, equation (6) simplifies to:

$$\mathrm{MS}_F^3 \approx 6\left(\sum_{j=1}^{n}\frac{\Delta F_j}{F_{j-1}} - \ \Omega_F\right) - 3VS_F.$$

The variance swap on futures can now be derived as given in equation (8) below:

$$\mathrm{V}S_F \approx 2\left(\sum_{j=1}^{n}\frac{\Delta F_j}{F_{j-1}} - \ \Omega_F\right) - \frac{1}{3}\mathrm{MS}_F^3. \tag{8}$$

Comparing equations (7) and (8) leads to the conclusion that taking an additional short position in 1/3 third moment swaps leads to an improvement of the hedging strategy of the variance swap.

4. The dynamics of some advanced models

To truly evaluate the effectiveness of the hedging strategies, the performance of the moment swap hedges must be compared to that of some advanced models' hedges. The Heston model and the Levy models with stochastic time and SV will be examined. These two classes of advanced models offer very acceptable global calibration fits and are well known and used in the world of derivatives practitioners. The dynamics of each advanced model is explicitly given by its characteristic function $\varphi(\omega,\ t)$ of the logarithm of the price process ln P_t, namely, by $\varphi(\omega,\ t) = E[exp(i\omega \ln(P_t))]$.

4.1 The Heston stochastic volatility model

The Heston SV model has a state variable (the stock price) process that follows the Black and Scholes (1973) stochastic differential equation, that is:

$$\mathrm{d}P_t/P_t = (\alpha - y)dt + \sigma_t dW_t, \quad P_0 \geq 0,$$

with the squared volatility driven by the Cox *et al.* (1985) process (Cox, Ingersoll and Ross [CIR] process):

$$\mathrm{d}\sigma_t^2 = \mathcal{K}\left(\delta - \sigma_t^2\right)dt + \beta\,\sigma_t dZ_t, \quad \sigma_0 \geq 0,$$

where $\mathrm{d}W_t$ and $\mathrm{d}Z_t$, $t \geq 0$, are increments to two correlated standard Brownian motions with $\mathrm{Cov}(dW_t,\ dZ_t) = \rho dt$.

The characteristic function $\varphi(\omega,\ t)$ for this advanced model is provided in Heston (1993) and in Bakshi *et al.* (1997) as follows:

$$\varphi(\omega,t)=E\left[exp(i\omega\ln(P_t))|P_0,\sigma_0^2\right]=exp\left(i\omega\left(\ln P_0+(\alpha-y)t\right)\right)(A)(B),\text{ with}$$
$$A=\exp\left(\delta\mathcal{K}\beta^{-2}\left((\mathcal{K}-\rho\beta\omega i-x)t-2\ln\left((1-ze^{-xt})/(1-z)\right)\right)\right),$$
$$B=\exp\left(\sigma_0^2\beta^{-2}(\mathcal{K}-\rho\beta i\omega-x)(1-e^{-xt})/(1-ze^{-xt})\right),$$

where

$$x=\left((\rho\beta\omega i-\mathcal{K})^2-\beta^2(-i\omega-\omega^2)\right)^{1/2} \quad (9)$$

$$z=(\mathcal{K}-\rho\beta\omega i-x)/(\mathcal{K}-\rho\beta\omega i+x). \quad (10)$$

4.2 Levy models with stochastic time and stochastic volatility

The BS model has well known limitations because its underlying source of randomness is a Brownian motion. One practical way to improve this model is to replace the Brownian motion by a Levy process. Some of these Levy models can also incorporate SV.

Two independent stochastic processes are used to build the Levy models with stochastic time considered here. This approach was initiated by Mandelbrot and Taylor (1967). The asset price dynamics is modeled by the exponential of the Levy process, Y = $\{Y_t,\ t\geq 0\}$ preferably time-changed and by a second process, a stochastic clock, which incorporates SV allowing time to be stochastic.

In t calendar time units, the economic time that has passed is given by the integral Γ_t of a positive process $\gamma=\{\gamma_t, t\geq 0\}$ such that $\Gamma_t=\int_0^t\gamma_s ds$ and $\Gamma=\{\Gamma_t, t\geq 0\}$.

If the characteristic function of Γ_t, given γ_0, is $\phi(\omega;t,\ \gamma_0)$, then the price process in the risk neutral world P = $\{P_t, t\geq 0\}$ can be modeled as:

$$P_t=P_0\frac{exp\left((\alpha-y)t\right)}{E\left[exp(Y_{\Gamma_t})|\gamma_0\right]}\exp(Y_{\Gamma_t}) \quad (11)$$

where $Y=\{Y_t,\ t\geq 0\}$ is a Levy process. The stock price process is modeled as the exponential of a time-changed Levy process which incorporates jumps and SV. The Levy process Y_t picks up the jumps while the SV is accounted for by the time change Γ_t. The characteristic function $\varphi(\omega,\ t)$ for the logarithm of the stock price is:

$$\varphi(\omega,t)=E\left[exp(i\omega\ln(P_t))|P_0,\ \gamma_0\right]$$
$$=exp\left(i\omega\left((\alpha-y)t+\ln P_0\right)\right)\frac{\phi\left(-i\ \Delta_Y(\omega);\quad t,\quad \gamma_0\right)}{\phi\left(-i\ \Delta_Y(-i);\quad t,\quad \gamma_0\right)^{i\omega}} \quad (12)$$

where $\Delta_Y(\omega)=\ln E\left[exp(i\omega Y_t)\right]$ is named the characteristic exponent of the Levy process. The subparagraphs that follow will examine:

- the Cox *et al.* (1985) process (CIR process) for γ; and
- two Levy processes which are the normal inverse Gaussian (NIG) process and the variance gamma (VG) process.

4.2.1 The normal inverse Gaussian levy process. The NIG process is derived from the NIG distribution or NIG($\eta,\ \mu,\ \vartheta$), with parameters $\eta>0$, $-\eta<\mu<\eta$, and $\vartheta>0$. The characteristic function is given as follows:

$$\varphi_{NIG}(\omega; \eta, \mu, \vartheta) = exp\left(-\vartheta\left(\sqrt{\eta^2 - (\mu + i\omega)^2} - \sqrt{\eta^2 - \mu^2}\right)\right).$$

This characteristic function is infinitely divisible and therefore can help define the NIG process $Y^{(NIG)} = \left\{Y_t^{(NIG)}, \quad t \geq 0\right\}$, with $Y_0^{(NIG)} = 0$, because the process has stationary and independent NIG distributed increments. Each increment satisfies the NIG$(\eta, \ \mu, \ \vartheta t)$ law over the time slot $[k, k+t]$.

4.2.2 The VG levy process. Madan *et al.* (1998) defines the characteristic function of the VG(*B, E, N*) distribution with parameters $B > 0$, $E > 0$ and $N > 0$ as follows:

$$\varphi_{VG}(\omega; B, E, N) = \left(\frac{EN}{EN + (N - E)i\omega + \omega^2}\right)^B.$$

Because this distribution is infinitely divisible, the VG-process can be defined as the one which begins at time zero with independent and stationary increments. The VG-process $Y^{(VG)} = \left\{Y_t^{(VG)}, \quad t \geq 0\right\}$ has increments $Y_{k+t}^{(VG)} - Y_k^{(VG)}$ that satisfy the VG(*Bt, E, N*) law over the time slot $[k, k+t]$.

4.2.3 The Cox, Ingersoll and Ross stochastic clock. The CIR process that solves the stochastic differential equation listed below is used as the rate of time change by Carr *et al.* (2003):

$$\mathrm{d}\gamma_t = \mathcal{K}(\delta - \gamma_t)dt + \epsilon\gamma_t^{1/2}\mathrm{d}Z_t,$$

where $\mathrm{d}Z_t$ is an increment to a standard Brownian motion, the parameter δ is the long run rate of time change, $\mathcal{K}$ is the rate of mean reversion and ϵ dictates the volatility of the time change. The process γ_t is the instantaneous rate of time change and the new clock is given by its integral $\Gamma_t = \int_0^t \gamma_s ds$. The characteristic function of Γ_t, given γ_0, is explicitly provided in Cox *et al.* (1985) as follows:

$$\begin{aligned}\phi_{CIR}(\omega; t, \mathcal{K}, \delta, \epsilon, \gamma_0) &= E\left[exp(i\omega \ \Gamma_t)|\gamma_0\right] \\ &= \frac{exp\left(\mathcal{K}^2\delta t/\epsilon^2\right)exp(2\gamma_0 i\omega/(\mathcal{K} + \lambda\coth(\lambda t/2)))}{(\cosh(\lambda t/2) + \mathcal{K}\sinh(\lambda t/2)/\lambda)^{2\mathcal{K}\delta/\epsilon^2}}\end{aligned}$$

where $\lambda = \sqrt{\mathcal{K}^2 - 2\epsilon^2 i\omega}$.

For the new clock to be increasing, the instantaneous rate of time change must be positive. Moreover, for the random time changes to persist, the rate of time change must be mean reverting. The square root process of CIR considered here is a classic example of a mean-reverting positive process.

5. The log-contract

The log-contract on the stock and the log-contract on the futures were already defined in Sections 2 and 3, respectively. Further, it was established that the log-contract plays a critical role in the hedging of moment swaps. Closed form solutions for the price of the log-contract are achieved under many models. For example, under the BS model, the log-contract price on the stock is given by:

$$\Omega_P = exp(-\alpha\Psi)E_Q\left[\ln(P_\Psi) - \ln P_0|\mathcal{T}_0\right] = \exp(-\alpha\Psi)E_Q\left[\left(\alpha - y - \sigma^2/2\right)\Psi + \sigma Z_\Psi|\mathcal{T}_0\right],$$

$$\Omega_P = \exp(-\alpha\Psi)\left(\alpha - y - \sigma^2/2\right)\Psi,$$

where σ is the volatility of the stock price P.

The log-contract price for more elaborate models can be directly obtained from the risk-neutral characteristic function of $\ln(P_\Psi)$, that is:

$$\varphi(\omega,t) = E_Q\left[exp(i\omega \ln P_\Psi)|\mathcal{T}_0\right].$$

Taking the partial derivative of $\varphi(\omega, t)$ with respect to ω leads to:

$$-i\frac{\partial\varphi(\omega,t)}{\partial\omega}|_{\omega=0} = E_Q\left[\ln(P_\Psi)|\mathcal{T}_0\right].$$

Therefore, the general expression for the price of the log-contract is given by:

$$\Omega_P = exp(-\alpha\Psi)E_Q\left[\ln(P_\Psi) - \ln P_0|\mathcal{T}_0\right] = exp(-\alpha\Psi)\left(-i\frac{\partial\varphi(0,t)}{\partial\omega} - \ln P_0\right).$$

In the case of the Heston model, the log-contract price on the stock takes the following expression:

$$\Omega_P^{HESTON} = \frac{exp(-\alpha\Psi)}{2\mathcal{K}}\left(2\mathcal{K}(\alpha-y)\Psi - \delta\mathcal{K}\Psi - \delta e^{-\mathcal{K}\Psi} + \delta - \sigma_0^2 + \sigma_0^2 e^{-\mathcal{K}\Psi}\right), \tag{13}$$

while in the case of the time-changed Levy models with a characteristic function $\varphi(\omega, t)$ shown in equation (12), the log-contract price on the stock is given by:

$$\begin{aligned}\Omega_P^{LM} &= \exp(-\alpha\Psi)\left(\Psi(\alpha-y) - \phi'(0)\Delta_Y'(0) - \ln\left(\phi\left(-i\ \Delta_Y(i)\right)\right)\right)\\ &= \exp(-\alpha\Psi)\left(\Psi(\alpha-y) + E[\ \Gamma_\Psi]E[Y_1] - \ln(E[exp(Y_{\Gamma_\Psi})])\right).\end{aligned} \tag{14}$$

6. Parameters estimation and results

The parameters to be estimated are those related to the Heston model, the NIG Levy process with CIR stochastic clock (NIG-L-CIR) and the VG Levy process with CIR stochastic clock (VG-L-CIR). For the Heston model, the parameters are $\sigma_0^2, \mathcal{K}, \delta, \beta$ and ρ. The parameters are $\eta, \mu, \vartheta, \mathcal{K}, \delta, \epsilon$ and γ_0 for the NIG-L-CIR model. Finally, the parameters for the VG-L-CIR model are $B, E, N, \mathcal{K}, \delta, \epsilon$, and γ_0. The calibration of the models follows the methods developed in Carr and Madan (1998) for pricing classical vanilla options. In general, these methods can be used when the characteristic function of the risk-neutral stock price process is well specified.

Carr and Madan (1998) shows that the price $C_E(E, \Psi)$ of a European call option with exercise price E and maturity date Ψ is given by:

$$C_E(E,\Psi) = \frac{exp(-\mathcal{E}\ln(E))}{\pi}\int_0^{+\infty} exp\left(-i\chi\ln(E)\right)g(\chi)d\chi \tag{15}$$

where ε is a positive constant such that the εth moment of the stock price exists and the function $g(\chi)$ is defined by:

$$g(\chi) = \frac{exp(-\alpha\Psi)E\left[exp\left(i\left(\chi-(\mathcal{E}+1)i\right)\ln(P_\Psi)\right)\right]}{\mathcal{E}^2+\mathcal{E}-\chi^2+i(2\mathcal{E}+1)\chi} = \frac{exp(-\alpha\Psi)\varphi\left(\chi-(\mathcal{E}+1)i,\Psi\right)}{\mathcal{E}^2+\mathcal{E}-\chi^2+i(2\mathcal{E}+1)\chi}, \tag{16}$$

with P being the stock price and α the compound interest rate defined earlier. The complete option surface can be computed using the fast Fourier transforms. This Carr and Madan computation methodology is applied in the calibration procedure to estimate the model parameters by minimizing the price differential (market observed price minus model price) in a least-squares configuration.

The sample data used consists of 432 plain vanilla call option prices with maturities from less than one month to 5.165 years. These prices are taken from the volatility surface of the euro Stoxx 50 Index. The models were tested on August 25, 2017, when the value of the index was €3,438.55. The risk-free interest rate is set at 3 per cent and the dividend yield at 0 per cent for the sake of simplicity and to focus on the essence of the stochastic behavior of the asset. The calibration procedure is like the one in Schoutens *et al.* (2004). The risk-neutral parameters for all three models are reported below:

$$\textit{Heston Model}: \quad \sigma_0^2 = 0.0648; \ \mathcal{K} = 0.6372; \ \delta = 0.0681; \ \beta = 0.2889; \ \rho = -0.7783$$

$$\begin{aligned} \textit{NIG} - L - \textit{CIR Model}: \quad & \eta = 16.5571; \quad \mu = -3.8402; \quad \vartheta = 1.1739; \\ & \mathcal{K} = 1.2709; \quad \delta = 0.5304 \quad \epsilon = 1.8013; \quad \gamma_0 = 1 \end{aligned}$$

$$\begin{aligned} \textit{VG} - L - \textit{CIR Model}: \quad & \mathrm{B} = 19.1064; \quad \mathrm{E} = 21.1449; \quad \mathrm{N} = 27.8697; \\ & \mathcal{K} = 1.2755; \quad \delta = 0.5298 \quad \epsilon = 1.8062; \quad \gamma_0 = 1 \end{aligned}$$

The out-of-sample performance of an option pricing model is measured by the stability of the parameters and accuracy of the parameter estimates. Testing the out-of-sample performance of these three models is in effect a test of how slowly the parameters change over time. The pricing biases of these models are computed and contrasted. These pricing biases are captured through four global measures of fit. These global measures of fit are:

(1) the root mean square pricing error (RMSPE);
(2) the mean absolute percentage pricing error (MAPPE);
(3) the mean absolute pricing error (MAPE); and
(4) the mean relative percentage pricing error (MRPPE).

These global measures of fit are defined as follows:

$$\mathrm{RMSPE} = \left(\sum_{i=1}^{n} \frac{\left(MP_i - \widehat{MP_i} \right)^2}{n} \right)^{1/2}$$

$$\mathrm{MAPPE} = \frac{1}{\overline{MP}} \sum_{i=1}^{n} \frac{|MP_i - \widehat{MP}_i|}{n}$$

$$\mathrm{MAPE} = \sum_{i=1}^{n} \frac{|MP_i - \widehat{MP}_i|}{n}$$

$$\mathrm{MRPPE} = \frac{1}{n} \sum_{i=1}^{n} \frac{|MP_i - \widehat{MP}_i|}{MP_i},$$

where n is the number of options, MP_i the market price of option i, $\widehat{MP}_i$ the model price of option i and $\overline{MP}$ the mean option price. The computations of these pricing errors are listed in Table I.

The prices of the log-contracts are computed from equations (13) and (14) using the parameters given above and are listed in Table II.

The pricing biases of the Heston model are compared to those of the NIG-L-CIR model and those of the VG-L-CIR model using the RMSPE, the MAPPE, the MAPE and the MRPPE statistics. These statistics are computed in Table I. Under the RMSPE measure, the NIG-L-CIR model performs better than the VG-L-CIR model which in turn performs better than the Heston model. Using the MAPPE measure, the NIG-L-CIR and the VG-L-CIR models show the same performance which is superior to that of the Heston model. Under the MAPE statistic, the NIG-L-CIR model outperforms the VG-L-CIR model which in turn outperforms the Heston model. Finally, the pricing errors obtained under the MRPPE statistic confirm the superior performance of the NIG-L-CIR model over that of the VG-L-CIR model and over that of the Heston model.

As indicated earlier, the prices listed in Table II are taken from the volatility surface of the euro Stoxx 50 Index. All models appear to fit this surface very well leading to almost similar vanilla prices under the different models. However, the fact that model risk does exist and is different from model to model explains why exotic prices under the various models can be significantly different. This difference is illustrated in Table II where the log-contract prices differ by 48.82 per cent between the Heston and NIG-L-CIR models and by 48.73 per cent between the Heston and VG-L-CIR models.

Moneyness, maturity and volatility effects on pricing bias can be further investigated using a regression analysis. The dependent variable is the percentage pricing error of a given call in the sample at a given date. The independent variables are the moneyness, the time to maturity and the volatility of the euro Stoxx index return. The regression equation is given by:

$$PPE_i(t) = a_0 + a_1 \frac{P(t)}{E_i} + a_2 TM_i + a_3 VOL(t-1) + \mathcal{O}_i(t), \quad i = 1, \ldots, n \tag{17}$$

where $PPE_i(t)$ is the percentage pricing error of call option i on date t; P/E_i and TM_i represent the moneyness and time to maturity, respectively, of the call option contract; $VOL(t-1)$ stands for the previous day's annualized standard deviation of the euro Stoxx 50 index return; and $\mathcal{O}_i(t)$ is the error term. This is a cross-sectional regression; therefore, the standard errors (in parentheses) are computed using the White (1980) heteroskedasticity consistent estimator. The regression is run for each of the three models considered and the results are summarized in Table III.

Model	RMSPE	MAPPE	MAPE	MRPPE
HESTON	2.9463	0.0042	2.3974	0.0169
NIG-L-CIR	2.2851	0.0033	1.8965	0.0096
VG-L-CIR	2.3179	0.0033	1.9106	0.0103

Table I. Overall pricing error measures

Model	Ω_P *times* 10,000
HESTON	−31.97
NIG-L-CIR	−47.58
VG-L-CIR	−47.55

Table II. Prices of the log-contract on the stock (Ω_P)

Each independent variable has statistically significant explanatory power of the remaining pricing errors regardless of the model examined. Hence, each model pricing errors have some maturity, intra-daily volatility and moneyness related biases with different magnitudes. The pricing errors have the same sign and therefore are biased in the same direction. The pricing errors relative to the euro Stoxx index's volatility on the previous day are almost stationary, confirming the importance of modeling SV. The mispricing of the NIG-L-CIR and VG-L-CIR models is lower than that of the Heston model, confirming that modeling both SV and jumps is important. The adjusted R^2 of 8.7 per cent for the Heston model, 4.6 per cent for the NIG-L-CIR model and 4.8 per cent for the VG-L-CIR model indicate that the collective explanatory power of these independent variables is quite low. The regression results confirm the results in Table I, that the NIG-L-CIR model is superior to the VG-L-CIR model which in turn is superior to the Heston model in terms of pricing accuracy.

7. Effectiveness of the variance swap hedge using moment swaps

To assess the effectiveness of the variance swap hedge using moment swaps, this paper will contrast the strategy that includes a third moment swap outlined in equation (8) with the one which excludes moment swaps as described in equation (7). This comparison is made for the different models discussed in Section 4. The parameters used are still the same listed earlier.

Exactly 20,000 variance swaps contracts are hedged. The profit and loss distributions of these hedges under the various models examined are obtained by running a Monte Carlo simulation with one million scenarios. To simulate a NIG process, a NIG($\eta, \ \mu, \ \vartheta$) random numbers are first simulated. The NIG random numbers are obtained by combining inverse Gaussian (IG) random numbers and standard normal numbers. An IG($c, \ d$) random variable Y has the following characteristic function:

$$E[exp(i\omega Y)] = exp\left(-c\sqrt{-2\omega i + d^2} - d\right).$$

First, the IG$\left(1, \vartheta\sqrt{\eta^2 - \mu^2}\right)$ random numbers i_m are simulated using for instance the IG generator of Michael, Schucany and Haas [Devroye (1986)]. Next, a series of standard normal random variables ω_m are sampled. The NIG random numbers n_m are then generated using the expression below:

$$n_m = \vartheta^2 \mu i_m + \vartheta\sqrt{i_m}\omega_m.$$

Finally, the sample paths of a NIG($\eta, \ \mu, \ \vartheta$) process $Y = \{Y_t, \ t \geq 0\}$ in the time slots $t_n = n\Delta t, \ n = 0, 1, 2, \ldots$ are generated using the independent NIG($\eta, \ \mu, \ \vartheta\Delta t$) random numbers n_m given by:

$$Y_0 = 0, \quad Y_{t_m} = Y_{t_{m-1}} + n_m, \quad m \geq 1.$$

Coefficients	Heston model	NIG-L-CIR model	VG-L-CIR model
Intercept	*0.214* (0.011)	*0.143* (0.011)	*0.146* (0.011)
P/E	*−0.172* (0.011)	*−0.114* (0.011)	*−0.116* (0.011)
TM	*0.061* (0.001)	*0.052* (0.001)	*0.054* (0.001)
VOL	*0.033* (0.011)	*0.021* (0.011)	*0.022* (0.011)
Adj. R^2	*0.087*	*0.046*	*0.048*

Note: The standard errors are in parentheses

Table III.
Regression analysis of pricing errors

The simulation of a VG process is straightforward because a VG process can be seen as the difference of two independent Gamma processes. Gamma numbers are generated using the Johnk's gamma generator or the Berman's gamma generator [Devroye (1986)].
The simulation of a CIR process $\gamma = \{\gamma_t, t \geq 0\}$ is also straightforward by discretizing the stochastic differential equation:

$$d\gamma_t = \mathcal{K}(\delta - \gamma_t)dt + \epsilon \gamma_t^{1/2} dZ_t, \ \gamma_0 \geq 0.$$

The sample path of the CIR process $\gamma = \{\gamma_t, t \geq 0\}$ in the time slots $t = n\Delta t$, $n = 0, 1, 2, \ldots$ is given by:

$$\gamma_{t_n} = \gamma_{t_{n-1}} + \mathcal{K}\left(\delta - \gamma_{t_{n-1}}\right) \Delta t + \epsilon \gamma_{t_{n-1}}^{1/2} \sqrt{\Delta t} \chi_n,$$

where $\{\chi_n, n = 1, 2, \ldots\}$ is a series of independent standard normally distributed random numbers.

The means and standard deviations of the profit and loss distributions for each of the three models are computed both under a hedging strategy without moment derivatives and under a hedging strategy with moment derivatives. The mean and standard deviation of the hedging without moment derivatives are respectively $\overline{m}_1$ and σ_1 while $\overline{m}_2$ and σ_2 are the same descriptive statistics of the hedging that uses a third moment swap. All these descriptive statistics are listed in Table IV.

Table IV clearly shows that the mean of the profit and loss distribution is closer to zero and its standard deviation is much smaller under the hedge with a third moment swap. Hence, consistent with the theory, the empirical results illustrated in Table IV confirm that the use of a third moment swap in hedging a variance swap dramatically improves the performance of the hedge. The tails of the basic hedge without the third moment swap are much fatter than those under the enhanced hedge, which incorporates the third moment swap. Under the basic hedge without the third moment swap, the standard deviation of the profit and loss distribution for the Heston model is already very low. This is justified by the fact that the Heston model exhibits continuous paths. The Taylor series expansion of the mth power of the logarithm function given by equation (4) is very accurate in a no-jump economy even if limited to the quadratic term. This is not the case in real markets where asset prices exhibit jumps of various magnitudes due of supply and demand shocks. These jumps are the main sources of large hedging errors.

8. Conclusions

This paper tests empirically the pricing accuracy of the Schoutens (2005) along with the Heston (1993) and the time-changed Levy models and their effectiveness in hedging moment swaps. Variance swaps which are now liquidly traded can be used to protect against changes in the volatility regime. Moments of higher order such as skewness and kurtosis are important, for example, in a jump economy. To protect against inaccurate estimates of

Model	$\overline{m}_1$	σ_1	$\overline{m}_2$	σ_2
HESTON	0.059	2.108	−0.0638	0.0051
NIG-L-CIR	6.341	24.893	−0.7252	2.8184
VG-L-CIR	6.101	18.867	−0.6609	1.4736

Table IV. Profit and loss descriptive statistics skewness and kurtosis, moment derivatives are needed. The results obtained here show that using moment swaps leads to a significant improvement of the standard hedge of the variance swap using only log-contracts. Each hedge is set up as a position in a log-contract combined with a dynamic trading strategy and a static position in moment swaps of lower order. Most specifically, SV models such as the Heston and time-changed Levy models are introduced and closed-form solutions for the price of the log-contract under these models are derived from the risk-neutral characteristic function. Then, the effectiveness of the variance swap hedge using a third-order moment swap is evaluated. The result shows that taking a position in third moment swaps can significantly improve the performance of the standard hedge of a variance swap based on a static position in the log-contract and a dynamic trading strategy. The position in third moment swaps is taken by running a Monte Carlo simulation under the various SV models considered. Overall, the pricing accuracy of the three SV models considered, measured by RMSPE, MAPPE, MAPE and MRPPE shown in Table I, is very satisfactory. A possible extension of this research would be to examine how effective these models are in pricing and hedging moment swaps of order n with $n \geq 3$, using moment derivatives of any order.

References

Bakshi, G., Cao, C. and Chen, Z. (1997), "Empirical performance of alternative option pricing models", *Journal of Finance*, Vol. 52 No. 5, pp. 2003-2049.

Bates, D.S. (1996), "Jumps and stochastic volatility: exchange rate processes implicit in deutsche mark options", *Review of Financial Studies*, Vol. 9 No. 1, pp. 69-107.

Black, F. and Scholes, M. (1973), "The pricing of options and corporate liabilities", *Journal of Political Economy*, Vol. 81 No. 3, pp. 637-654.

Carr, P. and Lewis, K. (2004), "Corridor variance swaps", *Risk*, Vol. 17, pp. 67-72.

Carr, P. and Madan, D. (1998), "Option valuation using the fast Fourier transform", *The Journal of Computational Finance*, Vol. 2 No. 4, pp. 61-73.

Carr, P., Geman, H., Madan, D. and Yor, M. (2003), "Stochastic volatility for levy processes", *Mathematical Finance*, Vol. 13 No. 3, pp. 345-382.

Cheang, G., Chiarella, C. and Mina, K. (2015), "Approximate hedging of options under jump-diffusion processes", *International Journal of Theoretical and Applied Finance*, Vol. 18, pp. 1-26.

Coleman, T., Forsyth, P., He, C., Kennedy, J., Lee, Y. and Vetzal, K. (2006), "Calibration and hedging under jump diffusion", *Review of Derivatives Research*, Vol. 9, pp. 1-35.

Corcuera, J.M., Nualart, D. and Schoutens, W. (2005), "Completion of a levy market by power-jump assets", *Finance and Stochastics*, Vol. 9 No. 1, pp. 109-127.

Cox, J., Ingersoll, J. and Ross, S. (1985), "A theory of the term structure of interest rates", *Econometrica*, Vol. 53 No. 2, pp. 385-408.

Devroye, L. (1986), *Non-Uniform Random Variate Generation*, Springer-Verlag, New York, NY.

Doffou, A. and Hilliard, J. (2001), "Pricing currency options under stochastic interest rates and jump-diffusion processes", *Journal of Financial Research*, Vol. 24 No. 4, pp. 565-585.

Heston, S. (1993), "A closed-form solution for options with stochastic volatility with applications to bond and currency options", *Review of Financial Studies*, Vol. 6 No. 2, pp. 327-343.

Madan, D., Carr, P. and Chang, E. (1998), "The variance gamma process and option pricing", *Review of Finance*, Vol. 2 No. 1, pp. 79-105.

Mandelbrot, B. and Taylor, H. (1967), "On the distribution of stock price differences", *Operations Research*, Vol. 15 No. 6, pp. 1057-1062.

Rompolis, L. and Tzavalis, E. (2017), "Pricing and hedging contingent claims using variance and higher order moment swaps", *Quantitative Finance*, Vol. 17 No. 4, pp. 531-550.

Schoutens, W. (2005), "Moment swaps", *Quantitative Finance*, Vol. 5 No. 6, pp. 525-530.

Schoutens, W., Simons, E. and Tistaert, J. (2004), "A perfect calibration! Now what?", *Wilmott Magazine*, Vol. 2004 No. 2, pp. 66-78.

Tankov, P., Cont, R. and Voltchkova, E. (2007), *Hedging with Options in Models with Jumps*, Springer-Verlag, Berlin Heidelberg.

Utzet, F., Leon, J., Sole, J. and Vives, J. (2002), "On levy processes: Malliavin calculus and market models with jumps", *Finance and Stochastics*, Vol. 6, pp. 197-225.

White, H. (1980), "A heteroskedasticity consistent covariance matrix estimator and a direct test for heteroskedasticity", *Econometrica*, Vol. 48 No. 4, pp. 817-838.

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