Inconsistency in intertemporal choice: a behavioral approach

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Abstract

Purpose – The purpose of this paper is to introduce the main measures of inconsistency in the context of intertemporal choice and to identify the relationships between them (more specifically, the measures by Prelec, Takahashi and Rohde). In effect, Thaler (1981), awarded the Nobel Prize in Economics 2017, argued that when a preference must be expressed between two reward options, some people may reverse their original preference when a significant delay is introduced before the reward is to be received. This anomaly is known as inconsistency in intertemporal choice.

Design/methodology/approach – After a revision of the existing literature and by using the methods from mathematical calculus, the authors have derived the logical relationships between the measures presented in this paper.

Findings – The main contribution of this paper is the proposal of a novel parameter, the so-defined ratio of two instantaneous discount rates, which the authors call the instantaneous variation rate, which allows relating some other measures of inconsistency, namely the measures described by Prelec and Rohde. A limitation of this paper is the unavailability of empirical information about the inconsistency measures needed to substantiate the theoretical findings. Indeed, this paper has social implications because recent behavioral and neuroeconomic studies have shown the existence of preference reversal or time inconsistency in other areas. The authors' models can be implemented in these fields in order to better analyze the situations of inconsistency.

Originality/value – The originality of this paper lies in the authors' aim to bring some order to the proposed measures of inconsistency which have arisen as a result of the different approaches adopted.

Keywords Preferences, Discount function, Impatience, Intertemporal choice, Time inconsistency

Paper type Research paper

1. Introduction

The process of intertemporal choice involves deciding between several alternatives whose monetary amounts or utilities take place at different moments in time. Decisions about savings, nutrition, exercise, and health are intertemporal choices. In economy, the study of intertemporal choice began in 1937 when Samuelson proposed his well-known Discounted Utility (DU) Model. The paradigm of the DU Model is exponential discounting which exhibits a constant instantaneous discount rate. This is based on the assumption that the behavior of people with respect to the choice does not change with the passage of time.

However, the latest behavioral and neuroeconomic studies have shown the existence of several limitations of the DU Model. In effect, a key concept in intertemporal choice is impatience which has been defined by different authors as a synonym of impulsivity. Thus, Takahashi *et al.* (2012) defined impatience as a strong preference for small immediate rewards over large delayed ones. But another question is the situation in which the subject changes his/her choice when the initial offered rewards are delayed over the same period of time. For instance, Thaler (1981) argued that some people may prefer one apple today over two apples tomorrow but, at the same time, they may prefer two apples in one year plus one

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day over one apple in one year. Another situation is described by the following example (Takeuchi, 2012) in which a person is requested to respond to the following two questions:

Q1. Which of the following reward options do you prefer?

(1) \$100 paid today.

(2) \$110 paid in 1 week.

Q2. Which of the following reward options do you prefer?

(3) \$100 paid in 52 weeks.

(4) \$110 paid in 53 weeks.

Table I summarizes the pairs of possible answers and the description of behavior.

Observe that, if subjects prefer smaller sooner rewards in the distant future, but prefer larger later rewards in the near future, their intertemporal choices are inconsistent, because their preferences reverse as time passes. On the other hand, if someone expresses a preference for smaller sooner rewards when responding to the above questions, his/her intertemporal choice is impulsive but consistent, because the preference does not reverse over time.

In other words, people may exhibit inconsistency when making intertemporal choices. This behavior is characterized by a variable (mainly, increasing or decreasing) instantaneous discount rate. The limitations of the DU Model allowed for the development of some alternatives to exponential discounting such as quasi-hyperbolic discounting (Phelps and Pollak, 1968) and generalized hyperbolic discounting (Loewenstein and Prelec, 1992), which replaced a constant with a variable discount factor. Additionally, other intertemporal choice models have been proposed in order to describe the inconsistency (increasing and decreasing impatience) in actual human behavior in intertemporal decision making.

This paper considers the subject of intertemporal choice which is of increasing interest within the field of behavioral finance. More specifically, it is devoted to the analysis of several measures of the paradox or anomaly labeled as "time inconsistency." In effect, many procedures have been proposed to tackle this phenomenon. The aim of this paper is therefore to present all the previously proposed measures of time inconsistency, trying to find the relationships between them. For example, in this paper the well-known measures proposed by Prelec (2004) and Rohde (2010) will be considered by using the ratio of instantaneous variation introduced by Cruz and Muñoz (2001). Table II shows some of these measures.

Figure 1 summarizes the framework of this paper.

This paper is focused on the field of behavioral finance where the importance of time inconsistency has been highlighted by several authors. For example, Hardisty and Pfeffer (2017) showed the relationship between temporal uncertainty and the preference for immediate or future gains. In this context, we can also mention the research by Sedghi and Gerayli (2015) about the inconsistency existing in the absence of timely payment of delayed receivables. On the other hand, the results obtained by Imas (2016) demonstrate that people take less risk after a realized loss and more risk if it is a paper loss, and this increase is due to the inconsistency in preferences. Very relevant to the last reference, although outside the economic-financial context, we must mention the work by Gneezy *et al.* (2014) who studied

Table I. Decreasing impatience, increasing impatience and consistent behavior

	(3) \$100 paid in 52 weeks	(4) \$110 paid in 53 weeks
(1) \$100 paid today	Consistent and impatient	Inconsistent and decreasingly impatient
(2) \$110 paid in 1 week	Inconsistent and increasingly impatient	Consistent and patient

Source: Takeuchi (2012)

the intertemporal inconsistency observed in moral and immoral attitudes of people. In this way, they concluded that a moral choice increases because of the temporal increase in guilt induced by a previous immoral choice.

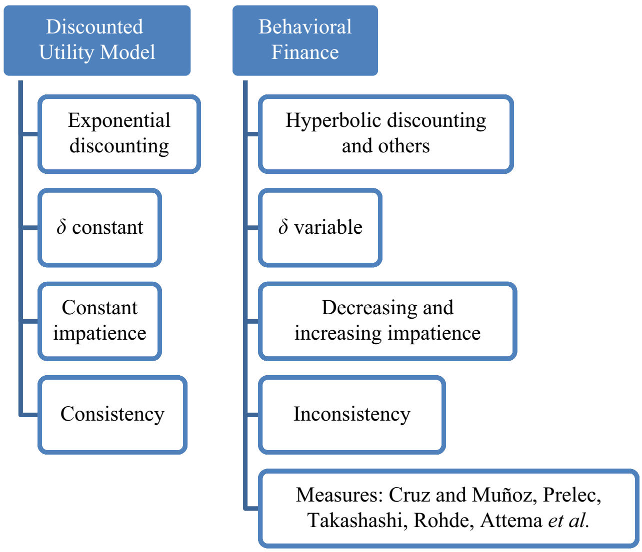
The issue of time inconsistency is not confined to behavioral finance. Thus, in the context of market strategies and marketing, we can highlight the work by Gilbert and Jonnalagedda (2011) who proposed manufacturers to make their durable products (e.g. printer) incompatible with contingent consumable product (e.g. ink) that are produced by other firms. This strategy forces consumers to reduce willingness to pay for the durable due to the higher consumables prices in the future. On the other hand, Feit *et al.* (2010) proposed a new model to describe the behavior of consumers when making decisions about the design and marketing of a given product.

This paper has been organized as follows. In the current section, we have contextualized this topic within the field of intertemporal choice and we have indicated the objectives we intend to achieve. In Section 2, we have introduced the concept of discount function as a necessary first step for the definition of impatience. In Section 3, we have introduced the concept of time inconsistency and some of the most relevant measures proposed by different authors. First, the instantaneous discount rate, denoted by $\delta(t)$, is a parameter whose behavior

Author	Measure of variation impatience
Cruz and Muñoz (2001)	$\frac{\delta(t)}{\delta(s)}$, with $s < t$
Prelec (2004)	Degree of convexity of $\ln F(t)$
Takahashi (2007)	$\delta_q'(t)$ (q-exponential discount function)
Rohde (2009)	$\frac{F''(t)}{F(t)}$
Rohde (2010)	Hyperbolic factor
Attema *et al.* (2010)	TTO sequence
Rohde (2015)	DI-index

Source: Authors' own elaboration

Table II. Several measures of impatience variation



Source: Authors' own elaboration

Figure 1. Discount utility model vs behavioral finance

(decrease or increase) is considered as a necessary and sufficient condition for inconsistency (decreasing or increasing impatience). In the same way, we propose the ratio of two instantaneous discount rates, which we call the instantaneous variation rate, denoted by $v(s,t)$, which will help us to relate some other measures of inconsistency. This section is divided into three subsections, one for each author under consideration: Prelec, Takahashi and Rohde, where the different parameters provided to measure the time inconsistency in intertemporal choice have been analyzed. Finally, Section 4 summarizes and concludes.

2. Impatience in intertemporal choice

The DU (Samuelson, 1937) is the classical model used in problems involving intertemporal choices, in which the present utility of a series of installments (hereinafter, called "stream" by using Rohde's terminology) is the sum of the discounted individual values over a given time horizon. It assumed that people discount future amounts with an exponential function, but Strotz (1956) demonstrated that they often display preference reversal (this concept will be seen in Section 3) as time passes, and so the exponential discount function fails to explain this fact. The necessity of modeling this change in individual time preference has made hyperbolic discounting an important tool in behavioral economics. Moreover, several types of hyperbolic discount functions have been introduced, such as quasi-hyperbolic discounting, proportional hyperbolic discounting and generalized hyperbolic discounting. In effect, most articles on behavioral neuroeconomics have used the simple hyperbolic discount function (Frederick *et al.*, 2002). However, Loewenstein and Prelec (1992) proposed the generalized hyperbolic discount function:

$$F(t) = \frac{1}{(1+kt)^{\alpha/k}}, \quad k>0, \quad \alpha>0,$$

which is similar to either the exponential function or the simple hyperbolic function, depending on the value of α. Recent papers have studied the q-exponential discount function whose parameter q indicates the deviation from exponential discounting. Table III summarizes some of the discount functions used in most papers (see Takahashi *et al.*, 2012).

At this point, it is necessary to introduce a general definition of discount function (Cruz and Muñoz, 2016) which indicates that, when a reward is subject to a delay, its present value decreases (Figure 2).

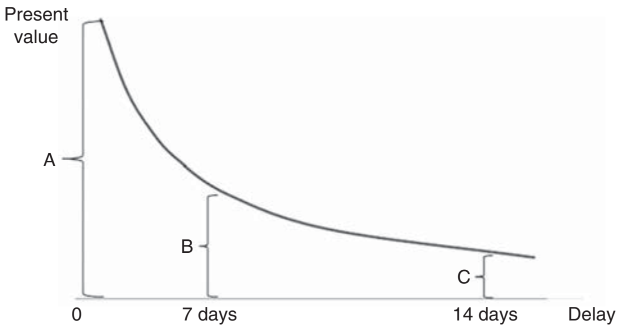
Definition 1. A stationary discount function $F(t)$ is a continuous real function $F: \mathfrak{R}^+ \to \mathfrak{R}^+$ such that $t \mapsto F(t)$, defined within an interval $[0,t_0[$ (t_0 can even be $+\infty$), where $F(t)$ represents the value at 0 of a \$1 reward available at instant t, satisfying the following conditions:

- $F(0) = 1$;
- $F(t) > 0$; and
- $F(t)$ is strictly decreasing.

Table III. Different discount functions alternative to the exponential

Discount Function	Equation
Simple exponential	$F(t) = \exp\{-kt\}$, $k > 0$
Simple hyperbolic	$F(t) = 1/(1 + kt)$, $k > 0$
q-exponential	$F(t) = 1/([1 + (1 - q)k_q t]^{1/(1-q)})$, $k_q > 0$

Source: Takahashi *et al.* (2012)



Source: Authors' own elaboration

Figure 2. Displaying a discount function

In order to formulate more accurately the DU model, an outcome stream (t_1: x_1, …, t_m: x_m) is defined as a sequence of dated amounts which yields outcome x_j at time point t_j, for $j = 1, \ldots, m$ and nothing at other time points. We assume $t_j \geqslant 0$ for all j. Time point $t = 0$ corresponds to the present moment. In this context, the DU is defined as (Rohde, 2015):

$$\mathrm{DU}(t_1 : x_1, \ldots, t_m : x_m) = \sum_{j=1}^{m} u(x_j) F\left(t_j\right),$$

where F is the discount function and u the (instant) utility function.

Alternatively, an intertemporal choice process can be treated by means of a preference relation $\succeq$ (at least as preferred as) over a set X of outcomes. Strict preference and indifference are denoted by $\succ$ and $\sim$, respectively. For two outcome streams x and y, we will write $x \succeq y$ (resp. $x \succ y$) if $u(x) \geqslant u(y)$ (resp. $u(x) > u(y)$), and $x \sim y$ if $u(x) = u(y)$.

The following statement (Cruz and Muñoz, 2016) relates the preference existing in an intertemporal choice and its associated discount function:

Theorem 1. A discount function $F(t)$ gives rise to the total preorder $\succeq$ defined by:

$$(t_1 : x_1) \sim (t_2 : x_2) \text{ if } x_1 F(t_1) \geqslant x_2 F(t_2),$$

satisfying the following conditions:

- if $t_1 \leqslant t_2$ then $(t_1{:}x) \succeq (t_2{:}x)$, and
- if $x_1 \geqslant x_2$ then $(t{:}x_1) \succeq (t{:}x_2)$.

Reciprocally, every total preorder $\succeq$ satisfying the former conditions gives rise to a discount function. ■

Once a discount function has been defined, it is easier to introduce the (im)patience corresponding to a time interval. In effect, a measure of the patience (resp. impatience) exhibited by the discount function associated with certain underlying intertemporal choice (Cruz and Muñoz, 2013) is presented in Definition 2 (resp. 3):

Definition 2. The patience associated with the discount function $F(t)$ in the interval $[t_1, t_2]$ is defined as the value of the discount factor $f(t_1, t_2)$ corresponding to this interval, namely:

$$f(t_1, t_2) := \frac{F(t_2)}{F(t_1)} = \exp\left\{ - \int_{t_1}^{t_2} \delta(x) \mathrm{d}x \right\},$$

where $\delta(x)$ is the instantaneous discount rate at time x:

$$\delta(x) = -\frac{\mathrm{d}\ln F(z)}{\mathrm{d}z}\bigg|_{z=x} = -\frac{F'(x)}{F(x)}.$$

Observe that the patience lies in the interval [0, 1]. Moreover, the greater the discount factor, the less sloped is the discount function in the interval $[t_1, t_2]$, and then the lesser is the preference for immediate over delayed rewards, i.e. people are more patient. On the other hand, the instantaneous discount rate represents the impatience of a decision maker at a given moment:

Definition 3. The impatience associated with $F(t)$ in the interval $[t_1, t_2]$ is defined as $1-f(t_1, t_2)$, which also lies within the interval [0, 1].

As F is strictly decreasing, the decision maker is always impatient. However, as time goes by, his/her impatience may increase or decrease. This variation will be analyzed in Section 3.

However, the instantaneous rate of discount has been the commonly used measure of impatience. This is logical because the difference $1-f(t_1, t_2)$ is directly related to the discount rate. In Frederick *et al.* (2002), we can find a résumé of the implicit discount rates from all the reviewed studies. On the other hand, some authors consider the term impatience as a synonym of impulsivity, e.g. Cheng and González-Vallejo (2014), and Takahashi *et al.* (2007). This term has been used not only in economics, but also appears in psychological studies. Takahashi (2007) showed some examples of impulsivity in intertemporal choice in smokers, addicts and attention-deficient hyperactivity-disorder patients. In the same way, Tanaka *et al.* (2010) related impatience with risk-aversion and household income, Nguyen (2011) analyzed the relationship between impatience and work environment, and finally Espín *et al.* (2015) studied the impatience in a game involving bargaining.

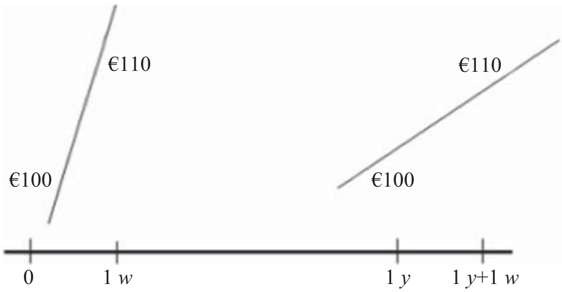
3. Variation of impatience: inconsistency in intertemporal choice

Based on the first study of inconsistency in intertemporal choice by Strotz (1956), many economists have defined the change of preference as a disagreement between the current subject and the same subject in the future. As an example, Kirby and Herrnstein (1995) offered subjects a choice between a small but earlier reward and a larger but later reward. After showing a preference for the small earlier reward when offered immediately, they then delayed both outcomes maintaining the temporal interval between them. Subjects typically switched to the larger later outcomes, even for very small amounts of added delay. This reversal in preferences or the fact that individuals show impulsivity and self-control at the same time is known as time inconsistency.

Example 1 (Frederick *et al.*, 2002): suppose one subject is asked to choose between receiving €100 now and receiving €110 in a week, and between receiving €100 in one year and receiving €110 in one year plus one week. If the subject chooses €100 now and €110 in one year plus one week, a choice reversal is detected in his/her decision. In Figure 3, we can observe the decrease in the slope of indifference lines which results in a convex discount function.

In the following paragraphs, we can consider either sequences or single values β and γ of money amounts in a given set of outcomes. In this case, $\beta \succ \gamma$ if $u(\beta) > u(\gamma)$, where u is a utility function. In the case of a single outcome, we will simply write $\beta > \gamma$. Many economic and psychological studies have found evidence for deviations from constant impatience such as decreasing impatience (Frederick *et al.*, 2002), but they have also determined the degrees of such deviations:

Definition 4. A decision maker exhibiting preferences $\succeq$ has decreasing impatience if for all $s < t$ and $\tau > 0$, $0 \prec \gamma \prec \beta$, and $(\gamma, s) \sim (\beta, t)$ imply $(\gamma, s+\tau) \preccurlyeq (\beta, t+\tau)$ (Rohde, 2009).



Source: Authors' own elaboration

Figure 3.
Preference reversal and indifference lines

Taking into account that the inconsistency shown by a subject at an instant is given by the instantaneous discount rate, in order to characterize the decreasing impatience it is sufficient to examine the behavior of $\delta(x)$. In effect, we can write the following statement:

P1. The time preference exhibits decreasing (resp. increasing) impatience if and only if the instantaneous discount rate $\delta(x)$ is decreasing (resp. increasing).

According to Rohde (2009), decreasing impatience means that a difference in timing is weighed less the further in the future it occurs, indicating that the subject is more willing to wait. This weight or the willingness to wait (WTT) is given by $-F'(t)$. Thus, decreasing impatience corresponds to an increasing $F'(t)$, which corresponds to the discount function being convex. This suggests that individuals can be compared in terms of their degrees of decreasing impatience by comparing the degrees of convexity of their discount functions. On the other hand, observe that the condition in *P1* is equivalent to saying that $-\ln F(t)$ is convex. This introduces a discussion between using the convexity of $-F(t)$ or $-\ln F(t)$ as a sufficient condition for decreasing impatience. Obviously, the convexity of $-\ln F(t)$ necessarily implies the convexity of $-F(t)$. So hereinafter, we will only consider the convexity of $-\ln F(t)$ as a condition stronger than the convexity of $-F(t)$.

Next, we are going to develop the characterizations of inconsistency which different authors have introduced starting from the concept of instantaneous discount rate. Prelec (2004) showed that the degree of decreasing impatience can be measured by the Arrow-Pratt degree of convexity of the logarithm of the discount function.

3.1 Inconsistency by Prelec

The objective of this subsection is to define the degree of inconsistency introduced by Prelec and to relate it with the hyperbolic factor (Rohde, 2010) with the help of the instantaneous variation rate (Cruz and Muñoz, 2001) which will be defined below. In effect, according to Prelec (2004), the degree of decreasing impatience is the rate of change of the instantaneous discount rate:

$$P(t) = -\frac{[\ln F(t)]''}{[\ln F(t)]'} = -[\ln \delta(t)]'.$$

This result has not yet been applied to empirical and experimental research because the discount function is hard to measure. Nevertheless, in recent years many economists have tried to obtain an approximation of Prelec's result. In order to resolve this difficulty, we are going to introduce the following three subsections.

3.1.1 Measure from Cruz and Muñoz (2001). Consider the indifference relationship $(\gamma, s) \sim (\beta, t)$, with $s < t$. If the availability of the reward γ is delayed until moment $s+\sigma$, with

$\sigma > 0$, the delay $\tau > 0$ for which the former indifference is preserved, $(\gamma, s+\sigma) \sim (\beta, t+\tau)$, satisfies the following equation:

$$\frac{\sigma}{\tau} = \frac{(F(t+\tau) - F(t))/\tau F(t)}{(F(s+\sigma) - F(s))/\sigma F(s)}.$$

Letting $\sigma \to 0$ (which implies $\tau \to 0$), one has:

$$\lim_{\sigma \to 0} \frac{\sigma}{\tau} = \frac{-\mathrm{d}\ln F(z)/\mathrm{d}z\big|_{z=t}}{-\mathrm{d}\ln F(z)/\mathrm{d}z\big|_{z=s}},$$

or equivalently:

$$\lim_{\tau \to 0} \frac{\sigma}{\tau} = \frac{\delta(t)}{\delta(s)}.$$

The left-hand side of the equation represents the instantaneous relative variation in the availability of rewards which will be denoted by $v(s, t)$ and will be called the instantaneous variation rate:

$$v(s,t) := \lim_{\sigma \to 0} \frac{\sigma}{\tau} = \frac{\delta(t)}{\delta(s)}.$$

Obviously, the instantaneous variation rate has the following properties:

(1) $v(s,t) = 1$ if and only if the discount function is exponential.

(2) $v(s,t) < 1$ (resp. $v(s,t) > 1$) if and only if:

- $\delta(t) < \delta(s)$ (resp. $\delta(t) > \delta(s)$);
- $\delta(z)$ is decreasing (resp. increasing); and
- the function $\ln F(z)$ is concave (resp. convex).

Therefore, the instantaneous variation rate is an indicator of the behavior of instantaneous discount rate as shown by the following corollary:

Corollary 1. The time preference exhibits decreasing impatience if and only if the instantaneous variation rate is less than 1.

3.1.2 Measure from Rohde. Rohde (2010, 2015) introduced two measures of decreasing impatience which can be easily calculated from experimental data without any knowledge of utility. So they can also be used as a measure when preferences cannot be represented by DU, instead of Prelec's measure which essentially needs a discount function. In effect, let us consider the indifference pair:

$$(\gamma, s) \sim (\beta, t)$$

and:

$$(\gamma, s+\sigma) \sim (\beta, t+\tau),$$

where γ and $\beta > 0$, with $s < t$ and $\tau > 0$. As indicated, time preference exhibits increasing impatience if $\sigma > \tau$. Obviously, as $s < t$, then $s\tau < t\sigma$.

Reciprocally, time preference exhibits decreasing impatience if $\sigma \leqslant \tau$, but nothing can be deduced about the relationship between $s\tau$ and $t\sigma$. Therefore, we will say that:

- Moderate decreasing impatience holds if $s\tau < t\sigma$. This condition is equivalent to require that:

$$0 < \tau - \sigma < \frac{\tau(t-s)}{t}.$$

- Strongly decreasing impatience holds if $s\tau \geqslant t\sigma$. This condition is equivalent to require that:

$$\tau - \sigma \geqslant \frac{\tau(t-s)}{t} > 0.$$

An immediate measure of decreasing impatience is $\tau - \sigma$ which is a parameter depending on s, t, γ, and β. In order to aggregate all these variables, Rohde (2010) proposed the following measure:

Definition 5. For every indifference pair, the hyperbolic factor is the function defined as:

$$H(s, t, \sigma, \tau) = \frac{\tau - \sigma}{t\sigma - s\tau}.$$

H is said to be regular if $t\sigma > s\tau$. Regularity implies the existence of an upper bound of the degree of decreasing impatience. Contrarily to regularity, H is infinite if $t\sigma = s\tau$, and negative if $t\sigma < s\tau$:

Theorem 2. Let regularity hold. Preferences $\succeq$ exhibit decreasing impatience if and only if $H \geqslant 0$.

Another question is to know whether a discount function changes from moderate to strongly decreasing impatience (or vice versa). To do this, we have to determine the poles of the function $H(s,t,\sigma,\tau)$, that is to say, it is necessary to solve the following equation for τ:

$$tF^{-1}\left[\frac{F(s)}{F(t)}F(t+\tau)\right] - s(t+\tau) = 0,$$

or equivalently:

$$\frac{F(s)}{F(t)}F(t+\tau) = F\left[\frac{s}{t}(t+\tau)\right].$$

The following paragraph describes an example in which the discount function changes from moderate to strongly decreasing impatience.

Example 2 (moderate and strongly decreasing impatience): assume that the intertemporal choice is ruled by the discount function:

$$F(t) = \exp\{\exp\{-kt\} - 1\},\ k > 0.$$

Let us consider $k = 0.3$ and a first time interval [1, 3]. The following indifferences hold:

$$(10, 1) \sim (13.97, 3)$$

and:

$$(10, 2) \sim (13.97, 5.13).$$

Observe that, in this case:

- $\tau = 2.13 > 1 = \sigma$;
- $\tau\sigma = 3 > 2.13 = s\tau$; and
- $H(1,3,1,2.13) = 1.299$.

Thus, in this interval a moderate decreasing impatience is observed. Now, let us consider the time interval [11,14]. The following indifferences hold:

$$(10, 11) \sim (10.22, 14)$$

and:

$$(10, 12) \sim (10.22, 27.98).$$

Observe that, in this case:

- $\tau = 13.98 > 1 = \sigma$;
- $\tau\sigma = 14 > 153.78 = s\tau$; and
- $H(11,14,1,13.98) = -0.0928$.

Thus, in this interval a strongly decreasing impatience is observed. Now we must ask what will be the characterization of a discount function exhibiting moderate decreasing impatience. For $t \geqslant 1$, the following necessary condition can be obtained:

$$\delta(t) \geqslant \frac{\delta(1)}{t}.$$

Rohde (2015) introduced another measure of decreasing impatience, the so-called DI-index, as defined below:

Definition 6. For every indifference pair, the decreasing impatience index is defined by:

$$\text{DI-index} = \frac{\tau - \sigma}{\sigma(t - s)}.$$

Thus, constant, decreasing, and increasing impatience correspond to a DI-index being zero, positive, or negative, respectively. The following theorem relates Prelec's degree of inconsistency with the DI-index by using the instantaneous variation rate as a tool. Observe that the proof of this theorem is much easier than that presented by Rohde (2015):

Theorem 3. The DI-index is an approximation of Prelec's degree of inconsistency.

Proof. In effect, by dividing the numerator and the denominator of the DI-index by τ, one has:

$$\text{DI-index} = \frac{1 - \sigma/\tau}{\sigma/\tau(t - s)}.$$

Recall that the instantaneous variation rate is:

$$v(s,t) := \lim_{\sigma\to 0}\frac{\sigma}{\tau} = \frac{\delta(t)}{\delta(s)}.$$

Thus:

$$\lim_{\substack{\tau\to 0\\ s\to t}} \text{DI-index} = \frac{1-\frac{\delta(t)}{\delta(s)}}{\frac{\delta(t)}{\delta(s)}(t-s)} = -[\ln\delta(t)]' = P(t). \qquad \blacksquare$$

The DI-index is obtained from similar indifferences as the hyperbolic factor (Rohde, 2010). However, the latter has a drawback, because it is a measure of impatience only for people who exhibit moderate decreasing impatience or increasing impatience. The DI-index does not have this problem and can also be computed for people who exhibit strongly decreasing impatience. Moreover, the DI-index approximates Prelec's measure of decreasing impatience, whilst the hyperbolic factor does not (Theorem 3).

Example 3 (strongly decreasing impatience, $\sigma < \tau$): if the intertemporal choice is ruled by the discount function:

$$F(t) = \exp\{-\arctan(t)\},$$

the following indifferences hold:

$$\left(10, \frac{\sqrt{3}}{3}\right) \sim (10\exp\{\pi/12\}, 1)$$

and:

$$\left(10, \sqrt{3}\right) \sim (10\exp\{\pi/2\}, 3.73).$$

Therefore, DI-index = 3.2278, but $H\left((\sqrt{3}/3), 1, (2\sqrt{3}/3), 2.73\right) = -3.7377$.

3.1.3 Attema et al.'s measure. Attema *et al.* (2010) gave another measure of time inconsistency, the so-called time-tradeoff (TTO) sequences:

Definition 7. A TTO sequence is a sequence $t_0, t_1, \ldots, t_n$ of time points such that there are two outcomes $\beta \succ \gamma$ with:

$$\begin{gathered}(\gamma, t_0) \sim (\beta, t_1),\\ (\gamma, t_1) \sim (\beta, t_2),\\ \vdots\\ (\gamma, t_{n-1}) \sim (\beta, t_n)\end{gathered}$$

that is, each delay between two consecutive time points exactly offsets the same outcome improvement. Such a delay, $t_i - t_{i-1}$, is called the WTW.

Thus, stationarity implies that the WTW is constant, so that points $t_0, t_1, \ldots, t_n$ are equally spaced in time units. Increasing and decreasing impatience correspond to decreasing and increasing WTW, respectively.

A TTO sequence is equally spaced in ln F units and also in the units of any renormalization of ln$F$. We consider a convenient renormalization, being ln F normalized at t_0 and t_n:

$$\varphi_{t_0,t_n}(t) = \frac{\ln F(t) - \ln F(t_n)}{\ln F(t_0) - \ln F(t_n)}.$$

This function will be called the TTO curve of the TTO sequence:

P2. The degree of convexity of a TTO curve determines the degree of decreasing impatience.

Proof. In effect, if φ is convex, then $\varphi' = F'(t)/F(t) = -\delta(t)$ is increasing. By *P1*, the time preference exhibits decreasing impatience. ■

From a mathematical point of view, the contribution of Attema *et al.* is elementary with respect to Prelec's result: replacing the convexity of ln$F$ by the equivalent convexity of φ. But, from an empirical perspective, *P2* is essential because φ is directly observable, whilst ln$F$ is not.

3.2 Inconsistency by Takahashi

Cajueiro (2006), based in Tsallis' statistics, introduced the so-called q-exponential discount function which was used by Takahashi (2007) to quantify the inconsistency in an intertemporal choice. In effect, if $F_q(t)$ denotes the q-exponential discount function:

$$F_q(t) = \frac{1}{\left[1+(1-q)k_q t\right]^{1/1-q}},$$

the parameter k_q quantifies the impulsivity whilst the parameter q represents a measure of consistency in the intertemporal choice process described by the discount function. In effect, when $q \to 1$, the q-exponential discounting tends to the exponential discounting and so the intertemporal choices are consistent, and when $q = 0$, the q-exponential discounting is equivalent to the hyperbolic discounting. In this latter case and in all the cases where $0 \leqslant q < 1$, the intertemporal choices are inconsistent. In effect, the q-exponential discount rate is given by:

$$\delta_q(t) = \frac{k_q}{1+(1-q)k_q t},$$

where at delay $t = 0$, $\delta_q(t) = k_q$, for any q. Therefore, at delay $t > 0$, the value of $\delta_q(t)$ indicates the impulsivity in the plan of future intertemporal choice behavior. It is important to examine how the inconsistency (or time-dependency of $\delta_q(t)$) depends on the parameters of the q-exponential discounting model. The time-dependency of $\delta_q(t)$ is given by:

$$\delta_q'(t) = \frac{-k_q^2(1-q)}{\left[1+(1-q)k_q t\right]^2}.$$

Thus:

- decreasing impatience holds if $\delta'_q(t) < 0$, that is, if $q < 1$;
- consistency holds if $\delta'_q(t) = 0$, that is, if $q = 1$; and
- increasing impatience holds if $\delta'_q(t) > 0$, that is, if $q > 1$.

The q-exponential discount function is mathematically equivalent to Loewenstein and Prelec's generalized hyperbolic function. The advantage of the q-exponential function over the generalized hyperbolic function is that the first model can parameterize the deviation of temporal discounting from the exponential function as the amount $1-q$. Takahashi (2007) showed that the q-exponential discounting model is well defined also for $q<0$, unlike Cajueiro (2006) who assumed that q is non-negative. Later Cruz and Muñoz (2013) demonstrated that the range of values of q in the q-exponential discount function can be extended to the joint interval $(-\infty,1)\cup(1,+\infty)$. Table IV summarizes the results.

Takahashi *et al.* (2007) demonstrated that the q-exponential discounting model explained human intertemporal choice behavior better than the exponential and simple hyperbolic discounting models.

The following result characterizes the q-exponential discount function according to the hyperbolic factor introduced by Rohde (2010):

P3. The hyperbolic factor corresponding to an intertemporal choice is constant if and only if the underlying discount function is the q-exponential.

Proof. Assume the discount function underlying the process of intertemporal choice is the q-exponential. Then:

$$\left(\frac{\gamma}{\beta}\right)^{1-q}=\frac{1+(1-q)k_q t}{1+(1-q)k_q s}$$

and:

$$\left(\frac{\gamma}{\beta}\right)^{1-q}=\frac{1+(1-q)k_q(t+\tau)}{1+(1-q)k_q(s+\sigma)}.$$

By identifying both fractions:

$$\frac{1+(1-q)k_q t}{1+(1-q)k_q s}=\frac{1+(1-q)k_q(t+\tau)}{1+(1-q)k_q(s+\sigma)},$$

from where:

$$H(s,t,\sigma,\tau)=\frac{\tau-\sigma}{t\sigma-s\tau}=(1-q)k_q.$$

Reciprocally, assume that $H(s,t,\sigma,\tau)=\alpha$, with α a constant. Then:

$$\frac{1-(\sigma/\tau)}{t(\sigma/\tau)-s}=\alpha.$$

q	$F_q(t)$	$\delta_q(t)$	Impatience
$q\to+\infty$	1	0	Zero
$q\in(1,+\infty)$	$1/([1+(1-q)k_q t]^{1/(1-q)})$	$k_q/(1+(1-q)k_q t)$	Increasing
$q\to1$	$\exp\{-k_q t\}$	k_q	Constant
$q\in(0,1)$	$1/([1+(1-q)k_q t]^{1/(1-q)})$	$k_q/(1+(1-q)k_q t)$	Decreasing
$q=0$	$1/(1+k_q t)$	$k_q/(1+k_q t)$	Decreasing
$q\in(-\infty,0)$	$1/([1+(1-q)k_q t]^{1/(1-q)})$	$k_q/(1+(1-q)k_q t)$	Decreasing
$q\to-\infty$	1	0	Zero

Source: Authors' own elaboration from Cruz and Muñoz (2013)

Table IV. Behavior of $F_q(t)$ according to different values of q

Letting $\sigma \to 0$ (which implies $\tau \to 0$):

$$\frac{1-\left(\delta(t)/\delta(s)\right)}{t\left(\delta(t)/\delta(s)\right)-s} = \alpha.$$

Then:

$$\frac{1+\alpha s}{1+\alpha t} = \frac{\delta(t)}{\delta(s)},$$

from where $\delta(t) = h\left(1/(1+\alpha t)\right)$. Thus:

$$-\ln F(t) = \int_0^t \delta(x)\mathrm{d}x = \frac{h}{\alpha}\ln(1+\alpha t).$$

Therefore:

$$F(t) = \frac{1}{(1+\alpha t)^{h/\alpha}},$$

which is the q-exponential discount function once suitable changes of the involved variables are introduced. ■

3.3 Inconsistency by Rohde

Rohde (2009) introduced the concept of decreasing relative impatience which can be measured by the convexity of the discount function itself, rather than its logarithm, as Prelec (2004) showed. This new notion of impatience considers sequences of two outcomes, unlike the concept of decreasing impatience, which considers single outcomes. Observe that when a subject is faced with a decision, he/she incurs costs today to receive a reward in the future. So, the sequence of two outcomes means "cost now" and "reward later."

First, we define the concept of relative impatience and then the concept of decreasing relative impatience:

Definition 8. Preferences $\succeq$ satisfy relative impatience if:

$$(\beta, 0; \gamma, s) \succ (\beta, 0; \gamma, t),$$

whenever $s < t$ and $\gamma > 0$.

That is to say, the delay in receiving γ from s to t is considered relative to the payment of β today.

Definition 9. Preferences $\succeq$ satisfy decreasing relative impatience if for all outcomes α,β,γ with $\alpha < \beta$ and $\gamma > 0$, all time points $s < t$, and $\sigma > 0$, $(\alpha,0;\gamma,s) \sim (\beta,0;\gamma,t)$ implies $(\alpha,0;\gamma,s+\sigma) \prec (\beta,0;\gamma,t+\sigma)$.

The indifference $(\alpha,0;\gamma,s) \sim (\beta,0;\gamma,t)$ means that to speed up the receipt of γ from t to s the decision maker is willing to pay $\beta-\alpha$. But, if γ is delayed at $t+\sigma$, the decision maker is no longer willing to pay $\beta-\alpha$. Thus, a delay of $t-s$ units of time becomes less important the further it lies in the future.

In the following theorem it is shown that the degree of decreasing relative impatience is indicated by the degree of convexity of the discount function:

Definition 10. Let F be a twice continuously differentiable discount function. The rate of decreasing relative impatience, denoted by $DRI(t)$, is defined as the degree of convexity of the discount function:

$$DRI(t) = -\frac{F''(t)}{F'(t)}.$$

Theorem 4. The degree of decreasing relative impatience at any point in time is the sum of the degree of decreasing impatience and the discount rate at that point in time:

$$DRI(t) = P(t) + \delta(t), \quad \text{for all } t. \qquad \blacksquare$$

Corollary 2. Decreasing impatience implies decreasing relative impatience. In other words, $P(t) \geqslant 0$ implies $DRI(t) \geqslant 0$. ■

This section finishes with the concept of spread-seeking introduced by Rohde (2009). Consider a decision maker who receives a reward at two different dates. If the decision maker wants to speed up the early receipt and delay the late receipt by the same temporal interval, we say that he/she is spread-seeking because he/she is increasing the spread of the two benefits:

Definition 11. Preferences $\succeq$ satisfy spread-seeking if, for all $\gamma > 0$ and all $0 \leqslant s < t$, $0 < \tau \leqslant s$:

$$(\gamma, s-\tau; \gamma, t+\tau) \succeq (\gamma, s; \gamma, t).$$

Spread-seeking can be characterized as follows:

Theorem 5. Preferences $\succeq$ satisfy spread-seeking if and only if F is convex. ■

4. Conclusions

This paper starts from the classical model of DU proposed by Samuelson (1937). Despite the fact that this model has been used in intertemporal choice, several recent studies have contributed empirical evidence which contradicts its principles, giving rise to the so-called anomalies in intertemporal choice. One of them is inconsistency, because people often display preference reversal as time passes. Thus, numerous alternative models have been studied over the years, such as hyperbolic discounting.

In order to tackle the inconsistency, we introduce the concept of impatience corresponding to a time interval. We observe that, as F is strictly decreasing, the decision maker is always impatient. However, as time goes by, the preferences may change; in other words, the impatience may increase or decrease. This behavior is called time inconsistency. Therefore, in this paper we study the variation of impatience, more specifically decreasing impatience. To do this, we introduce the instantaneous discount rate, $\delta(t)$, and its relation with decreasing impatience. In the same way, we propose the ratio of two instant discount rates, which we call the instantaneous variation rate, $v(s,t)$, for $s < t$, as an indicator of decreasing impatience (Cruz and Muñoz, 2001). From these results, we deduce that the convexity of the logarithm of the discount function is a strong condition for decreasing impatience.

We show different measures of time inconsistency which different authors have proposed in recent years. We highlight the following behavioral economists: Prelec, Rohde and Takahashi. The main measure was given by Prelec (2004) who considered that the degree of decreasing impatience is indicated by the convexity of the logarithm of the discount function. The drawback of this result is the difficulty to measure it, so we have shown three tools which approximate to Prelec's measure. Rohde (2010) derived the hyperbolic factor starting from an indifference pair, from which increasing impatience, moderate decreasing impatience and strongly decreasing impatience are defined. Moreover, we offer an example where we can observe a discount function changing from moderate to strongly decreasing impatience. Alternatively, Rohde (2015) provided another measure of decreasing impatience which approximates to Prelec's result: the DI-index. This tool is an improvement on the hyperbolic factor because it can be computed for people exhibiting strongly decreasing impatience. In the same way, Attema *et al.* (2010) proposed replacing the convexity of the logarithm of the discount function by the convexity of a curve which is directly observable.

On the other hand, Takahashi (2007) introduced the q-exponential discount function and the connection between the parameter q and the measure of the inconsistency in intertemporal choice. Rohde (2009) introduced some new concepts related to the study of time inconsistency: decreasing relative impatience and spread-seeking. The degree of decreasing relative impatience is defined as the degree of convexity of the discount function, and we highlight the result which relates this degree to the discount rate used in Prelec's measure.

Although this paper is focused on the field of behavioral finance, where the importance of time inconsistency has been highlighted by several authors (Hardisty and Pfeffer, 2017; Sedghi and Gerayli, 2015; and Imas, 2016), Section 1 has stressed the importance of this topic in the context of marketing and managerial decisions. We conclude by listing the main contributions of this paper. First, the importance of the concept of the instantaneous variation rate, represented by $v(s,t)$, to facilitate relating Prelec's measure with Rohde's parameter. Second, the establishment of a necessary (not sufficient) condition which allows us to know if the decreasing impatience is moderate or strong for a limited range of values, namely $\delta(t) \geqslant \left(\delta(1)/t\right)$ for $t \geqslant 1$. Third, the calculation of the poles of the function $H(s,t,\tau)$ to know whether a discount function changes from moderate to strongly decreasing impatience (or vice versa). Fourth, the generalization of the q-exponential discount function to the joint interval $(-\infty,1)\cup(1,+\infty)$, increasing the range of possible inconsistencies. Fifth, the statement of a necessary condition for a decision maker being spread-seeking has been included.

A limitation of this paper is the lack of empirical evidence to support the obtained theoretical results. Thus, our future research is addressed to the design of a survey to analyze the degree of inconsistency in intertemporal choice, using the most suitable measures.

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