On some nonlinear anisotropic elliptic equations in anisotropic Orlicz space

Omar Benslimane
Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar EL Mahraz, Sidi Mohamed Ben Abdellah University, Atlas Fez, Morocco

Ahmed Aberqi
Laboratory LAMA, National School of Applied Sciences Fez, Sidi Mohamed Ben Abdellah University, Fez, Morocco, and

Jaouad Bennouna
Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar EL Mahraz, Sidi Mohamed Ben Abdellah University, Atlas Fez, Morocco

Received 14 December 2020
Revised 2 March 2021
9 March 2021
Accepted 15 March 2021

Abstract

Purpose – In the present paper, the authors will discuss the solvability of a class of nonlinear anisotropic elliptic problems (*P*), with the presence of a lower-order term and a non-polynomial growth which does not satisfy any sign condition which is described by an *N*-uplet of *N*-functions satisfying the Δ_2-condition, within the fulfilling of anisotropic Sobolev-Orlicz space. In addition, the resulting analysis requires the development of some new aspects of the theory in this field. The source term is merely integrable.

Design/methodology/approach – An approximation procedure and some priori estimates are used to solve the problem.

Findings – The authors prove the existence of entropy solutions to unilateral problem in the framework of anisotropic Sobolev-Orlicz space with bounded domain. The resulting analysis requires the development of some new aspects of the theory in this field.

Originality/value – To the best of the authors' knowledge, this is the first paper that investigates the existence of entropy solutions to unilateral problem in the framework of anisotropic Sobolev-Orlicz space with bounded domain.

Keywords Anisotropic elliptic equation, Entropy solution, Sobolev-Orlicz anisotropic spaces

Paper type Research paper

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$. The aim behind this paper is the study of boundary value problems for a class of nonlinear anisotropic elliptic equations. More specifically, we consider the unilateral elliptical operators whose nonlinearity is given by a vector of *N*-functions like

The authors would like first to thank Miss Loulidi Wiame for her support and encouragement. Second, to be grateful to the anonymous referees for the valuable suggestions and comments which improved the quality of the presentation.

Arab Journal of Mathematical Sciences
Vol. 29 No. 1, 2023
pp. 29-51
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1108/AJMS-12-2020-0133

$$(\mathcal{P})\begin{cases} \mathfrak{A}(u) + \sum_{i=1}^{N} \mathfrak{b}_i(x, u, \nabla u) = f & \text{in } \Omega, \\ u \geq \zeta & \text{a.}e \text{ in } \Omega, \end{cases}$$

where $\mathfrak{A}(u) = \sum_{i=1}^{N} (\sigma_i(x, u, \nabla u))_{x_i}$ is a Leray–Lions operator defined in $\overset{\circ}{W}{}^1_M(\Omega)$ (defined as the adherence space $C_0^\infty(\Omega)$) into its dual (see assumptions (19), (20), (21) in Section 3); $M(t) = (M_1(t), \ldots, M_N(t))$ are N-uplet Orlicz functions that satisfy Δ_2–condition; the obstacle ζ is a measurable function that belongs to $L^\infty(\Omega) \cap \overset{\circ}{W}{}^1_M(\Omega)$; and for the $i = 1, \ldots, N$, $\mathfrak{b}_i(x, s, \xi)$: $\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions (measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω) that does not satisfy any sign condition and the growth which is described by the vector N-function $(M_1(t), \ldots, M_N(t))$ (see assumption (22)). As well as $f \in L^1(\Omega)$.

For several years great effort has been devoted to the study of nonlinear elliptic equations with an operator which was described by polynomial growth. For example, in the classical Sobolev space, Boccardo and Gallouêt in [1], proved the existence of a weak solution of $(\mathcal{P})$ in the case $\phi \equiv g \equiv 0$. Bénilan in [2] presented the idea of entropy solutions which were adjusted to the Boltzmann condition. For a deeper comprehension of these types of equations in this field, we refer the reader to [3–10] and references therein.

Next, in the Orlicz space, Benkirane and Bennouna in [11] demonstrated the existence of entropy solutions to the following nonlinear elliptic problem:

$$-\text{div}\, a(x, u, \nabla u) + \text{div}(\phi(u)) = f,$$

where $\phi \in (C^0(\mathbb{R}))^N$ and $f \in L^1(\Omega)$. For more results, we refer the reader to [12–24] and references therein.

And in the anisotropic Sobolev-Orlicz space, there are few results dealing with this topic. We will mention recent papers, and we are starting by the pertinent works of Korolev and Cianchi [25, 26] who proved the embeddings of this space. Then, Benslimane, Aberqi and Bennouna in [27] studied the existence and uniqueness of the solution in the following problem in an unbounded domain

$$(\mathcal{P})\begin{cases} A(u) + \sum_{i=1}^{N} b_i(x, u, \nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $A(u) = \sum_{i=1}^{N} (a_i(x, u, \nabla u))_{x_i}$ is a Leray-Lions operator defined from $\overset{\circ}{W}{}^1_B(\Omega)$ into its dual, $B(\theta) = (B_1(\theta), \ldots, B_N(\theta))$ are N-uplet Orlicz functions that satisfy the Δ_2–condition, and for $i = 1, \ldots, N$, $b_i(x, u, \nabla u) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are the Carathéodory functions that do not satisfy any sign condition and the growth described by the vector N-function $B(\theta)$. After that, Kozhevnikova in [28] established the existence of entropy solutions in an unbounded domain to the following problem:

$$\begin{cases} \sum_{i=1}^{N} (a_i(x, \nabla u))_{x_i} = a_0(x, u) & \text{in } \Omega, \\ u(x) = \psi(x) & \text{on } \partial\Omega, \end{cases}$$

where Ω is an arbitrary domain in $\mathbb{R}^N$, $N \geq 2$,

$$a_0(x,s_0) = a_0(x,\psi) + b(x,s_0),$$

with $a_0(x,\psi) \in L^1(\Omega)$, the function $b(x,s_0)$ satisfies the Carathéodory condition and decreases in $s_0 \in \mathbb{R}$, $b(x,\psi) = 0$ for all x in Ω; therefore, for any $x \in \Omega$, $s_0 \in \mathbb{R}$

$$b(x,s_0)\,(s_0 - \psi) > 0.$$

The author supposed two other conditions: the first one is

$$\sup_{|s_0|\le k} |b(x,s_0)| = G_k(x) \in L_{1,loc}(\Omega),$$

the second one, $\delta_0 > 0$ such as

$$b(x,\psi \pm \delta_0) \in L^1(\Omega).$$

For more results we refer the reader to [29–32] and the references therein.

This type of operator arises in a quite natural way in many different contexts, such as the study of fluid filtration in porous media, constrained heating, elasticity, electro-rheological fluids, optimal control, financial mathematics and other domains, see [33–36] and the references therein.

As far we know, no previous research has investigated the existence of entropy solutions to unilateral problem $(\mathcal{P})$ with the second term as an operator with growth described by an n-uplet of N-functions satisfying the Δ_2−condition, within the fulfilling of anisotropic Sobolev-Orlicz space with bounded domain, the function $\mathfrak{b}_i(x,u,\nabla u)$ does not satisfy any sign condition and the source f is merely integrable. Hence, motived by the aforementioned papers, our main work is to obtain the existence Theorem for unilateral problems corresponding to $(\mathcal{P})$ via an approximation procedure and some priori estimates.

The rest of this paper is organized as follows: In Section 2, we give some definitions and fundamental properties of anisotropic Sobolev-Orlicz spaces. In Section 3, we give our assumptions on data and the definition of entropy solutions to $(\mathcal{P})$. In Section 4, we will show the existence of entropy solutions, with the functions $\mathfrak{b}_i(x,u,\nabla u)$, $i = 1,\ldots,N$ which does not satisfy any sign condition. Finally, in the Appendix.

2. Definitions and preliminary tools

In this section, we recall the most important and relevant properties and notations about of anisotropic Sobolev-Orlicz space which we will need in our analysis of the problem $(\mathcal{P})$. A comprehensive presentation of Sobolev- Orlicz anisotropic space can be found in the work of M.A Krasnoselskii and Ja. B. Rutickii [25, 37].

Definition 1. *We say that* $M : \mathbb{R}^+ \to \mathbb{R}^+$ *is a N-function if M is continuous, convex, with* $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ when $t \to 0$ *and* $\frac{M(t)}{t} \to \infty$ when $t \to \infty$.

This N-function M admits the following representation: $M(t) = \int_0^t b(s)\, ds$, *with* $b : \mathbb{R}^+ \to \mathbb{R}^+$ *which is an increasing function on the right with* $b(0) = 0$ *in the case* $t > 0$ *and* $b(t) \to \infty$ *when* $t \to \infty$.

Its conjugate is noted by $\overline{M}(t) = \int_0^t q(s)\, ds$ *with q also satisfies all the properties already quoted from b, with*

$$\overline{M}(t) = \sup_{\mu \ge 0} (\mu t - M(\mu)), \qquad t > 0. \tag{1}$$

The Young's inequality is given as follows:

$$\forall t,\, \mu > 0 \qquad t\mu \le M(\mu) + \overline{M}(t). \tag{2}$$

Definition 2. *The N-function M(t) satisfies the Δ_2−condition if* $\exists c > 0$, $t_0 > 0$ *such as*

$$M(2t) \leq c\, M(t) \qquad t \geq t_0. \tag{3}$$

This definition is equivalent to $\forall k > 1$, $\exists\, c(k) > 0$ *such as*

$$M(k\,t) \leq c(k)\, M(t) \quad \text{for} \quad t \geq t_0. \tag{4}$$

Definition 3. *The N-function M(t) satisfies the Δ_2−condition as long as there exist positive numbers* $c > 1$ *and* $t_0 \geq 0$ *such as for* $t \geq t_0$ *we have*

$$t\, b(t) \leq c\, M(t). \tag{5}$$

Also, each N-function M(t) satisfies the inequality

$$M(\mu + t) \leq c\, M(t) + c\, M(\mu) \qquad t,\ \mu \geq 0. \tag{6}$$

We consider the Orlicz space $L_M(\Omega)$ provided with the norm of Luxemburg given by

$$\| u \|_{M,\,\Omega} = \inf \left\{ k > 0 \Big/ \int_\Omega M\left(\frac{|\, u(x)\,|}{k} \right) dx \leq 1 \right\}. \tag{7}$$

According to [37] we obtain the inequalities

$$\int_\Omega M\left(\frac{|\, u(x)\,|}{\| u \|_{M,\,\Omega}} \right) dx \leq 1, \tag{8}$$

and

$$\| u \|_{M,\,\Omega} \leq \int_\Omega M(|\, u\,|)\ dx + 1. \tag{9}$$

Moreover, the Hölder's inequality holds and we have for all $u \in L_M(\Omega)$ and $v \in L_{\overline{M}}(\Omega)$

$$\left| \int_\Omega u(x)\, v(x)\ dx \right| \leq 2\, \| u \|_{M,\Omega} \cdot \| v \|_{\overline{M},\,\Omega}. \tag{10}$$

In [25, 37], if $P(t)$ and $M(t)$ are two N-functions such as $P(t) \ll M(t)$ and meas $\Omega < \infty$, then $L_M(\Omega) \subset L_P(\Omega)$. Furthermore,

$$\| u \|_{P,\Omega} \leq A_0\, (\,\text{meas}\, \Omega\,)\, \| u \|_{M,\Omega} \qquad u \in L_M(\Omega). \tag{11}$$

And for all N-functions $M(t)$, if meas $\Omega < \infty$, then $L_\infty(\Omega) \subset L_M(\Omega)$ with

$$\| u \|_{M,\Omega} \leq A_1\, (\,\text{meas}\, \Omega\,)\, \| u \|_{\infty,\Omega} \qquad u \in L_M(\Omega). \tag{12}$$

Also for all N-functions $M(t)$, if meas $\Omega < \infty$, then $L_M(\Omega) \subset L^1(\Omega)$ with

$$\| u \|_{1,\Omega} \leq A_2\, \| u \|_{M,\Omega} \qquad u \in L_M(\Omega). \tag{13}$$

We define for all N-functions $M_1(t), \ldots, M_N(t)$ the space of Sobolev-Orlicz anisotropic $\overset{\circ}{W}{}^1_M(\Omega)$ as the adherence space $C_0^\infty(\Omega)$ under the norm

$$\| u \|_{\overset{\circ}{W}{}^1_M(\Omega)} = \sum_{i=1}^{N} \| u_{x_i} \|_{M_i,\,\Omega}. \tag{14}$$

Definition 4. *A sequence* $\{u_m\}$ *is said to converge modularly to* u *in* $\overset{\circ}{W}{}^1_M(\Omega)$ *if for some* $k > 0$ *we have*

$$\int_\Omega M\left(\frac{|u_m - u|}{k}\right) dx \to 0 \quad \text{as} \quad m \to \infty. \tag{15}$$

Remark 1. *Since* M *satisfies the* Δ_2-*condition, then the modular convergence coincides with the norm convergence.*

Proposition 1.

$$\theta M'(t) = \overline{M}(M'(t)) + M(t) \qquad t > 0, \tag{16}$$

with M' *is the right derivative of the N-function* $M(t)$.

Proof. By (2) we take $\mu = M'(t)$, then we obtain

$$M'(t)\,t \leq M(t) + \overline{M}(M'(t)),$$

and by Ch. I [37] we get the result. □

Since Ω is a bounded domain in $\mathbb{R}^N$. The following Lemmas are true:

Lemma 1. [8] *For all* $u \in \overset{\circ}{W}{}^1_{L_M}(\Omega)$ *with* meas $\Omega < \infty$, *we have*

$$\int_\Omega M\left(\frac{|u|}{\lambda}\right) dx \leq \int_\Omega M(|\nabla u|)\, dx,$$

where $\lambda = \operatorname{diam}(\Omega)$ *is the diameter of* Ω.

Note by $h(t) = \left(\prod_{i=1}^N \frac{B_i^{-1}(t)}{t}\right)^{\frac{1}{N}}$ and we assume that $\int_0^1 \frac{h(t)}{t}\, dt$ converge. So, we consider the N-functions $M^*(z)$ defined by $(M^*)^{-1}(z) = \int_0^{|z|} \frac{h(t)}{t}\, dt$.

Lemma 2. [26] *Let* $u \in \overset{\circ}{W}{}^1_M(\Omega)$. *If*

$$\int_1^\infty \frac{h(t)}{t}\, dt = \infty, \tag{17}$$

then, $\overset{\circ}{W}{}^1_M(\Omega) \subset L_{M^*}(\Omega)$ *and* $\|u\|_{M^*,\Omega} \leq \frac{N-1}{N}\|u\|_{\overset{\circ}{W}{}^1_M(\Omega)}$.

If

$$\int_1^\infty \frac{h(t)}{t}\, dt \leq \infty,$$

then, $\overset{\circ}{W}{}^1_M(\Omega) \subset L_\infty(\Omega)$ *and* $\|u\|_{\infty,\Omega} \leq \beta\|u\|_{\overset{\circ}{W}{}^1_M(\Omega)}$, *with* $\beta = \int_0^\infty \frac{h(t)}{t}\, dt$.

In the following, we will assume that for each N-function $M_i(z) = \int_0^{|z|} b_i(t)\, dt$ obeys the further condition:

$$\lim_{\alpha\to\infty} \inf_{\theta > 0} \frac{b_i(\alpha\theta)}{b_i(\alpha)} = \infty \quad i = 1, \ldots, N. \tag{18}$$

Remark 2. *The Following function:*

$$M(z) = |z|^b(|ln|z|| + 1),$$

with $b > 1$ *check the* Δ_2*-condition and* (18).

3. Assumptions on data and definition of solution

Statement of the problem: Suppose they have non-negative measurable functions ϕ, $\varphi \in L^1(\Omega)$ and positive constants $\overline{a}$ and $\tilde{a}$ such as:

$$\sum_{i=1}^{N} (\sigma_i(x,s,\xi) - \sigma_i(x,s,\xi')) . (\xi_i - \xi_i') > 0, \tag{19}$$

$$\sum_{i=1}^{N} \sigma_i(x,s,\xi) . \xi_i \geq \overline{a} \sum_{i=1}^{N} M_i(|\xi_i|) - \phi(x), \tag{20}$$

$$\sum_{i=1}^{N} | \sigma_i(x,s,\xi) | \leq \tilde{a} \sum_{i=1}^{N} \overline{M}_i^{-1} M_i(|\xi|) + \varphi(x), \tag{21}$$

and

$$\sum_{i=1}^{N} | \mathfrak{b}_i(x,s,\xi) | \leq h(x) + l(s) . \sum_{i=1}^{N} M_i(|\xi|), \tag{22}$$

with $\overline{M}(t)$ the complementary of $M(t)$, $h(x) \in L^1(\Omega)$ and $l : \mathbb{R} \to \mathbb{R}^+$ a positive continuous function such as: $l \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

3.1 Definition of entropy solutions:

Definition 5. *A measurable function u is said to be an entropy solution for the problem* $(\mathcal{P})$, *if* $u \in \mathring{W}^1_M(\Omega)$ *such that* $u \geq \zeta$ *a.e. in* Ω *and*

$$\sum_{i=1}^{N} \int_\Omega \sigma_i(x,u,\nabla u) . \nabla(u-v) \ dx + \sum_{i=1}^{N} \int_\Omega \mathfrak{b}_i(x,u,\nabla u) . (u-v) \ dx$$

$$+\int_\Omega m . T_m(u-\zeta)^- . sg_{\frac{1}{m}}(u) . (u-v) \ dx$$

$$\leq \int_\Omega f(x) . (u-v) \ dx \qquad \forall v \in K_\zeta \cap L^\infty(\Omega),$$

where, $K_\zeta = \{ u \in \mathring{W}^1_M(\Omega) \text{ such as } u \geq \zeta \text{ a.e. in } \Omega \}$, for $m \in \mathbb{N}^*$ $sg_m(s) = \frac{T_m(s)}{m}$. *We define the truncation at height m,* $T_m(u) : \mathbb{R} \to \mathbb{R}$ *by*

$$T_m(u) = \begin{cases} u & \text{if } |u| \leq m, \\ m & \text{if } |u| > m. \end{cases}$$

4 . Main result

In this section, we will show the existence of our problem $(\mathcal{P})$. We will assume that $f^m \to f$ in $L^1(\Omega)$, $m \to \infty$, $| f^m(x) | \leq | f(x) |$ and for $i = 1, \ldots, N$, $\sigma_i^m(x,u_m,\nabla u_m) : (\mathring{W}^1_M(\Omega))^N \to (\mathring{W}^{-1}_{\overline{M}}(\Omega))^N$ being Carathéodory functions with

$$\sigma_i^m(x,u,\nabla u) = \sigma_i(x, T_m(u), \nabla u),$$

and $\mathfrak{b}_i^m(x,u_m,\nabla u_m) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ again being Carathéodory functions not satisfying any sign condition, with

$$\mathfrak{b}^m(x,u,\nabla u) = \frac{\mathfrak{b}(x,u,\nabla u)}{1+\frac{1}{m}\,|\,\mathfrak{b}(x,u,\nabla u)\,|},$$

and
$$|\,\mathfrak{b}^m(x,u,\nabla u)\,| = |\,\mathfrak{b}(x,T_m(u),\nabla u)\,| \leq m \text{ for all } m\in\mathbb{N}^*, \tag{23}$$

Consider the penalized equations:

$$(\mathcal{P}_m)\ :\sum_{i=1}^{N}\int_\Omega \sigma_i^m(x,u_m,\nabla u_m).\nabla(u_m-v)\,dx+\sum_{i=1}^{N}\int_\Omega \mathfrak{b}_i^m(x,u_m,\nabla u_m).(u_m-v)\,dx$$
$$+\int_\Omega m.T_m(u_m-\zeta)^-.sg_{\frac{1}{m}}(u_m).(u_m-v)\,dx=\int_\Omega f^m(x).(u_m-v)\,dx\ \ \forall v\in \mathring{W}^1_M(\Omega).$$

Theorem 1. *Let's assume that conditions (19)–(22) and (18) hold true, then there exists at least one solution of the approximate problem* $(\mathcal{P}_m)$.

Proof. See Appendix. □

Now, we will show some results in the form of propositions that will be useful for the demonstration of existence Theorem 2, see below.

Proposition 2. *(see [27]) Suppose that conditions (19)–(22) are satisfied, and let* $(u_m)_{m\in\mathbb{N}}$ *be a sequence in* $\mathring{W}^1_B(\Omega)$ *such as.*

(a) $u_m \rightharpoonup u$ in $\mathring{W}^1_M(\Omega)$,

(b) $\sigma^m(x,u_m,\nabla u_m)$ is bounded in $L_{\overline{M}}(\Omega)$,

$(c)\int_\Omega(\sigma^m(x,u_m,\nabla u_m)-\sigma^m(x,u_m,\nabla u\chi_K)).\nabla(u_m-u\chi_K)\,dx\to 0$ *as* $K\to+\infty$ *(*χ_K *the characteristic function of* $\Omega_K=\{x\in\Omega;\,|\,\nabla u\,|\leq K\}$*).*

Then:

$$M(\,|\,\nabla u_m\,|\,)\to M(\,|\,\nabla u\,|\,)\quad\text{in}\quad L^1(\Omega).$$

Proposition 3. *(see [31]) Let's assume that conditions (19)–(22) and (18) hold true, then the generalized solution of the problems* $(\mathcal{P}_m)$ *satisfies the following estimate:*

$$\int_\Omega M(\,|\,\nabla T_K(u_m)\,|\,)\leq c=c(K)\ ,\quad K>0.$$

Proposition 4. *(see [31]) Suppose that conditions (19)–(22) and (18) are satisfied, and let* $(u_m)_{m\in\mathbb{N}}$ *be a solution of the problem* $(\mathcal{P}_m)$*, then there exists a measurable function u such as ∀ K > 0, we have for all subsequence noted again* u_n*,*

(a) $u_m\to u$ *a.e* in Ω,

$(b)T_K(u_m)\rightharpoonup T_K(u)$ weakly in $\mathring{W}^1_M(\Omega)$,

$(c)T_K(u_m)\to T_K(u)$ strongly in $\mathring{W}^{-1}_{\overline{M}}(\Omega)$.

Proposition 5. *Suppose that conditions (19)–(22) and (18) are satisfied, and let* $(u_m)_{m\in\mathbb{N}}$ *be a solution of the problem* $(\mathcal{P}_m)$*, then for any K > 0, we have*

(1) $\sigma^m(x,T_K(u_m),\nabla T_K(u_m))$ is bounded in $\mathring{W}^1_{\overline{M}}(\Omega)$,

(2) $M(\,|\,\nabla T_K(u_m)\,|\,)\to M(\,|\,\nabla T_K(u)\,|\,)$ *is strongly in* $L^1(\Omega)$,

Proof. 1)

$$\| \sigma^m(x, T_K(u_m), \nabla T_K(u_m)) \|_{\overline{M},\Omega} = \sum_{i=1}^{N} \| \sigma_i^m(x, T_K(u_m), \nabla T_K(u_m)) \|_{\overline{M_i},\Omega}$$
$$\leq \sum_{i=1}^{N} \int_\Omega M_i(|\nabla T_K(u_m)|)\, dx + \|\varphi\|_1 + N,$$

from Proposition 3 we obtain:

$$\| \sigma^m(x, T_K(u_m), \nabla T_K(u_m)) \|_{\overline{M},\Omega} \leq c(K) + \|\varphi\|_1 + N.$$

Hence, $\sigma^m(x, T_K(u_m), \nabla T_K(u_m))$ is bounded in $\overset{\circ}{W}{}^1_{\overline{M}}(\Omega)$.

2) Showing that $M(|\nabla T_K(u_m)|) \to M(|\nabla T_K(u)|)$ strongly in $L^1(\Omega)$ that's why, let's introduce the following functions of a variable K defined as $h_j(K) = \begin{cases} 1 & \text{if} |K| \leq \text{j}, \\ 0 & \text{if} |K| \geqslant \text{j}+1, \\ j+1+|K| & \text{if } j < |K| < j+1, \end{cases}$ with j as a non-negative real parameter, $\Omega_K = \{ \mathrm{x} \in \Omega : |\nabla \mathrm{T_K(u(x))}| \leq \mathrm{K} \}$ and we note that χ_K is a characteristic function of Ω_K. It's clear that $\Omega_K \subset \Omega_{K+1}$ and $\mathrm{meas}(\Omega \setminus \Omega_K) \to 0$ since $K \to \infty$ shows that the following assertions are true.

Assertion 1.

$$\lim_{j\to\infty} \lim_{m\to\infty} \int_{\{\Omega: j<|K|<j+1\}} \sigma^m(x, u_m, \nabla u_m) \,.\, \nabla u_m \, dx = 0.$$

Assertion 2.

$$T_K(u_m) \to T_K(u) \text{ modular convergence in } \overset{\circ}{W}{}^1_M(\Omega).$$

Proof of assertion 1. Let

$$v = u_m + \exp(G(|u_m|)) \,.\, T_1(u_m - T_j(u_m)),$$

with $G(s) = \int_0^s \frac{l(t)}{\bar{a}}\, dt$ as a test function in $(\mathcal{P}_m)$ then we get:

$$\sum_{i=1}^{N} \int_\Omega \sigma_i^m(x, u_m, \nabla u_m) \,.\, \nabla(\exp(G(|u_m|)) \,.\, T_1(u_m - T_j(u_m)))\, dx$$
$$+ \sum_{i=1}^{N} \int_\Omega \mathfrak{b}_i^m(x, u_m, \nabla u_m) \,.\, \exp(G(|u_m|)) \,.\, T_1(u_m - T_j(u_m))\, dx$$
$$+ \int_\Omega m \,.\, T_m(u_m - \zeta)^- \,.\, sg_{\frac{1}{m}}(u_m) \,.\, \exp(G(|u_m|)) \,.\, T_1(u_m - T_j(u_m))\, dx$$
$$= \int_\Omega f^m(x) \,.\, \exp(G(|u_m|)) \,.\, T_1(u_m - T_j(u_m))\, dx,$$

by (20) and (22) we obtain:

$$\sum_{i=1}^{N}\int_{\Omega}\sigma_i^m(x,u_m,\nabla u_m)\,.\exp(G(|u_m|))\,.\nabla(T_1(u_m-T_j(u_m)))\;dx$$

$$+\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\;dx$$

$$\leq\int_{\Omega}\left[f^m(x)+h(x)+\phi(x)\,.\,\frac{l(u_m)}{\overline{a}}\right].\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\;dx,$$

since $f,\phi\in L^1(\Omega), l\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R}), \exp(G(\pm\infty))\leq\exp\left(\frac{\|l(u_m)\|_{L^1(\mathbb{R})}}{\overline{a}}\right)$ and by proposition 4 we obtain

$$\lim_{m\to\infty}\lim_{j\to\infty}\int_{\Omega}\left[f^m(x)+h(x)+\phi(x)\,.\,\frac{l(u_m)}{\overline{a}}\right].\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\;dx=0.$$

Hence,

$$\lim_{m\to\infty}\lim_{j\to\infty}\sum_{i=1}^{N}\int_{\{\Omega:\,j<|u_m|<j+1\}}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\;dx=0,$$

and

$$\lim_{m\to\infty}\lim_{j\to\infty}\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\exp(G(|u_m|))\,.\,T_1(u_m-T_j(u_m))\;dx=0.$$

Proof of assertion 2. Let $j\geq K>0$, we consider

$$v=u_m+\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m),$$

as a test function in $(\mathcal{P}_m)$ we obtain:

$$\sum_{i=1}^{N}\int_{\Omega}\sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m))\;dx$$

$$+\sum_{i=1}^{N}\int_{\Omega}\mathfrak{b}_i^m(x,u_m,\nabla u_m)\,.\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\;dx$$

$$+\int_{\Omega}m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\;dx$$

$$=\int_{\Omega}f^m(x)\,.\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\;dx,$$

by (20) and (22) we have:

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\{|u_m|\le K\}}\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m)).(\nabla T_K(u_m)-\nabla T_K(u)).\exp(G(|u_m|))\,dx\\
&+\int_\Omega m.T_m(u_m-\zeta)^-.sg_{\frac{1}{m}}(u_m)\exp(G(|u_m|)).(T_K(u_m)-T_K(u)).h_j(u_m)\,dx\\
&\le\int_\Omega\left[f^m(x)+h(x)+\phi(x).\frac{l(u_m)}{\bar a}\right].\exp(G(|u_m|)).(T_K(u_m)-T_K(u)).h_j(u_m)\,dx\\
&+\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}|\sigma_i^m(x,T_{j+1}(u_m),\nabla T_{j+1}(u_m))|.|\nabla T_K(u_m)|.\exp(G(|u_m|))\,dx\\
&+\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}\sigma_i^m(x,u_m,\nabla u_m).\nabla u_m.|T_K(u_m)-T_K(u)|.\exp(G(|u_m|))\,dx,
\end{aligned}
\tag{24}
$$

and since $T_K(u_m)\rightharpoonup T_K(u)$ is weakly in $\mathring{W}^1_M(\Omega)$, we have:

$$
\int_\Omega\left[f^m(x)+h(x)+\phi(x).\frac{l(u_m)}{\bar a}\right].\exp(G(|u_m|)).(T_K(u_m)-T_K(u)).h_j(u_m)\,dx\to 0,
$$

and

$$
\int_\Omega m.T_m(u_m-\zeta)^-.sg_{\frac{1}{m}}(u_m)\exp(G(|u_m|)).(T_K(u_m)-T_K(u)).h_j(u_m)\,dx\to 0,
$$

since $|\sigma_i^m(x,T_{j+1}(u_m),\nabla T_j(u_m))|$ is bounded in $L_{\overline{M}}(\Omega)$, then there exist $\tilde\sigma^m\in L_{\overline{M}}(\Omega)$ such that

$$
|\sigma_i^m(x,T_{j+1}(u_m),\nabla T_{j+1}(u_m))|\rightharpoonup\tilde\sigma^m\ \text{in}\ L_{\overline{M}}(\Omega),
\tag{25}
$$

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}|\sigma_i^m(x,T_{j+1}(u_m),\nabla T_{j+1}(u_m))|.|\nabla T_K(u_m)|.\exp(G(|u_m|))\,dx\\
&\le\exp\left(\frac{\|l\|_{L^1(\mathbb{R})}}{\bar a}\right).\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}|\sigma_i^m(x,T_{j+1}(u_m),\nabla T_{j+1}(u_m))|.|\nabla T_K(u_m)|\,dx\\
&\to\exp\left(\frac{\|l\|_{L^1(\mathbb{R})}}{\bar a}\right).\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}\tilde\sigma^m.|\nabla T_K(u)|\,dx=0\ \text{with}\ m\to\infty,
\end{aligned}
\tag{26}
$$

according to Assertion 1, we get:

$$\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}\sigma_i^m(x,u_m,\nabla u_m).\nabla u_m.|T_K(u_m)-T_K(u)|.\exp(G(|u_m|))\,dx$$
$$\leq 2K.\exp\left(\frac{\|l\|_{L^1(\mathbb{R})}}{\overline{a}}\right).\sum_{i=1}^{N}\int_{\{\Omega:K<|u_m|<j+1\}}\sigma_i^m(x,u_m,\nabla u_m).\nabla u_m\,dx\to 0 \text{ with } j\to\infty, \tag{27}$$

combine (24)–(27) we obtain:

$$\sum_{i=1}^{N}\int_{\Omega}\left[\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m))-\sigma_i^m(x,T_K(u_m),\nabla T_K(u))\right].(\nabla T_K(u_m)-\nabla T_K(u))\,dx\to 0$$
$$\text{with } m\to\infty. \tag{28}$$

According to Proposition 2 we conclude that

$$M(|\nabla T_K(u_m)|)\to M(|\nabla T_K(u)|)\text{ in }L^1(\Omega).$$

Proposition 6. *(See [31]) Suppose that the conditions (19)–(22) and (18) are true, and* $u^j, u\in \overset{\circ}{W}{}^1_M(\Omega)$

$$\|u^j\|_{\overset{\circ}{W}{}^1_M(\Omega)}\leq c\,,\quad j=1,\ldots,\infty \tag{29}$$

$$u^j\to u \text{ in } L_M(\Omega), \tag{30}$$

with M(z) is a N-function. Let's assume the following functions:

$$\mathfrak{A}^j(x)=\sum_{i=1}^{N}\left(\sigma_i^m(x,u^j,\nabla u^j)-\sigma_i^m(x,u,\nabla u)\right)\nabla(u^j-u)$$
$$+\sum_{i=1}^{N}\left(\mathfrak{b}_i^m(x,u^j,\nabla u^j)-\mathfrak{b}_i^m(x,u,\nabla u)\right)(u^j-u),$$

$j=1,\cdots$ satisfying the condition

$$\int_\Omega \mathfrak{A}^j(x)\,dx\to 0,\qquad j\to\infty. \tag{31}$$

Then, there exists a sequence of natural numbers $J\subset\mathbb{N}$ such that as $j\to\infty$, $j\in J$

$$\sigma_i^m(x,u^j,\nabla u^j)\rightharpoonup\sigma_i^m(x,u,\nabla u)\text{ in } L_{\overline{M}_i}(\Omega)\qquad i=1,\ldots,N. \tag{32}$$

Theorem 2. *Under assumptions (19)–(22), the problem $(\mathcal{P})$ has at least one entropy solution.*

Proof of Theorem 2. We divide our proof in six steps:

Step 1: A priori estimate of $\{u^m\}$. We consider the following test function:

$$v = u_m + \eta \exp(G(|u_m|))\, T_1(u_m - T_j(u_m)),$$

with η small enough, we get:

$$\sum_{i=1}^{N} \int_\Omega \sigma_i^m(x, u_m, \nabla u_m) \,.\, \nabla[\exp(G(|u_m|))\, T_1(u_m - T_j(u_m))]\, dx$$

$$+ \sum_{i=1}^{N} \int_\Omega \mathfrak{b}_i^m(x, u_m, \nabla u_m) \,.\, \exp(G(|u_m|))\, T_1(u_m - T_j(u_m))\, dx$$

$$+ \int_\Omega m \,.\, T_m(u_m - \zeta)^- \,.\, sg_{\frac{1}{m}}(u_m) \,.\, \exp(G(|u_m|))\, T_1(u_m - T_j(u_m))$$

$$\leq \int_\Omega f^m(x) \,.\, \exp(G(|u_m|))\, T_1(u_m - T_j(u_m))\, dx,$$

according to (20) and (22) we obtain:

$$\sum_{i=1}^{N} \int_\Omega \sigma_i^m(x, u_m, \nabla u_m) \,.\, \exp(G(|u_m|))\, \nabla T_1(u_m - T_j(u_m))\, dx$$

$$+ \int_\Omega m \,.\, T_m(u_m - \zeta)^- \,.\, sg_{\frac{1}{m}}(u_m) \,.\, \exp(G(|u_m|))\, T_1(u_m - T_j(u_m))$$

$$\leq \int_\Omega \left[f^m(x) + h(x) + \phi(x) \,.\, \frac{l(u_m)}{\bar{a}} \right] .\, \exp(G(|u_m|))\, T_1(u_m - T_j(u_m))\, dx,$$

since f, h, $\phi \in L^1(\Omega)$, $l \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $\exp(G(\pm\infty)) \leq \exp\left(\frac{\| l \|_{L^1(\mathbb{R})}}{\bar{a}} \right)$ and the fact $T_1(u_m - T_j(u_m)) \to 0$ is weakly in $\mathring{W}^1_M(\Omega)$ as j → ∞ (proposition 4). We have:

$$\int_\Omega \left[f^m(x) + h(x) + \phi(x) \,.\, \frac{l(u_m)}{\bar{a}} \right] .\, \exp(G(|u_m|))\, T_1(u_m - T_j(u_m))\, dx \to 0 \text{ as } m \to \infty,$$

then,

$$\sum_{i=1}^{N} \int_\Omega \sigma_i^m(x, u_m, \nabla u_m) \,.\, \exp(G(|u_m|)) \,.\, \nabla T_1(u_m) - T_j(u_m)\, dx$$

$$+ \int_\Omega m \,.\, T_m(u_m - \zeta)^- \,.\, sg_{\frac{1}{m}}(u_m) \,.\, \exp(G(|u_m|)) \,.\, T_1(u_m) - T_j(u_m)\, dx$$

$$\leq 0.$$

Hence,

$$\lim_{m\to\infty} \lim_{j\to\infty} \sum_{i=1}^{N} \int_{\{\Omega:\, K < |u_m| < j+1\}} \sigma_i^m(x, u_m, \nabla u_m) \,.\, \nabla u_m\, dx = 0,$$

and

$$\lim_{m\to\infty}\int_\Omega m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\;dx=0. \tag{33}$$

Step 2. Convergence of the gradient:

In this step we consider again the following test function:

$$v=u_m+\eta\exp(G(|u_m|))\,(\,T_K(u_m)-T_K(u))\,h_j(u_m),$$

with, $h_j(u_m)=1-|\,T_1(u_m-T_j(u_m))\,|=\begin{cases}1 & \text{if } \{|u_m|\geq j\},\\ 0 & \text{if } \{|u_m|\geq j+1\},\\ j+1-|u_m| & \text{if } \{j<|u_m|<j+1\},\end{cases}$ and

$|T_K(u_m)-T_K(u)|$ at the same sign when $u_m\in\{\,|u_m|>K\}$ where $j\geq K>0$ and η are small enough, we obtain

$$\begin{aligned}&\sum_{i=1}^{N}\int_\Omega \sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla(\exp(G(|u_m|))\,(\,T_K(u_m)-T_K(u))\,h_j(u_m))\;dx\\&+\int_\Omega m\,.\,T_K(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\exp(G(|u_m|))\,(\,T_K(u_m)-T_K(u))\,h_j(u_m)\;dx\\&+\sum_{i=1}^{N}\int_\Omega \mathfrak{b}_i^m(x,u_m,\nabla u_m)\,.\exp(G(|u_m|))\,(\,T_K(u_m)-T_K(u))\,h_j(u_m)\;dx\\&\leq\int_\Omega f^m(x)\,.\exp(G(|u_m|))\,(\,T_K(u_m)-T_K(u))\,h_j(u_m)\;dx,\end{aligned}$$

by (20), (22) and the fact $j\geq K>0$ we have this:

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\{|u_m|\leq K\}}\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m)).\exp(G(|u_m|)).\nabla(T_K(u_m)-T_K(u))\,dx\\&+\sum_{i=1}^{N}\int_\Omega\sigma_i^m(x,u_m,\nabla u_m).\exp(G(|u_m|)).(T_K(u_m)-T_K(u)).\nabla h_j(u_m)\,dx\\&+\int_\Omega m.T_m(u_m-\zeta)^-.sg_{\frac{1}{m}}(u_m).\exp(G(|u_m|)).(T_K(u_m)-T_K(u)).h_j(u_m)\,dx\\&\leq\int_\Omega\left[f^m(x)+h(x)+\phi(x).\frac{l(|u_m|)}{\bar{a}}\right].\exp(G(|u_m|)).(T_K(u_m)-T_K(u)).h_j(u_m)\,dx,\end{aligned} \tag{34}$$

then, by the condition (c) in proposition 4 we have $T_K(u_m)\to T_K(u)$ weakly in $\mathring{W}_M^1(\Omega)$, and since f^m, $h(x)$, $\phi\in L^1(\Omega)$ we get

$$\int_\Omega\left[f^m(x)+h(x)+\phi(x).\frac{l(|u_m|)}{\bar{a}}\right].\exp(G(|u_m|)).(T_K(u_m)-T_K(u)).h_j(u_m)\,dx\to 0, \tag{35}$$

and

$$\int_\Omega m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,h_j(u_m)\,dx=0,$$

and

$$\sum_{i=1}^N \int_\Omega \sigma_i^m(x,u_m,\nabla u_m)\,.\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,.\,\nabla h_j(u_m)\,dx$$

$$=\sum_{i=1}^N \int_{\{\Omega:j<|u_m|<j+1\}} \sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,.\exp(G(|u_m|))\,.\,(T_K(u_m)-T_K(u))\,dx \tag{36}$$

$$\leq 2K\,.\exp\left(\frac{\|l\|_{L^1(\Omega)}}{\bar{a}}\right).\sum_{i=1}^N \int_{\{\Omega:j<|u_m|<j+1\}} \sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla u_m\,dx \to 0 \text{ as } j\to\infty,$$

combining (34) – (36)

$$\sum_{i=1}^N \int_{\{|u_m|\leq K\}} \sigma_i^m(x,T_K(u_m),\nabla T_K(u_m))\,.\exp(G(|u_m|))\,.\,\nabla(T_K(u_m)-T_K(u))\,dx \leq \epsilon(i,j,m),$$

thus,

$$\sum_{i=1}^N \int_\Omega \left[\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m))-\sigma_i^m(x,T_K(u_m),\nabla T_K(u))\right].\,\nabla(T_K(u_m)$$

$$-T_K(u))\,.\exp(G(|u_m|))\,dx \leq -\sum_{i=1}^N \int_\Omega \sigma_i^m(x,T_K(u_m),\nabla T_K(u))\,.\,\nabla(T_K(u_m)$$

$$-T_K(u))\,.\exp(G(|u_m|))\,dx - \sum_{i=1}^N \int_{\{|u_m|>K\}} \sigma_i^m(x,T_K(u_m),\nabla T_K(u_m))\,.\,\nabla(T_K(u_m)$$

$$-T_K(u))\,.\exp(G(|u_m|))\,dx+\epsilon(i,j,m),$$

letting i, j, m tend to infinity, we have:

$$\sum_{i=1}^N \int_\Omega \left[\sigma_i^m(x,T_K(u_m),\nabla T_K(u_m))-\sigma_i^m(x,T_K(u_m),\nabla T_K(u))\right].\,\nabla(T_K(u_m)$$

$$-T_K(u))\,.\exp(G(|u_m|))\,dx\to 0 \text{ as } m\to\infty, \tag{37}$$

which is implied by Proposition 2

$$M(|\nabla u_m|)\to M(|\nabla u|) \text{ in } L^1(\Omega). \tag{38}$$

Hence, we obtain for a subsequence:

$$\nabla u_m \to \nabla u \text{ a.e in } \Omega. \tag{39}$$

Step 3. The equi-integrability of $\mathfrak{b}_i^m(x,u_m,\nabla u_m)$:

In this section we will show that:

$$\mathfrak{b}_i^m(x, u_m, \nabla u_m) \to \mathfrak{b}_i(x, u, \nabla u). \tag{40}$$

Therefore, it is enough to show that $\mathfrak{b}_i^m(x, u_m, \nabla u_m)$ is uniformly equi-integrable. We take the following test function:

$$v = u_m + \eta \exp(G(|u_m|)) \int_{|u_m|}^0 \int_{\{\,|s|>j\,\}} l(s)\ ds\ dx,$$

we obtain:

$$\begin{aligned}\sum_{i=1}^N &\int_\Omega \sigma_i^m(x, u_m, \nabla u_m)\,.\,\nabla \exp(G(|u_m|)) \int_{|u_m|}^0 \int_{\{\,|s|>j\,\}} l(s)\ ds\ dx\\ &+ \sum_{i=1}^N \int_\Omega \mathfrak{b}_i^m(x, u_m, \nabla u_m)\,.\,\exp(G(|u_m|)) \int_{|u_m|}^0 \int_{\{\,|s|>j\,\}} l(s)\ ds\ dx\\ &+ \int_\Omega m\,.\,T_m(u_m - \zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|)) \int_{|u_m|}^0 \int_{\{\,|s|>j\,\}} l(s)\ ds\ dx\\ &= \int_\Omega f^m(x)\,.\,\exp(G(|u_m|)) \int_{|u_m|}^0 \int_{\{\,|s|>j\,\}} l(s)\ ds\ dx,\end{aligned}$$

by (20) and (22) we get:

$$\begin{aligned}\bar{a} \quad &\sum_{i=1}^N \int_\Omega M_i(\,|\nabla u_m|\,)\,.\,\exp(G(|u_m|))\,.\int_{\{\,|u_m|>j\,\}} l(u_m)\ dx\\ &+ \int_\Omega m\,.\,T_m(u_m - \zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|)) \int_{|u_m|}^0 \int_{\{\,|s|>j\,\}} l(s)\ ds\ dx\\ &\le \int_\Omega \left[f^m + h(x) + \phi(x)\,.\,\frac{l(|u_m|)}{\bar{a}}\right].\,\exp(G(|u_m|)) \int_{|u_m|}^0 \int_{\{\,|s|>j\,\}} l(s)\ ds\ dx\\ &+ \int_\Omega \phi(x)\,.\,\exp(G(|u_m|))\,.\int_{\{\,|u_m|>j\,\}} l(u_m)\ dx,\end{aligned}$$

which implies:

$$\bar{a} \sum_{i=1}^N \int_\Omega M_i(\,|\nabla u_m|\,)\,.\,\exp(G(|u_m|))\,.\,l(u_m)\,.\,\chi_{\{\,|u_m|>j\,\}}\ dx \le c_1 \int_{|u_m|}^0 l(u_m)\,.\,\chi_{\{\,|u_m|>j\,\}}\ dx.$$

Therefore,

$$\sum_{i=1}^N \int_{\{\,|u_m|>j\,\}} l(u_m)\,.\,M_i(|\nabla u_m|)\ dx \le c_2 \int_{|u_m|}^0 l(|u_m|)\,.\,\chi_{\{\,|u_m|>j\,\}}\ dx.$$

and

$$0 \le \int_\Omega m\,.\,T_m(u_m - \zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,\exp(G(|u_m|)) \int_{|u_m|}^0 \int_{\{\,|s|>j\,\}} l(s)\ ds\ dx \le c_3, \tag{41}$$

and since $l \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ we deduce that:

$$\lim_{j\to\infty}\sup_{m\in\{1,\dots,N\}}\sum_{i=1}^{N}\int_{\{|u_m|\}} l(|u_m|)\,.\,M_i(|\nabla u_m|)\;dx=0,$$

by (37) and (31) we conclude (30)

Step 4. Passing to the limit

Let $\varphi\in \mathring{W}_M^1(\Omega)\cap L^\infty(\Omega)$ we take the following test function:

$$v=u_m-\eta\, T_j(u_m-\varphi),$$

and $|u_m|-\|\varphi\|_\infty<|u_m-\varphi|\le j$. Then, $\{|u_m-\varphi|\le j\}\subset\{|u_m|\le j+\|\varphi\|_\infty\}$ we obtain:

$$\sum_{i=1}^{N}\int_\Omega \sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla T_j(u_m-\varphi)\;dx+\sum_{i=1}^{N}\int_\Omega \mathfrak{b}_i^m(x,u_m,\nabla u_m)\,.\,T_j(u_m-\varphi)\;dx$$

$$+\int_\Omega m\,.\,T_m(u_m-\zeta)^-\,.\,sg_{\frac{1}{m}}(u_m)\,.\,T_j(u_m-\varphi)\;dx$$

$$\le\int_\Omega f^m(x)\,.\,T_j(u_m-\varphi)\;dx,$$

which implies that:

$$\begin{aligned}\sum_{i=1}^{N}&\int_\Omega \sigma_i^m(x,u_m,\nabla u_m)\,.\,\nabla T_j(u_m-\varphi)\;dx\\ &=\sum_{i=1}^{N}\int_\Omega\left[\sigma_i^m(x,T_{j+\|\varphi\|_\infty}(u_m),\nabla T_{j+\|\varphi\|_\infty}(u_m))-\sigma_i^m(x,T_{j+\|\varphi\|_\infty}(u_m),\nabla\varphi)\right]\\ &\qquad\times\nabla T_{j+\|\varphi\|_\infty}(u_m-\varphi)\,.\,\chi_{\{|u_m-\varphi|<j\}}\;dx\\ &\quad+\sum_{i=1}^{N}\int_\Omega \sigma_i^m(x,T_{j+\|\varphi\|_\infty}(u_m),\nabla\varphi)\,\nabla T_{j+\|\varphi\|_\infty}(u_m-\varphi)\,.\,\chi_{\{|u_m-\varphi|<j\}}\;dx,\end{aligned}$$

by Fatou's Lemma we get:

$$\begin{aligned}\liminf_{m\to\infty}&\sum_{i=1}^{N}\int_\Omega \sigma_i^m(x,T_{j+\|\varphi\|_\infty}(u_m),\nabla\varphi)\,\nabla T_{j+\|\varphi\|_\infty}(u_m-\varphi)\,.\,\chi_{\{|u_m-\varphi|<j\}}\;dx\\ &=\sum_{i=1}^{N}\int_\Omega \sigma_i^m(x,T_{j+\|\varphi\|_\infty}(u),\nabla\varphi)\,\nabla T_{j+\|\varphi\|_\infty}(u-\varphi)\,.\,\chi_{\{|u-\varphi|<j\}}\;dx,\end{aligned}$$

and the fact that

$$\sum_{i=1}^{N}\sigma_i^m(x,T_{j+\|\varphi\|_\infty}(u_m),\nabla T_{j+\|\varphi\|_\infty}(u_m))\rightharpoonup\sum_{i=1}^{N}\sigma_i^m(x,T_{j+\|\varphi\|_\infty}(u),\nabla T_{j+\|\varphi\|_\infty}(u))\tag{42}$$

weakly in $\mathring{W}_M^1(\Omega)$. And since $T_j(u_m-\varphi)\rightharpoonup T_j(u-\varphi)$ weakly in $\mathring{W}_M^1(\Omega)$, and by (39) we obtain:

$$\sum_{i=1}^{N}\int_\Omega \mathfrak{b}_i^m(x,u_m,\nabla u_m)\;T_j(u_m-\varphi)\;dx\to\sum_{i=1}^{N}\int_\Omega \mathfrak{b}_i(x,u,\nabla u)\;T_j(u-\varphi)\;dx,$$

and

$$\int_\Omega f^m(x)\, T_j(u_m - \varphi)\ dx \to \int_\Omega f(x)\, T_j(u - \varphi)\ dx,$$

and

$$\int_\Omega m\, .\, T_m(u_m - \zeta)^-\, .\, sg_{\frac{1}{m}}(u_m)\, .\, T_j(u_m - \varphi)\ dx \to \int_\Omega m\, .\, T_m(u - \zeta)^-\, .\, sg_{\frac{1}{m}}(u)\, .\, T_j(u - \varphi)\ dx,$$

which completes the proof of Theorem 2.

Remark 3. *For the demonstration of the uniqueness solution to this problem* $(\mathcal{P})$ *in unbounded domain is obtained in* [31] *with the operator* $\mathfrak{b}_i(x, u, \nabla u) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *are strictly monotonic, at least for a broad class of lower order term, and in* [27] *with the operator* $\mathfrak{b}_i(x, u, \nabla u) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ *for* $i = 1, \ldots, N$ *are contraction Lipschitz continuous functions which do not satisfy any sign condition, and*

$$\sum_{i=1}^{N} [\,\sigma_i(x, \xi, \nabla\xi) - \sigma_i(x, \xi', \nabla\xi')\,]\, .\, (\nabla\xi - \nabla\xi')\ dx > 0.$$

References

1. Boccardo L, Gallouët T. Non-linear elliptic and parabolic equations involving measure data. J Funct Anal. 1989; 87: 149-69.
2. Bénilan P, Boccardo L, Gallouët T, Gariepy R, Pierre M, Vázquez JL. An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann della Scuola Norm Super Pisa - Cl Sci. 1995; 22: 241-273.
3. Aberqi A, Bennouna J, Mekkour M, Redwane H. Nonlinear parabolic inequalities with lower order terms. Hist Anthropol. 2017; 96: 2102-17.
4. Azroul E, Khouakhi M, Yazough C. Nonlinear p (x)-Elliptic equations in general domains. Differential Equations and Dynamical Systems. 2018; 1-24.
5. Benslimane O, Aberqi A, Bennouna J. Existence and Uniqueness of Weak solution of p(x)-laplacian in Sobolev spaces with variable exponents in complete manifolds. arXiv preprint arXiv: 2006.04763. 2020.
6. Boccardo L, Gallouët T, Vazquez JL. Nonlinear elliptic equations in $\mathbb{R}^N$ without growth restrictions on the data. J Differential Equations. 1993; 105: 334-63.
7. Boccardo L, Gallouët T. Nonlinear elliptic equations with right hand side measures. Commun Partial Differential Equations. 1992; 17: 89-258.
8. Gossez JP. Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients. Trans Am Math Soc. 1974; 190: 163-205.
9. Gushchin AK. The Dirichlet problem for a second-order elliptic equation with an Lp boundary function. Sb. Math. 2012; 203: 1.
10. Laptev GI. Existence of solutions of certain quasilinear elliptic equations in $\mathbb{R}^N$ without conditions at infinity. J Math Sci. 2008; 150: 2384-94.
11. Benkirane A, Bennouna J. Existence of entropy solutions for some nonlinear problems in Orlicz spaces. Abstr. Appl. Anal. 2002; 7: 85-102.
12. Aberqi A, Bennouna J, Elmassoudi M, Hammoumi M. Existence and uniqueness of a renormalized solution of parabolic problems in Orlicz spaces. Monatsh Math. 2019; 189: 195-219.

13. Aharouch L, Benkirane A, Rhoudaf M. Existence results for some unilateral problems without sign condition with obstacle free in Orlicz spaces. Nonlinear Anal Theor Methods Appl. 2008; 68: 2362-80.
14. Aharouch L, Bennouna J. Existence and uniqueness of solutions of unilateral problems in Orlicz spaces. Nonlinear Anal Theor Methods Appl. 2010; 72: 3553-65.
15. Bonanno G, Bisci GM, Rădulescu V. Quasilinear elliptic non-homogeneous Dirichlet problems through Orlicz–Sobolev spaces. Nonlinear Anal Theor Methods Appl. 2012; 75: 4441-56.
16. Bonanno G, Bisci GM, Rădulescu V. Arbitrarily small weak solutions for a nonlinear eigenvalue problem in Orlicz–Sobolev spaces. Monatsh. Math. 2012; 165: 305-18.
17. Bonanno G, Bisci GM, Rădulescu V. Infinitely many solutions for a class of nonlinear eigenvalue problem in Orlicz–Sobolev spaces, Compt. Rendus Math. 2011; 349: 263-68.
18. Bonanno G, Bisci GM, Rădulescu V. Existence of three solutions for a non-homogeneous Neumann problem through Orlicz–Sobolev spaces, Nonlinear Anal Theor Methods Appl. 2011; 74: 4785-95.
19. Chmara M, Maksymiuk J. Anisotropic Orlicz–Sobolev spaces of vector valued functions and Lagrange equations. J Math Anal Appl. 2017; 456: 457-75.
20. Chmara M, Maksymiuk J. Mountain pass type periodic solutions for Euler–Lagrange equations in anisotropic Orlicz–Sobolev space. J Math Anal Appl. 2019; 470: 584-98.
21. Dong GE, Fang X. Existence results for some nonlinear elliptic equations with measure data in Orlicz-Sobolev spaces. Bound. Value Probl. 2015; 2015: 18.
22. Lions JL. Quelques méthodes de résolution des problemes aux limites non linéaires. Dunod; 1969.
23. Mihăilescu M, Pucci P, Rădulescu V. Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent. J Math Anal Appl. 2008; 340: 687-98.
24. Ragusa MA, Tachikawa A. Regularity for minimizers for functionals of double phase with variable exponents. Adv Nonlinear Anal. 2019; 9: 710-28.
25. Cianchi A. A fully anisotropic Sobolev inequality. Pac J Mathematics. 2000; 196: 283-94.
26. Korolev AG. Embedding theorems for anisotropic Sobolev–Orlicz spaces. Vestn. Mosk. Univ. Seriya 1 Mat. Mekhanika. 1983: 32-37.
27. Benslimane O, Aberqi A, Bennouna J. Existence and uniqueness of entropy solution of a nonlinear elliptic equation in anisotropic Sobolev–Orlicz space. Rendiconti Del Circolo Matematico di Palermo Ser. 2020; 2: 1-30.
28. Kozhevnikova LM. On the entropy solution to an elliptic problem in anisotropic Sobolev–Orlicz spaces. Comput Mathematics Math Phys. 2017; 57: 434-52.
29. Alberico A, di Blasio G, Feo F. An eigenvalue problem for the anisotropic Φ-Laplacian. J Differential Equations. 2020; 269: 4853-83.
30. Barletta G. On a class of fully anisotropic elliptic equations. Nonlinear Anal. 2020; 197: 111838.
31. Benslimane O, Aberqi A, Bennouna J. The existence and uniqueness of an entropy solution to unilateral Orlicz anisotropic equations in an unbounded domain. Axioms. 2020; 9: 109.
32. Kozhevnikova LM. Existence of entropic solutions of an elliptic problem in anisotropic Sobolev–Orlicz spaces. J Math Sci. 2019; 241: 258-84.
33. Ball JM. Convexity conditions and existence theorems in nonlinear elasticity. Arch Ration Mech Anal. 1976; 63: 337-403.
34. Benkhira EH, Essoufi EH, Fakhar R. On convergence of the penalty method for a static unilateral contact problem with nonlocal friction in electro-elasticity. Eur J Appl Mathematics. 2016; 27: 1.
35. Chen Y, Levine S, Rao M. Variable exponent, linear growth functionals in image restoration. SIAM J Appl Mathematics. 2006; 66: 1383-406.

36. Růžička M. Modeling, mathematical and numerical analysis of electrorheological fluids. Appl Mathematics. 2004; 49: 565-609.
37. Krasnosel'skii MA, Rutickii JB. Convex functions and Orlicz spaces. Fizmatgiz: Moscow. 1958.

Appendix

Let

$$\mathfrak{A} : \overset{\circ}{W}{}^1_M(\Omega) \to (\overset{\circ}{W}{}^1_M(\Omega))'$$

$$v \to \ < \mathfrak{A}(u), v > = \int_\Omega \sum_{i=1}^N \left(\sigma_i(x, u, \nabla u) \, . \, \frac{\partial v}{\partial x_i} + \mathfrak{b}_i(x, u, \nabla u) \, . \, v \right) dx$$

$$- \int_\Omega f(x) \, . \, v \; dx,$$

and let denote $L_{\overline{M}}(\Omega) = \prod_{k=1}^N L_{\overline{M}_i}(\Omega)$ with the norm:

$$\| \, v \, \|_{L_{\overline{M}}(\Omega)} = \sum_{i=1}^N \| \, v_i \, \|_{\overline{M}_i,\Omega} \qquad v = (v_1, \ldots, v_N) \in L_{\overline{M}}(\Omega).$$

where $\overline{M_i}(t)$ are N-functions satisfying the Δ_2-conditions.

Sobolev-space $\overset{\circ}{W}{}^1_M(\Omega)$ is the completion of the space $C_0^\infty(\Omega)$.

$$\sigma(x, s, \xi) = (\, \sigma_1(x, s, \xi), \ldots, \sigma_N(x, s, \xi) \,)$$

and

$$\mathfrak{b}(x, s, \xi) = (\, \mathfrak{b}_1(x, s, \xi), \ldots, \mathfrak{b}_N(x, s, \xi) \,).$$

Let's show that operator $\mathfrak{A}$ is bounded. So, for $u \in \overset{\circ}{W}{}^1_M(\Omega)$, according to (9) and (23) we get:

$$\begin{aligned} \| \, \sigma(x, u, \nabla u) \, \|_{L_{\overline{M}}(\Omega)} &= \sum_{i=1}^N \| \, \sigma_i(x, u, \nabla u) \, \|_{L_{\overline{M}_i}(\Omega)} \\ &\leq \sum_{i=1}^N \int_\Omega \overline{M}_i(\, \sigma_i(x, u, \nabla u) \,) \; dx + N \\ &\leq \tilde{a}(\Omega) \, . \, \| \, M(u) \, \|_{1,\Omega} + \| \, \varphi \|_{1,\Omega} + N. \end{aligned} \tag{43}$$

Further, for $\sigma(x, u, \nabla u) \in L_{\overline{M}_i}(\Omega), \;\; v \in \overset{\circ}{W}{}^1_M(\Omega)$ using Hölder's inequality we have:

$$\begin{aligned} | < \mathfrak{A}(u), v >_\Omega | \leq \; & 2 \| \, \sigma(x, u, \nabla u) \, \|_{L_{\overline{M}}(\Omega)} \, . \, \| \, v \, \|_{\overset{\circ}{W}{}^1_M(\Omega)} \\ & + 2 \| \, \mathfrak{b}(x, u, \nabla u) \, \|_{L_M(\Omega)} \, . \, \| \, v \, \|_{\overset{\circ}{W}{}^1_M(\Omega)} + c_0 \, . \, \| \, v \, \|_{\overset{\circ}{W}{}^1_M(\Omega)}. \end{aligned} \tag{44}$$

Thus, $\mathfrak{A}$ is bounded.

And that $\mathfrak{A}$ is coercive. So, for $u \in \overset{\circ}{W}{}^1_M(\Omega)$

$$< \mathfrak{A}(u), u >_\Omega \; = \; \sum_{i=1}^N \int_\Omega \sigma_i(x, u, \nabla u) \, . \, \frac{\partial u}{\partial x_i} \; dx \; + \; \sum_{i=1}^N \int_\Omega \mathfrak{b}_i(x, u, \nabla u) \, . \, u \; dx$$

$$- \int_\Omega f(x) \, . \, u \; dx.$$

Then,

$$\frac{<\mathfrak{A}(u),\, u>_\Omega}{\| u \|_{\mathring{W}^1_M(\Omega)}} \geq \frac{1}{\| u \|_{\mathring{W}^1_M(\Omega)}} \cdot \Bigg[\bar{a} \sum_{i=1}^{N} \int_\Omega M_i\left(\left| \frac{\partial u}{\partial x_i} \right| \right) dx - c_1 - c_0$$

$$-l(u) \cdot \sum_{i=1}^{N} \int_\Omega M_i\left(\left| \frac{\partial u}{\partial x_i} \right| \right) dx - \int_\Omega h(x)\, dx \Bigg]$$

$$\geq \frac{1}{\| u \|_{\mathring{W}^1_M(\Omega)}} \cdot \Bigg[(\,\bar{a}(\Omega) - c_2\,) \cdot \sum_{i=1}^{N} \int_\Omega M_i\left(\left| \frac{\partial u}{\partial x_i} \right| \right) dx - c_0 - c_1$$

$$-c_3\,\Bigg].$$

According to (18) we have for all $k > 0$, $\exists\, \alpha_0 > 0$ such that:

$$\mathfrak{b}_i(\,| u_{x_i} |\,) > k\, \mathfrak{b}_i\left(\frac{| u_{x_i} |}{\| u_{x_i} \|_{M_i,\Omega}} \right), \qquad i = 1, \ldots, N.$$

We take $\| u_{x_i} \|_{M_i,\Omega} > \alpha_0 \quad i = 1, \ldots, N.$

Suppose that $\| u_{x_i} \|_{\mathring{W}^1_M(\Omega)} \to 0$ as $j \to \infty$, we can assume that:

$$\| u^j_{x_1} \|_{M_1,\Omega} + \cdots + \| u^j_{x_N} \|_{M_N,\Omega} \geq N\, \alpha_0.$$

According to (9) for $c > 1$, we have:

$$| u^j |\, \mathfrak{b}(\,| u^j |\,) < c\, M(u^j),$$

then, by (2.8) we obtain:

$$\frac{<\mathfrak{A}(u^j),\, u^j >_\Omega}{\| u^j \|_{\mathring{W}^1_M(\Omega)}} \geq \frac{\bar{a}(\Omega) - c_2}{N\, \alpha_0} \cdot \sum_{i=1}^{N} \int_\Omega M_i\left(\left| \frac{\partial u}{\partial x_i} \right| \right) dx - \frac{c_4}{N\, \alpha_0}$$

$$\geq \frac{\bar{a}(\Omega) - c_2}{N\, \alpha_0} \cdot \sum_{i=1}^{N} \int_\Omega | u^j_{x_i} |\, \mathfrak{b}(\,| u^j_{x_i} |\,)\, dx - \frac{c_4}{N\, \alpha_0}$$

$$\geq \frac{(\,\bar{a}(\Omega) - c_2\,) \cdot k}{c\, N \| u^j_{x_i} \|_{M_i}} \cdot \sum_{i=1}^{N} \int_\Omega | u^j_{x_i} |\, \mathfrak{b}_i\left(\frac{| u^j_{x_i} |}{\| u^j_{x_i} \|_{M_i,\Omega}} \right) dx - \frac{c_4}{N\, \alpha_0}$$

$$\geq \frac{(\,\bar{a}(\Omega) - c_2\,) \cdot k}{c\, N} \cdot \sum_{i=1}^{N} \int_\Omega M_i\left(\frac{| u^j_{x_i} |}{\| u^j_{x_i} \|_{M_i,\Omega}} \right) dx - \frac{c_4}{N\, \alpha_0}$$

$$\geq \frac{(\,\bar{a}(\Omega) - c_2\,) \cdot k}{c\, N} - \frac{c_4}{N\, \alpha_0},$$

which shows that $\mathfrak{A}$ is coercive because k is arbitrary.

And finally that $\mathfrak{A}$ is pseudo-monotonic. Following up this assumption and since the space $\mathring{W}^1_M(\Omega)$ is separable, then $\exists (u^j) \in C_0^\infty(\Omega)$ such as:

$$u^j \rightharpoonup u \text{ in } \mathring{W}^1_M(\Omega), \tag{45}$$

and

$$\mathfrak{A}(u^j) \rightharpoonup y \text{ in } \left(\mathring{W}^1_M(\Omega)\right)'; \tag{46}$$

according to (45), we have for all subsequences denoted again by u^j,

$$\| u^j \|_{\mathring{W}^1_M(\Omega)} \leq c_2, \; j \in \mathbb{N}$$

$(u^j)_{j\in\mathbb{N}}$ is bounded in $\mathring{W}^1_M(\Omega)$, and since $\mathring{W}^1_M(\Omega)$ is continuously and compactly injected into $L_M(\Omega)$

$$u^j \rightharpoonup u \text{ weakly in } L_M(\Omega),$$

$$u^j \to u \text{ a.e. in } \Omega, \; j \in \mathbb{N},$$

and according to (39), we have:

$$\sigma_i^m(x, u^j, \nabla u^j) \to \sigma_i^m(x, u, \nabla u) \text{ a.e. in } \Omega, \; j \in \mathbb{N}$$

and

$$\mathfrak{b}_i^m(x, u^j, \nabla u^j) \to \mathfrak{b}_i^m(x, u, \nabla u) \text{ a.e. in } \Omega, \; j \in \mathbb{N}$$

and

$$m \,.\, T_m(u^j - \zeta)^- \,.\, sg_{\frac{1}{m}}(u^j) \to m \,.\, T_m(u - \zeta)^- \,.\, sg_{\frac{1}{m}}(u) \text{ a.e. in } \Omega, \; j \in \mathbb{N}$$

from (45) and (46), there exist $\tilde{\sigma}^m$ in $L_{\overline{M}}(\Omega)$ such as:

$$\sigma_i^m(x, u^j, \nabla u^j) \rightharpoonup \tilde{\sigma}^m, \; j \in \mathbb{N} \tag{47}$$

and there exist $\tilde{\mathfrak{b}}^m$ in $L_M(\Omega)$ such as:

$$\mathfrak{b}_i^m(x, u^j, \nabla u^j) \rightharpoonup \tilde{\mathfrak{b}}^m, \; j \in \mathbb{N}. \tag{48}$$

By (33) it is clear that for any $v \in \mathring{W}^1_M(\Omega)$, we get:

$$\begin{aligned} < y, v > &= \lim_{j\to\infty} \sum_{i=1}^N \int_\Omega \sigma_i^m(x, u^j, \nabla u^j) \,.\, \nabla v \; dx + \lim_{j\to\infty} \sum_{i=1}^N \int_\Omega \mathfrak{b}_i^m(x, u^j, \nabla u^j) \,.\, v \; dx \\ &= \int_\Omega \tilde{\sigma}^m \,.\, \nabla v \; dx + \int_\Omega \tilde{\mathfrak{b}}^m \,.\, v \; dx, \end{aligned} \tag{49}$$

whereof:

$$\begin{aligned} \limsup_{j\to\infty} < \mathfrak{A}(u^j), u^j > &= \limsup_{j\to\infty} \left\{ \sum_{i=1}^N \int_\Omega \sigma_i^m(x, u^j, \nabla u^j) \nabla u^j \; dx \right. \\ &\left. + \lim_{j\to\infty} \sum_{i=1}^N \int_\Omega \mathfrak{b}_i^m(x, u^j, \nabla u^j) u^j \; dx \right\} \leq \int_\Omega \tilde{\sigma}^m \nabla u^j \; dx + \int_\Omega \tilde{\mathfrak{b}}^m u^j \; dx. \end{aligned} \tag{50}$$

By (48), we have:

$$\int_\Omega \mathfrak{b}^m(x,u^j,\nabla u^j)\,u^j\,dx \to \int_\Omega \tilde{\mathfrak{b}}^m\,u\,dx. \tag{51}$$

Consequently,

$$\limsup_{j\to\infty}\sum_{i=1}^N \int_\Omega \sigma_i^m(x,u^j,\nabla u^j)\,\nabla u^j\,dx \le \int_\Omega \tilde{\sigma}^m\,\nabla u^j\,dx. \tag{52}$$

On the other hand, we have by the condition of monotony:

$$\sum_{i=1}^N (\,\sigma_i^m(x,u^j,\nabla u^j\,) - \sigma_i^m(x,u^j,\nabla u\,)\,)\,.\,\nabla(u^j - u) \ge 0,$$

which implies

$$\sum_{i=1}^N (\,\sigma_i(x,T_m(u^j),\nabla u^j) - \sigma_i(x,T^j(u^j),\nabla u)\,)\,.\,\nabla(u^j - u) \ge 0, \tag{53}$$

then,

$$\sum_{i=1}^N \sigma_i(x,T_m(u^j),\nabla u^j)\,.\,\nabla u^j \ge \sum_{i=1}^N \sigma_i(x,T_m(u^j),\nabla u)\,.\,\nabla(u^j - u)$$
$$+\sum_{i=1}^N \sigma_i(x,T_m(u^j),\nabla u^j)\,.\,\nabla u,$$

and by Step 2, we get:

$$\sum_{i=1}^N \sigma_i(x,T_m(u^j),\nabla u) \to \sum_{i=1}^N \sigma_i(x,T_m(u),\nabla u)\text{in } L_{\overline{M}}(\Omega),$$

according to (47), we obtain:

$$\liminf_{j\to\infty}\sum_{i=1}^N \int_\Omega \sigma_i^m(x,u^j,\nabla u^j)\,.\,\nabla u^j\,dx \ge \int_\Omega \tilde{\sigma}^m\,.\,\nabla u^j\,dx. \tag{54}$$

Therefore, from (52), we have:

$$\lim_{j\to\infty}\sum_{i=1}^N \int_\Omega \sigma_i^m(x,u^j,\nabla u^j)\,.\,\nabla u^j\,dx = \int_\Omega \tilde{\sigma}^m\,.\,\nabla u^j\,dx, \tag{55}$$

According to (49), (51) and (54) we get:

$$< \mathfrak{A}(u^j),u^j > \;\to\; < y,u > \quad \text{as } j\to\infty.$$

Hence, from (55), and (39) we obtain:

$$\lim_{j\to\infty}\sum_{i=1}^N \int_\Omega (\,\sigma_i^m(x,u^j,\nabla u^j\,) - \sigma_i^m(x,u^j,\nabla u\,)\,)\,.\,\nabla(u^j - u)\,dx = 0.$$

By (49) we can conclude that

$$< y\,,\,u\ > = < \ \mathfrak{A}(u)\,,\,u\ > \forall\, u \in \mathring{W}^{1}_{M}(\Omega).$$

Corresponding author
Omar Benslimane can be contacted at: omar.benslimane@usmba.ac.ma