Three-dimensional trans-Sasakian manifolds and solitons

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Abstract

Purpose – The authors set the goal to find the solution of the Eisenhart problem within the framework of three-dimensional trans-Sasakian manifolds. Also, they prove some results of the Ricci solitons, η -Ricci solitons and three-dimensional weakly $\mathcal{Z}$ symmetric trans-Sasakian manifolds. Finally, they give a nontrivial example of three-dimensional proper trans-Sasakian manifold.

Design/methodology/approach – The authors have used the tensorial approach to achieve the goal.

Findings – A second-order parallel symmetric tensor on a three-dimensional trans-Sasakian manifold is a constant multiple of the associated Riemannian metric g .

Originality/value – The authors declare that the manuscript is original and it has not been submitted to any other journal for possible publication.

Keywords Trans-Sasakian manifolds, Second-order parallel tensors, Ricci solitons, Symmetric spaces, Generalized $\mathcal{Z}$ tensor, Curvature tensors

Paper type Research paper

1. Introduction

Let the product manifold $\tilde{M} = M \times \mathfrak{R}$, known as an almost Hermitian manifold, possesses an almost complex structure J and the product metric G , where M is a $(2n + 1)$ -dimensional almost contact metric manifold [1] and $\mathfrak{R}$, a real line. In 1980, Gray and Hervella [2] characterized the 16 classes of almost Hermitian manifolds and hence listed many classes of M , such as Sasakian structure, quasi-Sasakian structure, Kenmotsu structure and many structures. During the classification of M there appeared a class $\mathcal{W}_4$, which is close to the locally conformal Kaehler manifold. In view of the structure in the class $\mathcal{W}_4$ on $\tilde{M}$, there exists a class of almost contact metric structure, known as trans-Sasakian structure, which generalizes the Sasakian and Kenmotsu structures. The trans-Sasakian structure of type (α, β) coincides with the class $\mathcal{C}_6 \oplus \mathcal{C}_5$, where α and β are some smooth functions on M . A trans-Sasakian structure of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ is said to be a cosymplectic structure, α -Sasakian structure and β -Kenmotsu structure, respectively. Since then, many geometers characterized the geometrical properties of $(2n + 1)$ -dimensional trans-Sasakian

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manifolds. A trans-Sasakian manifold is said to be proper if α and β are nonzero. There was a natural question “Up to which dimension, the proper trans-Sasakian manifold exists”? Marrero [3], in 1992, gave an affirmative answer of this question. He proved that a trans-Sasakian manifold of dimension ≥ 5 is either a cosymplectic manifold, or an α -Sasakian manifold or a β -Kenmotsu manifold. Since then, the study of three-dimensional trans-Sasakian manifolds attract the researchers. For more details, we refer [4–10] and the references therein.

The Eisenhart problem of finding the parallel tensors (symmetric and skew-symmetric) is an important subject in the differential geometry and its allied areas. In 1923, Eisenhart showed that if a positive definite Riemannian manifold admits a second-order parallel symmetric covariant tensor other than a constant multiple of the metric tensor, then it is reducible [11]. In Ref. [12], Levy presented that a second-order parallel symmetric nondegenerate tensor in a space form is proportional to the metric tensor. The Eisenhart problems of finding the properties of second-order parallel tensors have been locally studied by Eisenhart and Levy, whereas Sharma [13] has solved the same problem globally on complex space form. Since then, many geometers studied the Eisenhart problems on different geometrical structures. For some deep results on this topic, we recommend [14–18] and the references therein.

A Ricci flow:

$$\frac{\partial}{\partial t}g = -2S,$$

introduced by Hamilton [19] on a Riemannian manifold M , is used to solve the celebrated Poincaré conjecture [20, 21] and differentiable structure theorem (extension of Hamilton’s sphere theorem) [22]. It has applications in the string theory, thermodynamics, general relativity, cosmology, quantum field theory, etc. Also, the uniformization theorem and geometrization conjecture can be solved with the help of Ricci flow. Here g and S are the Riemannian metric and the Ricci tensor of M , respectively. Let the Ricci flow be governed by a one parameter family of diffeomorphisms and scalings, then its solution is known as a Ricci soliton. A Ricci soliton (g, V, λ) on an n -dimensional Riemannian manifold M is an extension of Einstein soliton, and is defined by

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \quad (1.1)$$

where $\mathcal{L}_V$ denotes the Lie derivative along the vector field V , called the soliton vector of the Ricci soliton and λ is a real constant. The soliton (g, V, λ) with $\lambda > 0$, < 0 , or $= 0$ is said to be expanding, shrinking or steady, respectively.

A Riemannian metric g of a Riemannian manifold of dimension n is said to be an η -Ricci soliton [23] if it satisfies the equation

$$\mathcal{L}_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0$$

for some $\lambda, \mu \in \mathfrak{R}$, where $\mathfrak{R}$ is used for set of real numbers. If we choose λ and μ as smooth functions in the above equations, then the Ricci and η -Ricci solitons are called as almost Ricci and almost η -Ricci solitons, respectively. Many researchers have been studied the geometrical and physical properties of Ricci solitons and η -Ricci solitons. For readers, we suggest to cite [24–27] and the references therein.

The above deep studies motivate us to characterize the three-dimensional trans-Sasakian manifolds whose metrics are almost Ricci solitons and η -Ricci solitons. We also find the solutions of the Eisenhart problems and properties of three-dimensional weakly $\mathcal{Z}$ symmetric trans-Sasakian manifolds. We bind up our work with a nontrivial example of three-dimensional trans-Sasakian manifold.

2. Basic results of trans-Sasakian manifolds and some definitions

A triplet (ϕ, ξ, η) on a $(2n + 1)$ -dimensional differentiable manifold M is said to an almost contact structure on M if it satisfies

$$\phi^2 + I = \eta \otimes \xi, \quad \eta(\xi) = 1 \quad (2.1)$$

for the identity transformation I . Here ϕ , ξ , η and $\otimes$ denote a vector field of type $(1, 1)$, a vector field of type $(1, 0)$, a 1-form and tensor product, respectively. The manifold M equipped with the structure (ϕ, ξ, η) is known as an almost contact manifold [1]. From equation (2.1), we deduce

$$\phi\xi = 0, \quad \eta\phi = 0 \text{ and } \text{rank } \phi = 2n. \quad (2.2)$$

If M admits a Riemannian metric g such that

$$g(X_1, X_2) = g(\phi X_1, \phi X_2) + \eta(X_1)\eta(X_2), \quad g(X_1, \xi) = \eta(X_1) \quad (2.3)$$

for all $X_1, X_2 \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ represents the collection of all smooth vector fields of M , then it is known as an almost contact metric manifold. Additionally, if M holds the following relation for the smooth functions α and β ,

$$(\nabla_{X_1}\phi)(X_2) = \alpha\{g(X_1, X_2)\xi - \eta(X_2)X_1\} + \beta\{g(\phi X_1, X_2)\xi - \eta(X_2)\phi X_1\},$$

which gives

$$\nabla_{X_1}\xi = -\alpha\phi X_1 + \beta(X_1 - \eta(X_1)\xi) \quad (2.4)$$

for all $X_1, X_2 \in \mathfrak{X}(M)$, then M is said to be a trans-Sasakian manifold of type (α, β) [28].

The curvature tensor R of a three-dimensional trans-Sasakian manifold satisfies [29]

$$\begin{aligned} R(X_1, X_2)X_3 = & \left(\frac{r}{2} + 2\xi(\beta) - 2(\alpha^2 - \beta^2)\right)\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\} \\ & - g(X_2, X_3)\left\{\left(\frac{r}{2} + \xi(\beta) - 3(\alpha^2 - \beta^2)\right)\eta(X_1)\xi + [X_1(\beta) + \phi X_1(\alpha)]\xi \right. \\ & \left. - \eta(X_1)(\phi \text{grad } \alpha - \text{grad } \beta)\right\} - \left\{[X_3(\beta) + \phi X_3(\alpha)]\eta(X_2) + [X_2(\beta) \right. \\ & \left. + \phi X_2(\alpha)]\eta(X_3) + \left(\frac{r}{2} + \xi(\beta) - 3(\alpha^2 - \beta^2)\right)\eta(X_2)\eta(X_3)\right\}X_1 - \{1\} \\ & + \left\{[X_3(\beta) + \phi X_3(\alpha)]\eta(X_1) + [X_1(\beta) + \phi X_1(\alpha)]\eta(X_3) \right. \\ & \left. + \left(\frac{r}{2} + \xi(\beta) - 3(\alpha^2 - \beta^2)\right)\eta(X_1)\eta(X_3)\right\}X_2 - g(X_1, X_3)\left\{[X_2(\beta) + \phi X_2(\alpha)]\xi \right. \\ & \left. + \left(\frac{r}{2} + \xi(\beta) - 3(\alpha^2 - \beta^2)\right)\eta(X_2)\xi - \eta(X_2)(\phi \text{grad } \alpha - \text{grad } \beta)\right\} \end{aligned} \quad (2.5)$$

for all $X_1, X_2, X_3 \in \mathfrak{X}(M)$, where $X_1(\alpha) = g(X_1, \text{grad } \alpha)$ and grad stands for gradient. Contracting the above equation along the vector field X_1 , we find

$$\begin{aligned} S(X_2, X_3) = & \left(\frac{r}{2} + \xi(\beta) - (\alpha^2 - \beta^2)\right)g(X_2, X_3) - (X_3(\beta) + \phi X_3(\alpha))\eta(X_2) \\ & - \left(\frac{r}{2} + \xi(\beta) - 3(\alpha^2 - \beta^2)\right)\eta(X_2)\eta(X_3) - (X_2(\beta) + \phi X_2(\alpha))\eta(X_3). \end{aligned} \quad (2.6)$$

Setting $X_3 = \xi$ in Eqn (2.5) and making use of equations (2.1) and (2.2), we lead

$$R(X_1, X_2)\xi = (\alpha^2 - \beta^2)\{\eta(X_2)X_1 - \eta(X_1)X_2\} + 2\alpha\beta(\eta(X_2)\phi X_1 - \eta(X_1)\phi X_2) + X_2(\alpha)\phi X_1 - X_1(\alpha)\phi X_2 + X_2(\beta)\phi^2 X_1 - X_1(\beta)\phi^2 X_2,$$

which takes the form

$$R(\xi, X_1)\xi = (\alpha^2 - \beta^2 - \xi(\beta))(\eta(X_1)\xi - X_1). \quad (2.7)$$

Also, from equation (2.5) we have

$$\begin{aligned} R(\xi, X_1)X_2 &= (\alpha^2 - \beta^2)\{g(X_1, X_2)\xi - \eta(X_2)X_1\} + 2\alpha\beta\{g(\phi X_2, X_1)\xi + \eta(X_2)\phi X_1\} \\ &\quad + X_2(\alpha)\phi X_1 + g(\phi X_2, X_1)\text{grad } \alpha + X_2(\beta)(X_1 - \eta(X_1)\xi) \\ &\quad - g(\phi X_1, \phi X_2)\text{grad } \beta. \end{aligned} \quad (2.8)$$

Next, it can be easily verified that M satisfies

$$\begin{aligned} 2\alpha\beta + \xi(\alpha) &= 0, \\ S(X_1, \xi) &= \{2(\alpha^2 - \beta^2) - \xi(\beta)\}\eta(X_1) - X_1(\beta) - (\phi X_1)\alpha, \end{aligned} \quad (2.9)$$

which assumes the form

$$Q\xi = (2(\alpha^2 - \beta^2) - \xi(\beta))\xi - \text{grad } \beta + \phi(\text{grad } \alpha). \quad (2.10)$$

Here r , Q , S and R represents the scalar curvature, Ricci operator, Ricci tensor and the curvature tensor of the manifold, respectively.

The notion of quasi-conformal curvature tensor on a Riemannian manifold was introduced by Yano and Sewaki [30]. The quasi conformal curvature tensor $\tilde{C}$ on a $(2n+1)$ -dimensional trans-Sasakian manifold is defined by

$$\begin{aligned} \tilde{C}(X_1, X_2)X_3 &= aR(X_1, X_2)X_3 + b\{S(X_2, X_3)X_1 - S(X_1, X_3)X_2 + g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2\} \\ &\quad - \frac{r}{(2n+1)}\left(\frac{a}{2n} + 2b\right)\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\} \end{aligned} \quad (2.11)$$

for all $X_1, X_2, X_3 \in \mathfrak{X}(M)$, where a and b are constants. In particular, if we take $a = 1$ and $b = -\frac{1}{2n-1}$ then the quasi-conformal curvature tensor reduces to the Weyl-conformal curvature tensor C [31], defined by

$$\begin{aligned} C(X_1, X_2)X_3 &= R(X_1, X_2)X_3 - \frac{1}{2n-1}\{S(X_2, X_3)X_1 - S(X_1, X_3)X_2 + g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2\} \\ &\quad + \frac{r}{2n(2n-1)}\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\} \end{aligned} \quad (2.12)$$

for all $X_1, X_2, X_3 \in \mathfrak{X}(M)$. A trans-Sasakian manifold is said to be quasi-conformally (conformally) flat if $\tilde{C} = 0$ ($C = 0$).

In 2011, Mantica and Molinari [32] introduced the notion of generalized $\mathcal{Z}$ tensor on a semi-Riemannian manifold, and is defined by

$$\mathcal{Z} = S + \varphi g, \quad (2.13)$$

where φ is some smooth function. The choice of $\varphi = -\frac{r}{2}$ shows that the generalized $\mathcal{Z}$ tensor reduces to the classical $\mathcal{Z}$ tensor. The deep results of $\mathcal{Z}$ tensor have been noticed in [33–39].

A three-dimension trans-Sasakian manifold M is said to be weakly $\mathcal{Z}$ symmetric [40] if the nonvanishing $\mathcal{Z}$ tensor satisfies

$$(\nabla_{X_1}\mathcal{Z})(X_2, X_3) = A(X_1)\mathcal{Z}(X_2, X_3) + B(X_2)\mathcal{Z}(X_1, X_3) + D(X_3)\mathcal{Z}(X_1, X_2) \quad (2.14)$$

for all $X_1, X_2, X_3 \in \mathfrak{X}(M)$, where A, B and D are 1-forms. The weakly $\mathcal{Z}$ symmetric manifold with $\varphi = 0$ becomes a weakly Ricci symmetric manifold, introduced by Tamássy and Binh [40] within the context of Riemannian manifold. A weakly Ricci symmetric manifold with $A = B = D = 0$ is said to be a Ricci symmetric trans-Sasakian manifold.

A trans-Sasakian manifold M is said to be an η -Einstein manifold if its nonzero Ricci tensor S satisfies

$$S = ag + b\eta \otimes \eta,$$

where a and b are some smooth functions on M . Particularly, if we choose $b = 0$ and a is constant in the above equation, then we get the Einstein manifold.

3. Second-order parallel symmetric tensor on three-dimensional trans-Sasakian manifolds

Let δ be a symmetric tensor of type $(0, 2)$, then it is said to be parallel with respect to ∇ if $\nabla\delta = 0$. This equation together with Ricci identity $\nabla_{X_1, X_2}^2\delta(X_3, X_4) - \nabla_{X_1, X_2}^2\delta(X_4, X_3) = 0$ give

$$\delta(R(X_1, X_2)X_3, X_4) + \delta(X_3, R(X_1, X_2)X_4) = 0. \quad (3.1)$$

Setting $X_1 = X_4 = \xi$ in Eqn (3.1) and making use of equations (2.7) and (2.8), we obtain

$$\begin{aligned} (\alpha^2 - \beta^2 - \xi(\beta))\delta(X_2, X_3) = & \{(\alpha^2 - \beta^2)g(X_2, X_3) + 2\alpha\beta g(\phi X_3, X_2) \\ & - X_3(\beta)\eta(X_2)\}\delta(\xi, \xi) + \{2\alpha\beta\eta(X_3) + X_3(\alpha)\}\delta(\phi X_2, \xi) \\ & + \{X_3(\beta) - (\alpha^2 - \beta^2)\eta(X_3)\}\delta(X_2, \xi) + g(\phi X_3, X_2)\delta(\text{grad } \alpha, \xi) \\ & - g(\phi X_3, \phi X_2)\delta(\text{grad } \beta, \xi) + (\alpha^2 - \beta^2 - \xi(\beta))\eta(X_2)\delta(X_3, \xi). \end{aligned}$$

Again changing X_3 with ξ in the above equation and using equations (2.1) and (2.2), we have

$$\delta(X_2, \xi) = g(X_2, \xi)\delta(\xi, \xi), \quad (3.2)$$

provided $\alpha^2 - \beta^2 - \xi(\beta) \neq 0$. Differentiating Eqn (3.2) covariantly along the vector field X_1 , we infer

$$\delta(X_2, \nabla_{X_1}\xi) = g(X_2, \nabla_{X_1}\xi)\delta(\xi, \xi) + 2\delta(\nabla_{X_1}\xi, \xi)g(X_2, \xi). \quad (3.3)$$

Replacing X_2 with $\nabla_{X_1}\xi$ in Eqn (3.2) and then using the foregoing equation in (3.3), we find

$$\delta(X_2, \nabla_{X_1}\xi) = [g(X_2, \nabla_{X_1}\xi) + 2g(\nabla_{X_1}\xi, \xi)g(X_2, \xi)]\delta(\xi, \xi).$$

In consequence of Eqns (2.2)–(2.4) and (3.2), the above equation assumes the form

$$-\alpha\{\delta(X_2, \phi X_1) - g(X_2, \phi X_1)\delta(\xi, \xi)\} + \beta\{\delta(X_1, X_2) - g(X_1, X_2)\delta(\xi, \xi)\} = 0. \quad (3.4)$$

Changing X_1 with ϕX_1 in Eqn (3.4) and then using equations (2.1) and (3.2), we conclude that

$$\alpha[\delta(X_1, X_2) - g(X_1, X_2)\delta(\xi, \xi)] + \beta[\delta(\phi X_1, X_2) - g(\phi X_1, X_2)\delta(\xi, \xi)] = 0. \quad (3.5)$$

We suppose that α and β are nonzero and $\alpha \neq \pm\beta$ on M , then from equations (3.4) and (3.5) we lead to

$$\delta(X_1, X_2) = g(X_1, X_2)\delta(\xi, \xi), \forall X_1, X_2 \in \mathfrak{X}(M). \quad (3.6)$$

This fact together with [equation \(3.2\)](#) reflect that $\nabla_{X_3}\delta(\xi, \xi) = 2\delta(\xi, \xi)g(\nabla_{X_3}\xi, \xi) = 0$. Since X_3 is an arbitrary vector field of M and therefore $\delta(\xi, \xi)$ is constant on M . Hence we are in position to state the following:

Theorem 3.1. A second-order parallel symmetric tensor on a three-dimensional trans-Sasakian manifold is a constant multiple of the associated Riemannian metric g . In other words, the almost contact metric on a regular three-dimensional trans-Sasakian manifold is irreducible.

Next, we consider that the regular three-dimensional trans-Sasakian manifold is Ricci symmetric, that is, $\nabla S = 0$, and therefore from [Theorem 3.1](#) we obtain $S = S(\xi, \xi)g$. In view of [equations \(2.1\), \(2.2\)](#) and [\(2.9\)](#), we conclude that $S(\xi, \xi) = 2\{\alpha^2 - \beta^2 - \xi(\beta)\}$. Thus, we have

$$S(X_1, X_2) = 2(\alpha^2 - \beta^2 - \xi(\beta))g(X_1, X_2). \quad (3.7)$$

This shows that the three-dimensional Ricci symmetric trans-Sasakian manifold is an Einstein manifold, provided $\alpha^2 - \beta^2 - \xi(\beta)$ is nonzero. Thus we state:

Corollary 3.2. Every three-dimensional regular Ricci symmetric trans-Sasakian manifold M is an Einstein manifold, provided the ξ -sectional curvature tensor of M is non-zero. Particularly, if $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$ then the trans-Sasakian manifolds reduce to the Sasakian and the Kenmotsu manifolds, respectively. Hence, we can state:

Corollary 3.3. Every three-dimensional Ricci symmetric Sasakian (or Kenmotsu) manifold is Einstein.

For a three-dimensional Kenmotsu manifold, [Corollary 3.3](#) has been proved by De and Pathak [\[41\]](#).

If possible, we suppose that the three-dimensional trans-Sasakian manifold (M, g) is Ricci symmetric and therefore it satisfies [equation \(3.7\)](#). Hence we have

$$Q\xi = 2\{\alpha^2 - \beta^2 - \xi(\beta)\}\xi.$$

This equation together with [equation \(2.10\)](#) reflect that

$$\text{grad } \beta - \phi(\text{grad } \alpha) = \xi(\beta)\xi,$$

which is equivalent to

$$X_1(\beta) + \phi X_1(\alpha) = \xi(\beta)\eta(X_1) \quad (3.8)$$

for all vector field X_1 on M . This tell us the following lemma.

Lemma 3.4. Every three-dimensional Ricci symmetric trans-Sasakian manifold satisfies $\xi(\beta)\xi = \text{grad } \beta - \phi(\text{grad } \alpha)$.

If we take $\alpha = 0$ and $\beta \neq 0$ (or $\alpha \neq 0$ and $\beta = 0$), then the trans-Sasakian manifold reduces to β -Kenmotsu (or α -Sasakian) manifold. Thus from [Lemma 3.4](#), we have the following corollary.

Corollary 3.5. A three-dimensional Ricci symmetric β -Kenmotsu (or α -Sasakian) manifold holds the relation $\text{grad}(\beta) = \xi(\beta)\xi$ (or $\phi(\text{grad } \alpha) = 0$).

Let the three-dimensional trans-Sasakian manifold be Ricci symmetric, then it satisfies [equation \(3.8\)](#). In consequence of [equations \(2.5\)](#) and [\(3.8\)](#), we have

$$\begin{aligned} R(X_1, X_2)X_3 &= \left(\frac{r}{2} - 2(\alpha^2 - \beta^2) + 2\xi(\beta)\right)\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\} \\ &\quad - \left(\frac{r}{2} - 3(\alpha^2 - \beta^2 - \xi(\beta))\right)\{g(X_2, X_3)\eta(X_1) - g(X_1, X_3)\eta(X_2)\}\xi \\ &\quad - \left(\frac{r}{2} - 3(\alpha^2 - \beta^2 - \xi(\beta))\right)\{\eta(X_2)X_1 - \eta(X_1)X_2\}\eta(X_3) \end{aligned} \quad (3.9)$$

for all vector fields X_1, X_2 and X_3 on M . If $\{e_i\}_{i=1}^3$ denotes a set of orthonormal vector fields of M , then for $X_1 = X_2 = e_i$, $1 \leq i \leq 3$, [equation \(3.7\)](#) assumes the form

$$r = 6(\alpha^2 - \beta^2 - \xi(\beta)). \quad (3.10)$$

In consequence of [equations \(3.9\)](#) and [\(3.10\)](#), we find that

$$R(X_1, X_2)X_3 = (\alpha^2 - \beta^2 - \xi(\beta))\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\}. \quad (3.11)$$

This shows that the trans-Sasakian manifold M of dimension three possesses a space of constant scalar curvature $\alpha^2 - \beta^2 - \xi(\beta)$.

Next, we suppose that M has a space of constant scalar curvature $\alpha^2 - \beta^2 - \xi(\beta)$ and therefore [equation \(3.11\)](#) holds on M . The contraction of [\(3.11\)](#) along the vector field X_1 gives [equation \(3.7\)](#) and we have

$$QX_2 = 2(\alpha^2 - \beta^2 - \xi(\beta))X_2. \quad (3.12)$$

The covariant derivative of [\(3.7\)](#) gives $\nabla S = 0$, provided $\alpha^2 - \beta^2 - \xi(\beta) = \text{constant}$. Thus we can state:

Theorem 3.6. A three-dimensional trans-Sasakian manifold M is Ricci symmetric if and only if it is a space of constant scalar curvature $\alpha^2 - \beta^2 - \xi(\beta)$.
Again from [equation \(3.11\)](#), we can state the following:

Corollary 3.7. A three-dimensional Ricci symmetric Kenmotsu manifold is locally isometric to the hyperbolic space $\mathbb{H}^3(-1)$.

Corollary 3.8. A Ricci symmetric Sasakian manifold of dimension three is locally isometric to the sphere $\mathbb{S}^3(1)$.

In the light of [equations \(2.11\)](#), [\(3.7\)](#) and [\(3\)-\(3.12\)](#), we conclude that $\tilde{C} = 0$. This shows that M is quasi-conformally flat. Thus, we can say that every three-dimensional Ricci symmetric trans-Sasakian manifold is quasi-conformally flat. From the straight forward calculations, we can easily show that a three-dimensional quasi-conformally flat trans-Sasakian manifold M is not Ricci symmetric. Hence, we have

Corollary 3.9. A three-dimensional Ricci symmetric trans-Sasakian manifold M is quasi-conformally flat, but the converse is not true.

We suppose that a three-dimensional trans-Sasakian manifold is quasi-conformally flat, that is, $\tilde{C} = 0$. Hence, from [equation \(2.11\)](#), for a three-dimensional trans-Sasakian manifold, we have

$$\begin{aligned} R(X_1, X_2)X_3 &= \frac{b}{a}\{S(X_2, X_3)X_1 - S(X_1, X_3)X_2 + g(X_2, X_3)QX_1 - g(X_1, X_3)QX_2\} \\ &\quad + \frac{r}{3a}\left(\frac{a}{2} + 2b\right)\{g(X_2, X_3)X_1 - g(X_1, X_3)X_2\}, \end{aligned}$$

which is equivalent to

$$\begin{aligned} g(R(X_1, X_2)X_3, X_4) = & -\frac{b}{a}\{S(X_2, X_3)g(X_1, X_4) - S(X_1, X_3)g(X_2, X_4) + g(X_2, X_3)S(X_1, X_4) \\ & - g(X_1, X_3)S(X_2, X_4)\} + \frac{r}{3a}\left(\frac{a}{2} + 2b\right)\{g(X_2, X_3)g(X_1, X_4) \\ & - g(X_1, X_3)g(X_2, X_4)\}, \end{aligned}$$

provided $a \neq 0$. Changing X_1 and X_3 with the characteristic vector field ξ in the above equation and then using Eqns (2.1)–(2.3), (2.7) and (2.9), we obtain

$$\begin{aligned} bS(X_2, X_4) = & \left(\frac{r}{6}(a + 4b) - (a + 2b)(\alpha^2 - \beta^2 - \xi(\beta))\right)g(X_2, X_4) \\ & + \left(\left(a + 4b\right)(\alpha^2 - \beta^2) - (a + 2b)\xi(\beta) - \frac{r}{6}(a + 4b)\right)\eta(X_2)\eta(X_4) \\ & - b\{(X_2(\beta) + \phi X_2(\alpha))\eta(X_4) + (X_4(\beta) + \phi X_4(\alpha))\eta(X_2)\}. \end{aligned} \quad (3.13)$$

With the help of equations (2.6) and (3.13), we find

$$(a + b)[r - 6(\alpha^2 - \beta^2 - \xi(\beta))]\{g(X_2, X_4) - \eta(X_2)\eta(X_4)\} = 0.$$

Since $g(\phi X_2, \phi X_4) \neq 0$ on an almost contact metric manifold, therefore the above equation reflects that either $b = -a (\neq 0)$ or $r = 6(\alpha^2 - \beta^2 - \xi(\beta))$. If $b = -a$ and $r \neq 6(\alpha^2 - \beta^2 - \xi(\beta))$, then from equations (2.11) and (2.12), we have $\tilde{C} = aC$. This shows that the quasi-conformal and Weyl conformal curvature tensors on a three-dimensional trans-Sasakian manifold are linearly dependent. Next, we assume that $a + b \neq 0$ and $r = 6(\alpha^2 - \beta^2 - \xi(\beta))$, then equation (3.13) takes the form

$$\begin{aligned} S(X_2, X_4) = & 2(\alpha^2 - \beta^2 - \xi(\beta))g(X_2, X_4) + 2(\xi(\beta))\eta(X_4)\eta(X_2) \\ & - [(X_2(\beta) + \phi X_2(\alpha))\eta(X_4) + (X_4(\beta) + \phi X_4(\alpha))\eta(X_2)]. \end{aligned} \quad (3.14)$$

By considering equations (2.3), (2.4), $g(\phi X_1, X_2) + g(X_1, \phi X_2) = 0$ and the definition of Lie derivative, we conclude that

$$(\mathcal{L}_\xi g)(X_1, X_2) = g(\nabla_{X_1}\xi, X_2) + g(X_1, \nabla_{X_2}\xi) = 2\beta\{g(X_1, X_2) - \eta(X_1)\eta(X_2)\}. \quad (3.15)$$

From equations (3.14) and (3.15), we notice that

$$\begin{aligned} \frac{1}{2}(\mathcal{L}_\xi g)(X_1, X_2) + S(X_1, X_2) + \lambda g(X_1, X_2) + \mu\eta(X_1)\eta(X_2) = & 2\xi(\beta)\eta(X_1)\eta(X_2) \\ - [(X_2(\beta) + \phi X_2(\alpha))\eta(X_1) + (X_1(\beta) + \phi X_1(\alpha))\eta(X_2)], \end{aligned} \quad (3.16)$$

where $\lambda = 2(\beta^2 + \xi(\beta) - \alpha^2) - \beta$ and $\mu = \beta$. Hence, with help of equations (2.9), (3.14) and (3.16), we can state the following results.

Theorem 3.10. Let M be a three-dimensional quasi-conformally flat trans-Sasakian manifold, then either M

- (1) is conformally flat or
 - (2) possesses an almost η -Ricci soliton (g, ξ, λ, μ) if and only if $\xi(\beta)\xi = \text{grad } \beta - \phi(\text{grad } \alpha)$.
- In consequence of [Theorem 3.10](#), we can state the following corollary.

Corollary 3.11. A three-dimensional quasi-conformally flat trans-Sasakian manifold M with $a + b \neq 0$ possesses the constant scalar curvature.

Again, in view of [equation \(3.16\)](#) and [Theorem 3.10](#), we can state the following:

Corollary 3.12. The metric of a three-dimensional quasi-conformally flat β -Kenmotsu manifold with $a + b \neq 0$ is an almost η -Ricci soliton (g, ξ, λ, μ) if and only if $\xi(\beta)\xi = \text{grad } \beta$.

Corollary 3.13. A three-dimensional quasi-conformally flat α -Sasakian manifold with $a + b \neq 0$ admits an almost η -Ricci soliton (g, ξ, λ, μ) if and only if $\phi \text{grad } \alpha = 0$.

Corollary 3.14. A three-dimensional quasi-conformally flat Sasakian manifold satisfying $a + b \neq 0$ is a shrinking Ricci soliton with $\lambda = -2$.

Corollary 3.15. Every quasi-conformally flat Kenmotsu manifold of dimension three with $a + b \neq 0$ possesses an expanding type η -Ricci soliton.

In view of [equation \(3.14\)](#), we can state:

Corollary 3.16. Suppose a three-dimensional quasi-conformally flat trans-Sasakian manifold M satisfies $a + b \neq 0$. Then M is an η -Einstein manifold if and only if $\text{grad } \beta = \phi(\text{grad } \alpha)$.

From [equation \(3.16\)](#), we have $\lambda = 2(\beta^2 + \xi(\beta) - \alpha^2) - \beta$. This shows that the almost η -Ricci soliton under consideration is expanding, shrinking or steady if $\beta^2 + \xi(\beta) >$, $<$, or $= \alpha^2 + \frac{\beta}{2}$ respectively. Thus, we can state:

Theorem 3.17. Let a three-dimensional quasi-conformally flat trans-Sasakian manifold M satisfies $a + b \neq 0$ and $\text{grad } \beta - \phi(\text{grad } \alpha) = \xi(\beta)\xi$. Then the almost η -Ricci soliton (g, ξ, λ, μ) on M is shrinking, expanding or steady if $\beta^2 + \xi(\beta) <$, $>$, or $= \alpha^2 + \frac{\beta}{2}$ respectively.

With the help of [Theorem 3.17](#), we can state the following:

Corollary 3.18. If a three-dimensional quasi-conformally flat β -Kenmotsu manifold M satisfies $a + b \neq 0$ and $\text{grad } \beta = \xi(\beta)\xi$, then the η -Ricci soliton (g, ξ, λ, μ) on M is shrinking, expanding or steady if $\beta^2 + \xi(\beta) <$, $>$, or $= \frac{\beta}{2}$ respectively.

Corollary 3.19. Suppose a three-dimensional quasi-conformally flat α -Sasakian manifold M satisfies $a + b \neq 0$ and $\text{grad } \alpha + \xi(\alpha)\xi = 0$. Then the almost η -Ricci soliton (g, ξ, λ, μ) on M is always shrinking.

Let us suppose that $\delta_1 = \mathcal{L}_\xi g + 2S$, $\mathcal{L}_\xi g$ is parallel and the regular three-dimensional trans-Sasakian manifold is Ricci symmetric. Then in view of [Eqns \(3.6\)](#), [\(3.15\)](#) and [Theorem 3.1](#), we find

$$\delta_1(X_1, X_2) = \delta_1(\xi, \xi)g(X_1, X_2), \quad (3.17)$$

where $\delta_1(\xi, \xi) = (\mathcal{L}_\xi g)(\xi, \xi) + 2S(\xi, \xi) = 2(\alpha^2 - \beta^2 - \xi(\beta))$. In consequence of [equations \(1.1\)](#) and [\(3.17\)](#), we observe that $\lambda = -\frac{1}{2}\delta_1(\xi, \xi) = -(\alpha^2 - \beta^2 - \xi(\beta))$. Thus, we discuss our results in the following corollary:

Theorem 3.20. Let $\mathcal{L}_\xi g$ is parallel on a three-dimensional Ricci symmetric trans-Sasakian manifold M . Then the almost Ricci soliton (ξ, g, λ) on M is shrinking, expanding, or steady if the ξ -sectional curvature of M is positive, negative or zero, respectively.

It is well-known that a three-dimensional trans-Sasakian manifold of type (α, β) reduces to a Sasakian; Kenmotsu or cosymplectic manifolds if $\alpha = 1, \beta = 0$; $\alpha = 0, \beta = 1$ and $\alpha = 0, \beta = 0$, respectively. Thus, we can state the following corollaries.

Corollary 3.21. If $\mathcal{L}_{\xi}g$ is parallel on a three-dimensional Ricci symmetric Sasakian manifold M , then the Ricci soliton (ξ, g, λ) on M is always shrinking.

Corollary 3.22. A Ricci soliton (ξ, g, λ) on a three-dimensional Ricci symmetric Kenmotsu manifold with a parallel vector field $\mathcal{L}_{\xi}g$ is expanding.

Corollary 3.23. Suppose $\mathcal{L}_{\xi}g$ is parallel on a three-dimensional Ricci symmetric cosymplectic manifold M , then (ξ, g, λ) on M is steady.

Now, we define the following definitions as:

Definition 3.24. A vector field $X_1 \in \mathfrak{X}(M)$ on a Riemannian manifold is said to be an affine Killing vector field if $\nabla \mathcal{L}_{X_1}g = 0$.

Definition 3.25. A vector field $X_1 \in \mathfrak{X}(M)$ on a Riemannian manifold is said to be a Killing vector field if $\mathcal{L}_{X_1}g = 0$.

From the Definition 3.24 and Definition 3.25, it is clear that if a vector field is Killing then it is affine Killing, but converse is not, in general, true. Here we prove that the converse is true in a three-dimensional trans-Sasakian manifold, provided $\alpha^2 - \beta^2$ is nonzero constant.

Let us suppose that X_1 is an affine Killing vector field on a three-dimensional trans-Sasakian manifold M , then Theorem 3.1 and Definition 3.24 reflect that

$$(\mathcal{L}_{X_1}g)(X_2, X_3) = cg(X_2, X_3),$$

where $c = (\mathcal{L}_{X_1}g)(\xi, \xi)$. The covariant derivative of $Q\xi = [2(\alpha^2 - \beta^2) - \xi(\beta)]\xi$ along the vector field X_1 gives $(\mathcal{L}_{X_1}Q)(\xi) = 0$ and hence $(\mathcal{L}_{X_1}S)(\xi, \xi) = 0$, provided $\alpha^2 - \beta^2 = \text{constant}$. Also, we have $(\mathcal{L}_{X_1}S)(\xi, \xi) = -2S(\mathcal{L}_{X_1}\xi, \xi) = -2[2(\alpha^2 - \beta^2) - \xi(\beta)]g(\mathcal{L}_{X_1}\xi, \xi)$. Since $(\mathcal{L}_{X_1}S)(\xi, \xi) = 0$ and therefore $g(\mathcal{L}_{X_1}\xi, \xi) = 0$ on M . By considering this fact, we have $c = (\mathcal{L}_{X_1}g)(\xi, \xi) = -2g(\mathcal{L}_{X_1}\xi, \xi) = 0$. Thus, with the help of Definition 3.25, we can say that the vector field X_1 is Killing on a three-dimensional trans-Sasakian manifold, provided $\alpha^2 - \beta^2$ is a nonzero constant. Thus, we can state the following:

Corollary 3.26. An affine Killing vector field on a three-dimensional Kenmotsu (or Sasakian) manifold is Killing.

It is obvious that the metric tensor g is covariantly constant, that is, $\nabla g = 0$, which implies that $\nabla 2\lambda g = 0$, where λ is a real constant. This fact together with equation (1.1) give $\nabla(\mathcal{L}_Vg + 2S) = 0$ and therefore Theorem 3.1 reflects that $\mathcal{L}_Vg + 2S = ag$, where $a = (\mathcal{L}_Vg)(\xi, \xi) + 2S(\xi, \xi) = -2\lambda$ is constant. Thus we can state:

Corollary 3.27. A Ricci soliton (g, V, λ) on a three-dimensional trans-Sasakian manifold is expanding, shrinking or steady if $a < 0, > 0$ or $= 0$, respectively.

4. Three-dimensional weakly $\mathcal{Z}$ symmetric trans-Sasakian manifolds and Ricci flow

This section is dedicated to study the properties of three-dimensional weakly $\mathcal{Z}$ symmetric trans-Sasakian manifold M . Since the Ricci tensor S and the Riemannian metric g of M are parallel with respect to the Levi-Civita connection, therefore $\mathcal{Z}$ is also symmetric. From equation (2.14), we conclude that

$$E(X_3)\mathcal{Z}(X_1, X_2) = E(X_2)\mathcal{Z}(X_1, X_3), \quad (4.1)$$

where $E = B - D \neq 0$. In view of equations (2.13) and (4.1), we observe that

$$E(X_3)S(X_1, X_2) + \varphi E(X_3)g(X_1, X_2) - E(X_2)S(X_1, X_3) - \varphi E(X_2)g(X_1, X_3) = 0.$$

Substituting $X_3 = \xi$ in the above equation, we find

$$E(\xi)S(X_1, X_2) + \varphi E(\xi)g(X_1, X_2) - E(X_2)S(X_1, \xi) - \varphi E(X_2)g(X_1, \xi) = 0. \quad (4.2)$$

Again, replacing X_1 with ξ in equation (4.2) and making use of equations (2.1), (2.3) and (2.9), we lead to

$$E(X_2) = \frac{E(\xi)[(2(\alpha^2 - \beta^2) - \xi(\beta) + \varphi)\eta(X_2) - X_2(\beta) - \phi X_2(\alpha)]}{\varphi + 2(\alpha^2 - \beta^2 - \xi(\beta))}, \quad (4.3)$$

provided $\varphi + 2(\alpha^2 - \beta^2 - \xi(\beta)) \neq 0$. Substituting the value of E from Eqn (4.3) in equation (4.2), we observe that

$$S(X_1, X_2) = \frac{[(2(\alpha^2 - \beta^2) - \xi(\beta) + \varphi)\eta(X_2) - X_2(\beta) - \phi X_2(\alpha)]}{\varphi + 2(\alpha^2 - \beta^2 - \xi(\beta))} \times \\ [(2(\alpha^2 - \beta^2) - \xi(\beta) + \varphi)\eta(X_1) - X_1(\beta) - \phi X_1(\alpha)] - \varphi g(X_1, X_2). \quad (4.4)$$

By considering the above discussions, we can state the following:

Theorem 4.1. The Ricci tensor of a three-dimensional weakly $\mathcal{Z}$ symmetric trans-Sasakian manifold with $B - D \neq 0$ is given by (4).

If possible, we suppose that $X_1(\beta) + \phi X_1(\alpha) = 0 \Leftrightarrow \text{grad } \beta = \phi \text{ grad } \alpha$, then equation (4) assumes the form

$$S(X_1, X_2) = b\eta(X_2)\eta(X_1) + ag(X_1, X_2), \quad (4.5)$$

where $b = \frac{(2(\alpha^2 - \beta^2) - \xi(\beta) + \varphi)^2}{\varphi + 2(\alpha^2 - \beta^2 - \xi(\beta))} \neq 0$ and $a = -\varphi$. The above equation shows that the three-dimensional weakly $\mathcal{Z}$ symmetric manifold is an η -Einstein manifold, provided $B \neq D$ and $\text{grad } \beta = \phi \text{ grad } \alpha$. Thus, we can state the following:

Corollary 4.2. A three-dimensional weakly $\mathcal{Z}$ symmetric manifold with $B \neq D$ and $\text{grad } \beta = \phi \text{ grad } \alpha$ is η -Einstein.

We know that a weakly $\mathcal{Z}$ symmetric trans-Sasakian manifold with $\varphi = 0$ is a weakly Ricci symmetric trans-Sasakian manifold. Now, we suppose that the weakly Ricci symmetric trans-Sasakian manifold of dimension three satisfies $\text{grad } \beta = \phi \text{ grad } \alpha$, then equation (4.5) turns into the equation

$$S(X_1, X_2) = r\eta(X_2)\eta(X_1), \quad (4.6)$$

where $r = b = \frac{(2(\alpha^2 - \beta^2) - \xi(\beta))^2}{2(\alpha^2 - \beta^2 - \xi(\beta))} \neq 0$. It is well-known that a semi-Riemannian manifold is said to be Ricci simple if its nonvanishing Ricci tensor satisfies the relation $S = r\eta \otimes \eta$. Thus, we have the following:

Corollary 4.3. Every weakly Ricci symmetric trans-Sasakian manifold of dimension three with $B \neq D$ and $\text{grad } \beta = \phi \text{ grad } \alpha$ is Ricci simple.

In view of equations (3.15) and (4.5), we obtain

$$\mathcal{L}_\xi g + 2S + (2\varphi - 2\beta)g - (2b - 2\beta)\eta \otimes \eta = 0,$$

which shows that the three-dimensional weakly $\mathcal{Z}$ symmetric manifold under consideration is an almost η -Ricci soliton (g, ξ, λ, μ) with $\lambda = \varphi - \beta$ and $\mu = \beta - b \neq 0$. It is obvious that the soliton (g, ξ, λ, μ) is expanding, shrinking or steady if $\varphi >, <, \text{ or } = \beta$, respectively. Thus, we conclude our result as:

Corollary 4.4. The metric of a three-dimensional weakly $\mathcal{Z}$ symmetric manifold with $B \neq D$ and $\text{grad } \beta = \phi \text{ grad } \alpha$ is an almost η -Ricci soliton. Also, the soliton (g, ξ, λ, μ) is shrinking, expanding or steady if $\varphi <, >, \text{ or } = \beta$, respectively.

Again from equations (3.15) and (4.6), we have

$$\mathcal{L}_\xi g + 2S - 2\beta g - 2(r - \beta)\eta \otimes \eta = 0.$$

This reflects that a three-dimension weakly Ricci symmetric trans-Sasakian manifold with $B \neq D$ and $\text{grad } \beta = \phi \text{ grad } \alpha$ is an η -Ricci soliton and $\lambda = -\beta$. Thus, we can write the following:

Corollary 4.5. A three-dimensional weakly Ricci symmetric trans-Sasakian manifold with $B \neq D$ and $\text{grad } \beta = \phi \text{ grad } \alpha$ is an almost η -Ricci soliton.

5. Example of proper three-dimensional trans-Sasakian manifold

Example 5.1. Let $M^3 = \{(x, y, z) \in \mathfrak{R}^3 : |z| > 0\}$ be a differentiable manifold of dimension three with the standard coordinate system (x, y, z) of $\mathfrak{R}^3$. Define a $(1, 1)$ -tensor ϕ on M^3 as: $\phi(q_1) = -q_2, \phi(q_2) = q_1, \phi(q_3) = 0$, where $q_1 = z\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), q_2 = z\frac{\partial}{\partial y}, q_3 = \frac{\partial}{\partial z} = \xi$ are the linear independent vector fields at each point of M^3 and therefore it forms a basis of tangent space at each point of M^3 . Now, if the Riemannian metric g of M^3 is defined by $g(q_i, q_j) = \delta_{ij}, \forall i, j = 1, 2, 3$, then we can easily verify that the relations

$$\eta(q_3) = 1, \quad \phi^2 q_i = -q_i + \eta(q_i)q_3$$

hold for $i = 1, 2, 3$ and $\eta(q_i) = g(q_i, q_3)$, where η is the 1-form. By the definition of Lie bracket $([X_1, X_2]f = X_1X_2(f) - X_2X_1(f))$ for some smooth function f , we get the following nonzero components of Lie bracket as:

$$[q_1, q_2] = yq_2 - z^2q_3, \quad [q_3, q_2] = \frac{1}{z}q_2, \quad [q_3, q_1] = \frac{1}{z}q_1.$$

The Koszul's formula along with above results give

$$\begin{aligned} \nabla_{q_1} q_1 &= \frac{1}{z}q_3, & \nabla_{q_1} q_2 &= \frac{-z^2}{2}q_3, & \nabla_{q_1} q_3 &= \frac{-1}{z}q_1 + \frac{z^2}{2}q_2, \\ \nabla_{q_2} q_1 &= \frac{z^2}{2}q_3 - yq_2, & \nabla_{q_2} q_2 &= yq_1 + \frac{1}{z}q_3, & \nabla_{q_2} q_3 &= \frac{-1}{z}q_2 - \frac{z^2}{2}q_1, \\ \nabla_{q_3} q_1 &= \frac{z^2}{2}q_2, & \nabla_{q_3} q_2 &= \frac{-z^2}{2}q_1, & \nabla_{q_3} q_3 &= 0. \end{aligned}$$

By the straightforward calculations, we can write

$$\nabla q_3 = -\alpha \phi I + \beta(I - \eta \otimes q_3),$$

where $\alpha = \frac{\varepsilon^2}{2}$ and $\beta = \frac{-1}{z}$. This shows that the structure $(\phi, \varrho_3, \eta, g)$ is a trans-Sasakian structure on M^3 and the manifold M^3 equipped with $(\phi, \varrho_3, \eta, g)$ is a trans-Sasakian manifold with $\alpha = \frac{\varepsilon^2}{2}$ and $\beta = \frac{-1}{z}$.

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