On the α -connections and the α -conformal equivalence on statistical manifolds

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Abstract

Purpose – In this paper, we give some properties of the α -connections on statistical manifolds and we study the α -conformal equivalence where we develop an expression of curvature $\bar{R}$ for $\bar{\nabla}$ in relation to those for ∇ and $\bar{\nabla}$.

Design/methodology/approach – In the first section of this paper, we prove some results about the α -connections of a statistical manifold where we give some properties of the difference tensor K and we determine a relation between the curvature tensors; this relation is a generalization of the results obtained in [1]. In the second section, we introduce the notion of α -conformal equivalence of statistical manifolds treated in [1, 3], and we construct some examples.

Findings – We give some properties of the difference tensor K and we determine a relation between the curvature tensors; this relation is a generalization of the results obtained in [1]. In the second section, we introduce the notion of α -conformal equivalence of statistical manifolds, we give the relations between curvature tensors and we construct some examples.

Originality/value – We give some properties of the difference tensor K and we determine a relation between the curvature tensors; this relation is a generalization of the results obtained in [1]. In the second section, we introduce the notion of α -conformal equivalence of statistical manifolds, we give the relations between curvature tensors and we construct some examples.

Keywords Statistical manifold, α -connections, α -conformal equivalence

Paper type Research paper

1. Introduction

Let (M^m, g) be a Riemannian manifold and ∇ a torsion free linear connection on M . The triple (M^m, ∇, g) is called a statistical manifold if ∇g is symmetric and the pair (∇, g) is called a statistical structure. For a statistical manifold (M^m, ∇, g) , let ∇^* be an affine connection on M such that,

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g\left(Y, \nabla_X^* Z\right),$$

for all X, Y and Z in $\Gamma(TM)$. The affine connection ∇^* is torsion free, and ∇g symmetric. Then ∇^* is called the dual connection of ∇ , the triple (M^m, ∇^*, g) is called the dual statistical manifold of (M^m, ∇, g) and (∇, ∇^*, g) is the dualistic structure. Denoted by $\hat{\nabla}$ the Levi-Civita

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connection associated with g , the difference tensor K is the $(1, 2)$ -tensor defined by (see [1]).

$$K(X, Y) = \nabla_X^* Y - \nabla_X Y.$$

The difference tensor K satisfies for any vector fields X, Y, Z and any smooth function f on M the following properties:

$$\nabla_X Y = \widehat{\nabla}_X Y - \frac{1}{2}K(X, Y), \quad \nabla_X^* Y = \widehat{\nabla}_X Y + \frac{1}{2}K(X, Y),$$

$$\begin{aligned} K(X, Y) &= K(Y, X), \\ K(X, Y + Z) &= K(X, Y) + K(X, Z) \end{aligned}$$

and

$$K(fX, Y) = K(X, fY) = fK(X, Y).$$

Moreover, we have,

$$g(K(X, Y), Z) = g(K(X, Z), Y).$$

A statistical structure is called trace-free if $\nabla v_g = 0$ where v_g is the volume form determined by g . This condition is equivalent to the condition $Tr_g K = 0$. A statistical structure (∇, g) is said to be of constant curvature $k \in \mathbb{R}$ if the curvature tensor field R of ∇ satisfies,

$$R(X, Y)Z = k\{g(Y, Z)X - g(X, Z)Y\}.$$

If $k = 0$, (∇, g) is called a Hessian structure. The concept of α -conformally equivalence was treated in [1] where the author develops an expression of the curvature $R^{(\alpha)}$. In [2], the authors studied a 1-conformally flat statistical submanifold of a flat statistical manifold; they proved that a 1-conformally flat statistical manifold with a Riemannian metric can be locally realized as a submanifold of a flat statistical manifold. The author in [3] gives a procedure to realize a statistical manifold, which is α -conformally equivalent to a manifold with an α -transitively flat connection, as a statistical submanifold and in [4], he describe a method to obtain α -conformally equivalent connections from the relation between tensors and the symmetric cubic form. In [5], the authors studied the statistical hypersurfaces of some types of the statistical manifolds, which enable to construct a structure of a constant curvature. The divergence of 1-conformally flat statistical manifolds is studied in [6] where the authors prove that the generalized Pythagorean theorem holds if the statistical manifold has a constant curvature. In the first section of this paper, we prove some results about the α -connections of a statistical manifolds where we give some properties of the difference tensor K and we determine a relation between the curvature tensors $R^{(\alpha)}$ and $R^{(\beta)}$; this relation is a generalization of the results obtained in [1]. In the second section, we introduce the notion of α -conformal equivalence of statistical manifolds treated in [1, 3], and we give the relations between $\bar{R}$, R and $\bar{R}$ and we construct some examples.

2. Some results on the α -connections of a statistical manifolds

Let (M^m, ∇, g) a statistical manifold with a dualistic structure $\left(\nabla, \nabla^*, g \right)$. For $\alpha \in \mathbb{R}$, we define a family of torsion-free connections $\nabla^{(\alpha)}$ by,

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*.$$

$\nabla^{(\alpha)}$ is called an α -connection of (M^m, ∇, g) . The triple $(M^m, \nabla^{(\alpha)}, g)$ is also a statistical manifold, and $\nabla^{(-\alpha)}$ is the dual connection of $\nabla^{(\alpha)}$. In particular,

$$\nabla^{(0)} = \widehat{\nabla}, \quad \nabla^{(1)} = \nabla, \quad \nabla^{(-1)} = \nabla^*.$$

Moreover, we have the following equality,

$$\nabla_X^{(\alpha)} Y = \widehat{\nabla}_X Y - \frac{\alpha}{2} K(X, Y).$$

In general, for any $\alpha, \beta \in \mathbb{R}$, it is easy to see that,

$$\nabla_X^{(\alpha)} Y = \nabla_X^{(\beta)} Y - \frac{\alpha - \beta}{2} K(X, Y) \quad (1)$$

Proposition 1. Let (M^m, ∇, g) a statistical manifold with a dualistic structure (∇, ∇^*, g) .
For all vector fields X, Y, Z on M , we have,

$$\begin{aligned} \left(\nabla_X^{(\alpha)} K \right) (Y, Z) &= \left(\nabla_X^{(\beta)} K \right) (Y, Z) - \frac{\alpha - \beta}{2} K(X, K(Y, Z)) \\ &\quad + \frac{\alpha - \beta}{2} K(K(X, Y), Z) + \frac{\alpha - \beta}{2} K(Y, K(X, Z)). \end{aligned} \quad (2)$$

Proof of Proposition 1. Let $X, Y, Z \in \Gamma(TM)$, by definition, we have,

$$\left(\nabla_X^{(\alpha)} K \right) (Y, Z) = \nabla_X^{(\alpha)} K(Y, Z) - K\left(\nabla_X^{(\alpha)} Y, Z \right) - K\left(Y, \nabla_X^{(\alpha)} Z \right).$$

The properties of the difference tensor K gives us,

$$\nabla_X^{(\alpha)} K(Y, Z) = \nabla_X^{(\beta)} K(Y, Z) - \frac{\alpha - \beta}{2} K(X, K(Y, Z)),$$

$$K\left(\nabla_X^{(\alpha)} Y, Z \right) = K\left(\nabla_X^{(\beta)} Y, Z \right) - \frac{\alpha - \beta}{2} K(K(X, Y), Z)$$

and

$$K\left(Y, \nabla_X^{(\alpha)} Z \right) = K\left(Y, \nabla_X^{(\beta)} Z \right) - \frac{\alpha - \beta}{2} K(Y, K(X, Z)),$$

then

$$\begin{aligned} \left(\nabla_X^{(\alpha)} K \right) (Y, Z) &= \nabla_X^{(\beta)} K(Y, Z) - \frac{\alpha - \beta}{2} K(X, K(Y, Z)) - K\left(\nabla_X^{(\beta)} Y, Z \right) \\ &\quad + \frac{\alpha - \beta}{2} K(K(X, Y), Z) - K\left(Y, \nabla_X^{(\beta)} Z \right) + K(Y, K(X, Z)) \end{aligned}$$

Finally, using the fact that,

$$\left(\nabla_X^{(\beta)} K \right) (Y, Z) = \nabla_X^{(\beta)} K(Y, Z) - K\left(\nabla_X^{(\beta)} Y, Z \right) - K\left(Y, \nabla_X^{(\beta)} Z \right),$$

we deduce that,

$$\begin{aligned} \left(\nabla_X^{(\alpha)} K \right) (Y, Z) &= \left(\nabla_X^{(\beta)} K \right) (Y, Z) - \frac{\alpha - \beta}{2} K(X, K(Y, Z)) + \frac{\alpha - \beta}{2} K(K(X, Y), Z) \\ &\quad + \frac{\alpha - \beta}{2} K(Y, K(X, Z)). \end{aligned}$$

Remark 1. As particular cases of Eqn (2), we have

$$\begin{aligned}
 \left(\nabla_X^{(\alpha)} K\right)(Y, Z) &= \left(\widehat{\nabla}_X K\right)(Y, Z) - \frac{\alpha}{2} K(X, K(Y, Z)) + \frac{\alpha}{2} K(K(X, Y), Z) \\
 &\quad + \frac{\alpha}{2} K(Y, K(X, Z)) \\
 &= (\nabla_X K)(Y, Z) - \frac{\alpha-1}{2} K(X, K(Y, Z)) + \frac{\alpha-1}{2} K(K(X, Y), Z) \\
 &\quad + \frac{\alpha-1}{2} K(Y, K(X, Z)) \\
 &= \left(\nabla_X^* K\right)(Y, Z) - \frac{\alpha+1}{2} K(X, K(Y, Z)) + \frac{\alpha+1}{2} K(K(X, Y), Z) \\
 &\quad + \frac{\alpha+1}{2} K(Y, K(X, Z))
 \end{aligned}$$

For a statistical structure $\left(\nabla, \nabla^*, g\right)$, we denote $R, R^*, \widehat{R}$ the curvature tensors for $\nabla, \nabla^*, \widehat{\nabla}$, respectively, and $R^{(\alpha)}$ the curvature tensor for $\nabla^{(\alpha)}$. In the first results, we give the relation between $R^{(\alpha)}$ and $R^{(\beta)}$ for any $\alpha, \beta \in \mathbb{R}$.

Theorem 1. Let (M^m, ∇, g) a statistical manifold. The relation between $R^{(\alpha)}$ and $R^{(\beta)}$ is given by,

$$\begin{aligned}
 R^{(\alpha)}(X, Y)Z &= R^{(\beta)}(X, Y)Z + \frac{\beta-\alpha}{2} \left(\nabla_X^{(\beta)} K\right)(Y, Z) - \frac{\beta-\alpha}{2} \left(\nabla_Y^{(\beta)} K\right)(X, Z) \\
 &\quad + \frac{(\beta-\alpha)^2}{4} K(X, K(Y, Z)) - \frac{(\beta-\alpha)^2}{4} K(Y, K(X, Z)),
 \end{aligned} \tag{3}$$

for all $X, Y, Z \in \Gamma(TM)$.

Proof of Theorem 1. Let $X, Y, Z \in \Gamma(TM)$, By definition we have,

$$R^{(\alpha)}(X, Y)Z = \nabla_X^{(\alpha)} \nabla_Y^{(\alpha)} Z - \nabla_X^{(\alpha)} \nabla_Y^{(\alpha)} Z - \nabla_{[X, Y]}^{(\alpha)} Z. \tag{4}$$

For the first term $\nabla_X^{(\alpha)} \nabla_Y^{(\alpha)} Z$, we have,

$$\nabla_Y^{(\alpha)} Z = \nabla_Y^{(\beta)} Z + \frac{\beta-\alpha}{2} K(Y, Z),$$

then

$$\nabla_X^{(\alpha)} \nabla_Y^{(\alpha)} Z = \nabla_X^{(\alpha)} \widehat{\nabla}_Y Z - \frac{\alpha}{2} \nabla_X^{(\alpha)} K(Y, Z).$$

It is simple to see that,

$$\nabla_X^{(\alpha)} \nabla_Y^{(\beta)} Z = \nabla_X^{(\beta)} \nabla_Y^{(\beta)} Z + \frac{\beta-\alpha}{2} K(X, \nabla_Y^{(\beta)} Z)$$

and

$$\nabla_X^{(\alpha)} K(Y, Z) = \nabla_X^{(\beta)} K(Y, Z) + \frac{\beta-\alpha}{2} K(X, K(Y, Z)),$$

which gives us

$$\begin{aligned}\nabla_X^{(\alpha)} \nabla_Y^{(\alpha)} Z &= \nabla_X^{(\beta)} \nabla_Y^{(\beta)} Z + \frac{\beta - \alpha}{2} K(X, \nabla_Y^{(\beta)} Z) \\ &+ \frac{\beta - \alpha}{2} \nabla_X^{(\beta)} K(Y, Z) + \frac{(\beta - \alpha)^2}{4} K(X, K(Y, Z)).\end{aligned}\tag{5}$$

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A similar calculation gives,

$$\begin{aligned}\nabla_Y^{(\alpha)} \nabla_X^{(\alpha)} Z &= \nabla_Y^{(\beta)} \nabla_X^{(\beta)} Z + \frac{\beta - \alpha}{2} K(Y, \nabla_X^{(\beta)} Z) \\ &+ \frac{\beta - \alpha}{2} \nabla_Y^{(\beta)} K(X, Z) + \frac{(\beta - \alpha)^2}{4} K(Y, K(X, Z)).\end{aligned}\tag{6}$$

Finally, we have,

$$\nabla_{[X, Y]}^{(\alpha)} Z = \nabla_{[X, Y]}^{(\beta)} Z + \frac{\beta - \alpha}{2} K([X, Y], Z)\tag{7}$$

If we replace (5), (6) and (7) in (4), we deduce that,

$$\begin{aligned}R^{(\alpha)}(X, Y)Z &= R^{(\beta)}(X, Y)Z + \frac{\beta - \alpha}{2} \nabla_X^{(\beta)} K(Y, Z) - \frac{\beta - \alpha}{2} \nabla_Y^{(\beta)} K(X, Z) \\ &+ \frac{\beta - \alpha}{2} K(X, \nabla_Y^{(\beta)} Z) - \frac{\beta - \alpha}{2} K(Y, \nabla_X^{(\beta)} Z) - \frac{\beta - \alpha}{2} K([X, Y], Z) \\ &+ \frac{(\beta - \alpha)^2}{4} K(X, K(Y, Z)) - \frac{(\beta - \alpha)^2}{4} K(Y, K(X, Z))\end{aligned}$$

Using the fact that,

$$\begin{aligned}\nabla_X^{(\beta)} K(Y, Z) &= \left(\nabla_X^{(\beta)} K \right)(Y, Z) + K\left(\nabla_X^{(\beta)} Y, Z \right) + K\left(Y, \nabla_X^{(\beta)} Z \right), \\ \nabla_Y^{(\beta)} K(X, Z) &= \left(\nabla_Y^{(\beta)} K \right)(X, Z) + K\left(\nabla_Y^{(\beta)} X, Z \right) + K\left(X, \nabla_Y^{(\beta)} Z \right)\end{aligned}$$

and

$$K([X, Y], Z) = K\left(\nabla_X^{(\beta)} Y, Z \right) - K\left(\nabla_Y^{(\beta)} X, Z \right),$$

we get

$$\begin{aligned}R^{(\alpha)}(X, Y)Z &= R^{(\beta)}(X, Y)Z + \frac{\beta - \alpha}{2} \left(\nabla_X^{(\beta)} K \right)(Y, Z) - \frac{\beta - \alpha}{2} \left(\nabla_Y^{(\beta)} K \right)(X, Z) \\ &+ \frac{(\beta - \alpha)^2}{4} K(X, K(Y, Z)) - \frac{(\beta - \alpha)^2}{4} K(Y, K(X, Z)).\end{aligned}$$

As particular cases of [Theorem 1](#), we get the following Corollary:

Corollary 1. Let (M^m, ∇, g) a statistical manifold with a dualistic structure $\left(\nabla, \nabla^*, g \right)$.
The relations between $R^{(\alpha)}$, $\bar{R}$, R and R^* are given by

$$R^{(\alpha)}(X, Y)Z = \widehat{R}(X, Y)Z - \frac{\alpha}{2} \left(\widehat{\nabla}_X K \right) (Y, Z) + \frac{\alpha}{2} \left(\widehat{\nabla}_Y K \right) (X, Z) + \frac{\alpha^2}{4} K(X, K(Y, Z)) - \frac{\alpha^2}{4} K(Y, K(X, Z)), \quad (8)$$

$$R^{(\alpha)}(X, Y)Z = R(X, Y)Z + \frac{1-\alpha}{2} (\nabla_X K)(Y, Z) - \frac{1-\alpha}{2} (\nabla_Y K)(X, Z) + \frac{(1-\alpha)^2}{4} K(X, K(Y, Z)) - \frac{(1-\alpha)^2}{4} K(Y, K(X, Z)) \quad (9)$$

and

$$R^{(\alpha)}(X, Y)Z = R^*(X, Y)Z - \frac{1+\alpha}{2} \left(\nabla_X^* K \right) (Y, Z) + \frac{1+\alpha}{2} \left(\nabla_Y^* K \right) (X, Z) + \frac{(1+\alpha)^2}{4} K(X, K(Y, Z)) - \frac{(1+\alpha)^2}{4} K(Y, K(X, Z)), \quad (10)$$

for all $X, Y, Z \in \Gamma(TM)$.

Remark 2. From [Theorem 1](#), we can give other relations:

- (1) The relation between $R^{(\alpha)}$ and $R^{(-\alpha)}$ is given by (see [\[1\]](#)).

$$R^{(\alpha)}(X, Y)Z = R^{(-\alpha)}(X, Y)Z + \alpha(R(X, Y)Z - R^*(X, Y)Z).$$

- (2) The relation between $R^{(\alpha)}$, R and R^* is given by (see [\[1\]](#)).

$$R^{(\alpha)}(X, Y)Z = \frac{1+\alpha}{2} R(X, Y)Z + \frac{1-\alpha}{2} R^*(X, Y)Z - \frac{1-\alpha^2}{4} K(X, K(Y, Z)) + \frac{1-\alpha^2}{4} K(Y, K(X, Z))$$

Corollary 2. Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame field on (M^m, g) , for a statistical structure $\left(\nabla, \nabla, g \right)^*$, if we denote

$$Ricci^{(\alpha)}(X) = Tr_g R^{(\alpha)}(X, \cdot) \cdot = R^{(\alpha)}(X, e_i) e_i$$

and

$$Ricci^{(\beta)}(X) = Tr_g R^{(\beta)}(X, \cdot) \cdot = R^{(\beta)}(X, e_i) e_i,$$

for any $X \in \Gamma(TM)$, then the relation between $Ricci^{(\alpha)}(X)$ and $Ricci^{(\beta)}(X)$ is given by the following formula :

$$Ricci^{(\alpha)}(X) = Ricci^{(\beta)}(X) + \frac{(\beta - \alpha)^2}{4} K(X, E) + \frac{\beta - \alpha}{2} Tr_g \left(\nabla_X^{(\beta)} K \right) (\cdot, \cdot) - \frac{\beta - \alpha}{2} Tr_g \left(\nabla^{(\beta)} K \right) (X, \cdot) - \frac{(\beta - \alpha)^2}{4} Tr_g K(K(X, \cdot), \cdot),$$

where

$$Tr_g \left(\nabla_X^{(\beta)} K \right) (\cdot, \cdot) = \left(\nabla_X^{(\beta)} K \right) (e_i, e_i),$$

$$Tr_g \left(\nabla^{(\beta)} K \right) (X, \cdot) = \left(\nabla_{e_i}^{(\beta)} K \right) (X, e_i),$$

and

$$Tr_g K(K(X, \cdot), \cdot) = K(K(X, e_i), e_i),$$

$$E = Tr_g K = K(e_i, e_i).$$

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In particular for $\beta \in \{-1, 0, 1\}$, we obtain,

$$Ricci^{(\alpha)}(X) = \widehat{Ricci}(X) + \frac{\alpha^2}{4} K(X, E) - \frac{\alpha}{2} Tr_g(\widehat{\nabla}_X K)(\cdot, \cdot)$$

$$+ \frac{\alpha}{2} Tr_g(\widehat{\nabla} \cdot K)(X, \cdot) - \frac{\alpha^2}{4} Tr_g K(K(X, \cdot), \cdot),$$

$$Ricci^{(\alpha)}(X) = Ricci(X) + \frac{(1-\alpha)^2}{4} K(X, E) + \frac{1-\alpha}{2} Tr_g(\nabla_X K)(\cdot, \cdot)$$

$$- \frac{1-\alpha}{2} Tr_g(\nabla \cdot K)(X, \cdot) - \frac{(1-\alpha)^2}{4} Tr_g K(K(X, \cdot), \cdot)$$

and

$$Ricci^{(\alpha)}(X) = Ricci^*(X) + \frac{(1+\alpha)^2}{4} K(X, E) - \frac{1+\alpha}{2} Tr_g\left(\nabla_X^* K\right)(\cdot, \cdot)$$

$$+ \frac{1+\alpha}{2} Tr_g\left(\nabla^* \cdot K\right)(X, \cdot) - \frac{(1+\alpha)^2}{4} Tr_g K(K(X, \cdot), \cdot)$$

Example 1. Let $(\mathbb{R}^2, g)$ be a statistical manifold with Riemannian metric $g = dx^2 + dy^2$ and ∇ an affine connection defined by

$$\nabla_{e_1} e_1 = e_2, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = e_1$$

where $\left\{e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}\right\}$ is an orthonormal frame field. A simple calculation gives,

$$\nabla_{e_1}^* e_1 = -e_2, \quad \nabla_{e_2}^* e_2 = 0, \quad \nabla_{e_1}^* e_2 = \nabla_{e_2}^* e_1 = -e_1.$$

We deduce that,

$$K(e_1, e_1) = -2e_2, \quad K(e_2, e_2) = 0, \quad K(e_1, e_2) = K(e_2, e_1) = -2e_1,$$

then,

$$E = Tr_g K = K(e_1, e_1) + K(e_2, e_2) = -2e_2.$$

In this case, we have,

$$\nabla_{e_1}^{(\alpha)} e_1 = \alpha e_2, \quad \nabla_{e_2}^{(\alpha)} e_2 = 0, \quad \nabla_{e_1}^{(\alpha)} e_2 = \nabla_{e_2}^{(\alpha)} e_1 = \alpha e_1,$$

$$R^{(\alpha)}(e_1, e_2)e_2 = -\alpha^2 e_1, \quad R^{(\alpha)}(e_2, e_1)e_1 = -\alpha^2 e_2.$$

and

$$Ricci^{(\alpha)}(X) = -\alpha^2 X, \quad Ric^{(\alpha)}(X, Y) = -\alpha^2 g(X, Y), \quad S_g^{(\alpha)} = -2\alpha^2.$$

Then $(\mathbb{R}^2, \nabla^{(\alpha)}, g)$ is a statistical manifold of constant curvature $-\alpha^2$ and it is a Hessian structure if and only if $\alpha = 0$.

Example 2. Let $(H^2 = \{(x, y) \in \mathbb{R}^2, y > 0\}, g)$ be a statistical manifold with Riemannian metric $g = \frac{1}{y^2}(dx^2 + dy^2)$ and ∇ an affine connection defined by

$$\nabla_{e_1} e_1 = \nabla_{e_2} e_2 = 2e_2, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_1 = e_1,$$

where $\left\{ e_1 = y \frac{\partial}{\partial x}, e_2 = y \frac{\partial}{\partial y} \right\}$ is an orthonormal frame field. A simple calculation gives

$$\overset{*}{\nabla}_{e_1} e_1 = 0, \quad \overset{*}{\nabla}_{e_2} e_2 = -2e_2, \quad \overset{*}{\nabla}_{e_1} e_2 = -2e_1, \quad \overset{*}{\nabla}_{e_2} e_1 = -e_1.$$

Then,

$$\nabla_{e_1}^{(\alpha)} e_1 = (1 + \alpha)e_2, \quad \nabla_{e_2}^{(\alpha)} e_2 = 2\alpha e_2, \quad \nabla_{e_1}^{(\alpha)} e_2 = -(1 - \alpha)e_1, \quad \nabla_{e_2}^{(\alpha)} e_1 = \alpha e_1.$$

We deduce that,

$$R^{(\alpha)}(e_1, e_2)e_2 = (\alpha^2 - 1)e_1, \quad R^{(\alpha)}(e_2, e_1)e_1 = (\alpha^2 - 1)e_2,$$

it follows that,

$$Ricci^{(\alpha)}(X) = (\alpha^2 - 1)X, \quad Ric^{(\alpha)}(X, Y) = (\alpha^2 - 1)g(X, Y), \quad S_g^{(\alpha)} = 2(\alpha^2 - 1).$$

In this case, $(H^2, \nabla^{(\alpha)}, g)$ is a statistical manifold of constant curvature $\alpha^2 - 1$ and it is a Hessian structure if and only if $\alpha = \pm 1$.

3. The α -conformal equivalence

For a real number α , statistical manifolds (M^m, ∇, g) and $(M^m, \bar{\nabla}, \bar{g})$ are said to be α -conformally equivalent if there exists a function γ on M such that the Riemannian metrics $\bar{g}$ and g and $\bar{\nabla}$ and ∇ are related by the following relation,

$$\bar{g}(X, Y) = e^{2\gamma}g(X, Y) \quad (11)$$

and the connection $\bar{\nabla}$ is given by,

$$\bar{\nabla}_X Y = \nabla_X Y + (1 - \alpha)Y(\gamma)X + (1 - \alpha)X(\gamma)Y - (1 + \alpha)g(X, Y)grad\gamma, \quad (12)$$

for all $X, Y, Z \in \Gamma(TM)$. Using the fact that $\nabla_X Y = \hat{\nabla}_X Y - \frac{1}{2}K(X, Y)$, we obtain,

$$\bar{\nabla}_X Y = \hat{\nabla}_X Y + (1 - \alpha)Y(\gamma)X + (1 - \alpha)X(\gamma)Y - (1 + \alpha)g(X, Y)grad\gamma - \frac{1}{2}K(X, Y). \quad (13)$$

Theorem 2.

$$\begin{aligned} \bar{R}(X, Y)Z &= \hat{R}(X, Y)Z - (1 + \alpha)g(Y, Z)\hat{\nabla}_X grad\gamma + (1 + \alpha)g(X, Z)\hat{\nabla}_Y grad\gamma \\ &\quad + (1 + \alpha)^2 g(Y, Z)X(\gamma)grad\gamma - (1 + \alpha)^2 g(X, Z)Y(\gamma)grad\gamma + (1 - \alpha)^2 Y(\gamma)Z(\gamma)X \\ &\quad - (1 - \alpha^2)g(Y, Z)|grad\gamma|^2 X - \frac{1 - \alpha}{2}K(Y, Z)(\gamma)X - (1 - \alpha)g(\hat{\nabla}_Y grad\gamma, Z)X \\ &\quad - (1 - \alpha)^2 X(\gamma)Z(\gamma)Y + (1 - \alpha)g(\hat{\nabla}_X grad\gamma, Z)Y + (1 - \alpha^2)g(X, Z)|grad\gamma|^2 Y \\ &\quad + \frac{1 - \alpha}{2}K(X, Z)(\gamma)Y - \frac{1}{2}(\hat{\nabla}_X K)(Y, Z) + \frac{1}{2}(\hat{\nabla}_Y K)(X, Z) \\ &\quad + \frac{1 + \alpha}{2}g(Y, Z)K(X, grad\gamma) - \frac{1 + \alpha}{2}g(X, Z)K(Y, grad\gamma) \\ &\quad + \frac{1}{4}K(X, K(Y, Z)) - \frac{1}{4}K(Y, K(X, Z)). \end{aligned} \quad (14)$$

Proof of Theorem 2. By definition, we have,

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z. \quad (15)$$

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We will study the right side of this equation term by term. By (13), we obtain,

$$\bar{\nabla}_Y Z = \hat{\nabla}_Y Z + (1 - \alpha)Z(\gamma)Y + (1 - \alpha)Y(\gamma)Z - (1 + \alpha)g(Y, Z)grad\gamma - \frac{1}{2}K(Y, Z),$$

which gives us,

$$\begin{aligned} \bar{\nabla}_X \bar{\nabla}_Y Z &= \bar{\nabla}_X \hat{\nabla}_Y Z + (1 - \alpha)\bar{\nabla}_X Z(\gamma)Y + (1 - \alpha)\bar{\nabla}_X Y(\gamma)Z \\ &\quad - (1 + \alpha)\bar{\nabla}_X g(Y, Z)grad\gamma - \frac{1}{2}\bar{\nabla}_X K(Y, Z). \end{aligned}$$

Using Eqn (13), we deduce that,

$$\begin{aligned} \bar{\nabla}_X \hat{\nabla}_Y Z &= \hat{\nabla}_X \hat{\nabla}_Y Z + (1 - \alpha)\left(\hat{\nabla}_Y Z\right)(\gamma)X + (1 - \alpha)X(\gamma)\hat{\nabla}_Y Z \\ &\quad - (1 + \alpha)g\left(X, \hat{\nabla}_Y Z\right)grad\gamma - \frac{1}{2}K\left(X, \hat{\nabla}_Y Z\right), \end{aligned}$$

$$\begin{aligned} \bar{\nabla}_X Z(\gamma)Y &= Z(\gamma)\hat{\nabla}_X Y + (1 - \alpha)Y(\gamma)Z(\gamma)X + (1 - \alpha)X(\gamma)Z(\gamma)Y \\ &\quad - (1 + \alpha)g(X, Y)Z(\gamma)grad\gamma - \frac{1}{2}Z(\gamma)K(X, Y) + g\left(\hat{\nabla}_X grad\gamma, Z\right)Y \\ &\quad + g\left(grad\gamma, \hat{\nabla}_X Z\right)Y, \end{aligned}$$

$$\begin{aligned} \bar{\nabla}_X Y(\gamma)Z &= Y(\gamma)\hat{\nabla}_X Z + (1 - \alpha)Y(\gamma)Z(\gamma)X + (1 - \alpha)X(\gamma)Y(\gamma)Z \\ &\quad - (1 + \alpha)g(X, Z)Y(\gamma)grad\gamma - \frac{1}{2}Y(\gamma)K(X, Z) + g\left(\hat{\nabla}_X grad\gamma, Y\right)Z \\ &\quad + g\left(grad\gamma, \hat{\nabla}_X Y\right)Z, \end{aligned}$$

$$\begin{aligned} \bar{\nabla}_X g(Y, Z)grad\gamma &= g(Y, Z)\hat{\nabla}_X grad\gamma + (1 - \alpha)g(Y, Z)|grad\gamma|^2 X - 2\alpha g(Y, Z)X(\gamma)grad\gamma \\ &\quad - \frac{1}{2}g(Y, Z)K(X, grad\gamma) + g\left(\hat{\nabla}_X Y, Z\right)grad\gamma + g\left(Y, \hat{\nabla}_X Z\right)grad\gamma \end{aligned}$$

and

$$\begin{aligned} \bar{\nabla}_X K(Y, Z) &= \hat{\nabla}_X K(Y, Z) + (1 - \alpha)K(Y, Z)(\gamma)X + (1 - \alpha)X(\gamma)K(Y, Z) \\ &\quad - (1 + \alpha)g(X, K(Y, Z))grad\gamma - \frac{1}{2}K(X, K(Y, Z)). \end{aligned}$$

It follows that,

$$\begin{aligned}
\overline{\nabla}_X \overline{\nabla}_Y Z &= \widehat{\nabla}_X \widehat{\nabla}_Y Z - (1+\alpha)g(Y, Z)\widehat{\nabla}_X \text{grad}\gamma + \frac{1+\alpha}{2}g(X, K(Y, Z))\text{grad}\gamma \\
&\quad - (1-\alpha^2)g(X, Y)Z(\gamma)\text{grad}\gamma - (1-\alpha^2)g(X, Z)Y(\gamma)\text{grad}\gamma \\
&\quad + 2\alpha(1+\alpha)g(Y, Z)X(\gamma)\text{grad}\gamma - (1+\alpha)g\left(X, \widehat{\nabla}_Y Z\right)\text{grad}\gamma \\
&\quad - (1+\alpha)g\left(\widehat{\nabla}_X Y, Z\right)\text{grad}\gamma - (1+\alpha)g\left(Y, \widehat{\nabla}_X Z\right)\text{grad}\gamma \\
&\quad - (1-\alpha^2)g(Y, Z)|\text{grad}\gamma|^2 X + 2(1-\alpha)^2 Y(\gamma)Z(\gamma)X + (1-\alpha)\left(\widehat{\nabla}_Y Z\right)(\gamma)X \\
&\quad + (1-\alpha)^2 X(\gamma)Z(\gamma)Y + (1-\alpha)g\left(\widehat{\nabla}_X \text{grad}\gamma, Z\right)Y + (1-\alpha)g\left(\text{grad}\gamma, \widehat{\nabla}_X Z\right)Y \\
&\quad + (1-\alpha)^2 X(\gamma)Y(\gamma)Z + (1-\alpha)g\left(\widehat{\nabla}_X \text{grad}\gamma, Y\right)Z + (1-\alpha)g\left(\text{grad}\gamma, \widehat{\nabla}_X Y\right)Z \\
&\quad + (1-\alpha)X(\gamma)\widehat{\nabla}_Y Z - \frac{1}{2}K\left(X, \widehat{\nabla}_Y Z\right) + (1-\alpha)Z(\gamma)\widehat{\nabla}_X Y + (1-\alpha)Y(\gamma)\widehat{\nabla}_X Z \\
&\quad - \frac{1-\alpha}{2}Y(\gamma)K(X, Z) - \frac{1-\alpha}{2}Z(\gamma)K(X, Y) + \frac{1+\alpha}{2}g(Y, Z)K(X, \text{grad}\gamma) \\
&\quad - \frac{1}{2}\widehat{\nabla}_X K(Y, Z) - \frac{1-\alpha}{2}K(Y, Z)(\gamma)X - \frac{1-\alpha}{2}X(\gamma)K(Y, Z) + \frac{1}{4}K(X, K(Y, Z)).
\end{aligned} \tag{16}$$

A similar calculation gives us,

$$\begin{aligned}
\overline{\nabla}_Y \overline{\nabla}_X Z &= \widehat{\nabla}_Y \widehat{\nabla}_X Z - (1+\alpha)g(X, Z)\widehat{\nabla}_Y \text{grad}\gamma + \frac{1+\alpha}{2}g(Y, K(X, Z))\text{grad}\gamma \\
&\quad - (1-\alpha^2)g(X, Y)Z(\gamma)\text{grad}\gamma - (1-\alpha^2)g(Y, Z)X(\gamma)\text{grad}\gamma \\
&\quad + 2\alpha(1+\alpha)g(X, Z)Y(\gamma)\text{grad}\gamma - (1+\alpha)g\left(Y, \widehat{\nabla}_X Z\right)\text{grad}\gamma \\
&\quad - (1+\alpha)g\left(\widehat{\nabla}_Y X, Z\right)\text{grad}\gamma - (1+\alpha)g\left(X, \widehat{\nabla}_Y Z\right)\text{grad}\gamma \\
&\quad - (1-\alpha^2)g(X, Z)|\text{grad}\gamma|^2 Y + 2(1-\alpha)^2 X(\gamma)Z(\gamma)Y + (1-\alpha)\left(\widehat{\nabla}_X Z\right)(\gamma)Y \\
&\quad + (1-\alpha)^2 Y(\gamma)Z(\gamma)X + (1-\alpha)g\left(\widehat{\nabla}_Y \text{grad}\gamma, Z\right)X + (1-\alpha)g\left(\text{grad}\gamma, \widehat{\nabla}_Y Z\right)X \\
&\quad + (1-\alpha)^2 X(\gamma)Y(\gamma)Z + (1-\alpha)g\left(\widehat{\nabla}_X \text{grad}\gamma, Y\right)Z + (1-\alpha)g\left(\text{grad}\gamma, \widehat{\nabla}_Y X\right)Z \\
&\quad + (1-\alpha)Y(\gamma)\widehat{\nabla}_X Z - \frac{1}{2}K\left(Y, \widehat{\nabla}_X Z\right) + (1-\alpha)Z(\gamma)\widehat{\nabla}_Y X + (1-\alpha)X(\gamma)\widehat{\nabla}_Y Z \\
&\quad - \frac{1-\alpha}{2}X(\gamma)K(Y, Z) - \frac{1-\alpha}{2}Z(\gamma)K(X, Y) + \frac{1+\alpha}{2}g(X, Z)K(Y, \text{grad}\gamma) \\
&\quad - \frac{1}{2}\widehat{\nabla}_Y K(X, Z) - \frac{1-\alpha}{2}K(X, Z)(\gamma)Y - \frac{1-\alpha}{2}Y(\gamma)K(X, Z) + \frac{1}{4}K(Y, K(X, Z)).
\end{aligned} \tag{17}$$

Finally, it is easy to see that,

$$\begin{aligned}\bar{\nabla}_{[X,Y]}Z &= \hat{\nabla}_{[X,Y]}Z - (1+\alpha)g(\hat{\nabla}_X Y, Z)grad\gamma + (1+\alpha)g(\hat{\nabla}_Y X, Z)grad\gamma \\ &\quad + (1-\alpha)(\hat{\nabla}_X Y)(\gamma)Z - (1-\alpha)(\hat{\nabla}_Y X)(\gamma)Z + (1-\alpha)Z(\gamma)\hat{\nabla}_X Y \\ &\quad - (1-\alpha)Z(\gamma)\hat{\nabla}_Y X - \frac{1}{2}K(\hat{\nabla}_X Y, Z) + \frac{1}{2}K(\hat{\nabla}_Y X, Z).\end{aligned}\quad (18)$$

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By replacing (16), (17) and (18) in (15), we conclude that,

$$\begin{aligned}\bar{R}(X, Y)Z &= \hat{R}(X, Y)Z - (1+\alpha)g(Y, Z)\hat{\nabla}_X grad\gamma + (1+\alpha)g(X, Z)\hat{\nabla}_Y grad\gamma \\ &\quad + (1+\alpha)^2g(Y, Z)X(\gamma)grad\gamma - (1+\alpha)^2g(X, Z)Y(\gamma)grad\gamma + (1-\alpha)^2Y(\gamma)Z(\gamma)X \\ &\quad - (1-\alpha^2)g(Y, Z)|grad\gamma|^2X - \frac{1-\alpha}{2}K(Y, Z)(\gamma)X - (1-\alpha)g(\hat{\nabla}_Y grad\gamma, Z)X \\ &\quad - (1-\alpha)^2X(\gamma)Z(\gamma)Y + (1-\alpha)g(\hat{\nabla}_X grad\gamma, Z)Y + (1-\alpha^2)g(X, Z)|grad\gamma|^2Y \\ &\quad + \frac{1-\alpha}{2}K(X, Z)(\gamma)Y - \frac{1}{2}(\hat{\nabla}_X K)(Y, Z) + \frac{1}{2}(\hat{\nabla}_Y K)(X, Z) \\ &\quad + \frac{1+\alpha}{2}g(Y, Z)K(X, grad\gamma) - \frac{1+\alpha}{2}g(X, Z)K(Y, grad\gamma) + \frac{1}{4}K(X, K(Y, Z)) \\ &\quad - \frac{1}{4}K(Y, K(X, Z)).\end{aligned}$$

The same method of calculation used in Theorem 2 and the following equations,

$$\begin{aligned}\hat{\nabla}_X Y &= \nabla_X Y + \frac{1}{2}K(X, Y), \\ \hat{R}(X, Y)Z &= R(X, Y)Z + \frac{1}{2}(\nabla_X K)(Y, Z) - \frac{1}{2}(\nabla_Y K)(X, Z) \\ &\quad + \frac{1}{4}K(X, K(Y, Z)) - \frac{1}{4}K(Y, K(X, Z))\end{aligned}$$

gives us the following theorem

Theorem 3.

$$\begin{aligned}\bar{R}(X, Y)Z &= R(X, Y)Z - (1+\alpha)g(Y, Z)\nabla_X grad\gamma + (1+\alpha)g(X, Z)\nabla_Y grad\gamma \\ &\quad - (1+\alpha)^2g(X, Z)Y(\gamma)grad\gamma + (1+\alpha)^2g(Y, Z)X(\gamma)grad\gamma \\ &\quad - (1-\alpha^2)g(Y, Z)|grad\gamma|^2X + (1-\alpha^2)g(X, Z)|grad\gamma|^2Y \\ &\quad - (1-\alpha)g(\nabla_Y grad\gamma, Z)X + (1-\alpha)g(\nabla_X grad\gamma, Z)Y \\ &\quad + (1-\alpha)^2Y(\gamma)Z(\gamma)X - (1-\alpha)^2X(\gamma)Z(\gamma)Y \\ &\quad - (1-\alpha)g(K(Y, Z), grad\gamma)X + (1-\alpha)g(K(X, Z), grad\gamma)Y\end{aligned}\quad (19)$$

Corollary 3. Let us choose $\{e_i\}_{1 \leq i \leq m}$ to be an orthonormal frame on (M^m, ∇, g) , an orthonormal frame on $(M^m, \bar{\nabla}, \bar{g} = e^{2\gamma}g)$ is given by $\{\bar{e}_i = e^{-\gamma}e_i\}_{1 \leq i \leq m}$. For any $X, Y \in \Gamma(TM)$, we define

$$\begin{aligned} Ricci(X) &= Tr_g R(X, \cdot) \cdot = R(X, e_i) e_i, & \overline{Ricci}(X) &= Tr_{\bar{g}} \bar{R}(X, \cdot) \cdot = \bar{R}(X, \bar{e}_i) \bar{e}_i, \\ Ric(X, Y) &= g(Ricci(X), Y), & \bar{Ric}(X, Y) &= \bar{g}(\bar{Ricci}(X), Y) \end{aligned}$$

and

$$S_g = Tr_g Ric = Ric(e_i, e_i), \quad S_{\bar{g}} = Tr_{\bar{g}} \bar{Ric} = \bar{Ric}(\bar{e}_i, \bar{e}_i).$$

Using [Theorem 3](#), we obtain the following relations,

$$\begin{aligned} \overline{Ricci}(X) &= e^{-2\gamma} Ricci(X) + ((m-2)\alpha^2 + 2m\alpha + m-2)e^{-2\gamma} X(\gamma) grad\gamma \\ &\quad - (m\alpha + m-2)e^{-2\gamma} \nabla_X grad\gamma + (m\alpha^2 - 2\alpha - m+2)e^{-2\gamma} |grad\gamma|^2 X \\ &\quad - (1-\alpha)e^{-2\gamma} (\hat{\Delta}\gamma) X - \frac{(1-\alpha)}{2} e^{-2\gamma} E(\gamma) X + (1-\alpha)e^{-2\gamma} K(X, grad\gamma), \\ \bar{Ric}(X, Y) &= Ric(X, Y) + ((m-2)\alpha^2 + 2m\alpha + m-2)X(\gamma)Y(\gamma) \\ &\quad + (m\alpha^2 - 2\alpha - m+2)|grad\gamma|^2 g(X, Y) - (1-\alpha)(\hat{\Delta}\gamma)g(X, Y) \\ &\quad - (m\alpha + m-2)g(\nabla_X grad\gamma, Y) - \frac{(1-\alpha)}{2} E(\gamma)g(X, Y) \\ &\quad + (1-\alpha)g(K(X, Y), grad\gamma) \end{aligned}$$

and

$$\begin{aligned} S_{\bar{g}} &= e^{-2\gamma} S_g + (m-1)((m+2)\alpha^2 - m+2)e^{-2\gamma} |grad\gamma|^2 \\ &\quad - 2(m-1)e^{-2\gamma} (\hat{\Delta}\gamma) + (m-1)\alpha e^{-2\gamma} E(\gamma) \end{aligned}$$

Corollary 4. [Theorem 3](#) and [Corollary 3](#) gives us two particular cases:

(1) If $\alpha = 1$, we obtain,

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z - 2g(Y, Z)\nabla_X grad\gamma + 2g(X, Z)\nabla_Y grad\gamma \\ &\quad - 4g(X, Z)Y(\gamma)grad\gamma + 4g(Y, Z)X(\gamma)grad\gamma, \\ \overline{Ricci}(X) &= e^{-2\gamma} (Ricci(X) + 4(m-1)X(\gamma)grad\gamma - 2(m-1)\nabla_X grad\gamma), \\ \bar{Ric}(X, Y) &= Ric(X, Y) + 4(m-1)X(\gamma)Y(\gamma) - 2(m-1)g(\nabla_X grad\gamma, Y) \end{aligned}$$

and

$$S_{\bar{g}} = e^{-2\gamma} \left(S_g + 4(m-1)|grad\gamma|^2 - 2(m-1)(\hat{\Delta}\gamma) + (m-1)E(\gamma) \right).$$

(2) If $\alpha = -1$, we obtain,

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z - 2g(\nabla_Y grad\gamma, Z)X + 2g(\nabla_X grad\gamma, Z)Y \\ &\quad + 4Y(\gamma)Z(\gamma)X - 4X(\gamma)Z(\gamma)Y - 2g(K(Y, Z), grad\gamma)X \\ &\quad + 2g(K(X, Z), grad\gamma)Y, \\ \overline{Ricci}(X) &= e^{-2\gamma} Ricci(X) - 4e^{-2\gamma} X(\gamma)grad\gamma + 2e^{-2\gamma} \nabla_X grad\gamma \\ &\quad + 4e^{-2\gamma} |grad\gamma|^2 X - 2e^{-2\gamma} (\hat{\Delta}\gamma) X - e^{-2\gamma} E(\gamma) X \\ &\quad + 2e^{-2\gamma} K(X, grad\gamma), \end{aligned}$$

$$\begin{aligned}\overline{Ric}(X, Y) &= Ric(X, Y) - 4X(\gamma)Y(\gamma) + 2g(\nabla_X grad\gamma, Y) \\ &\quad + 4|grad\gamma|^2 g(X, Y) - 2(\widehat{\Delta}\gamma)g(X, Y) \\ &\quad - E(\gamma)g(X, Y) + 2g(K(X, Y), grad\gamma)\end{aligned}$$

and

$$S_{\overline{g}} = e^{-2\gamma} S_g + (m-1)e^{-2\gamma} \left(4|grad\gamma|^2 - 2(\widehat{\Delta}\gamma) + E(\gamma) \right),$$

where

$$\widehat{\Delta}\gamma = g\left(\widehat{\nabla}_e grad\gamma, e_i\right) = e_i(e_i(\gamma)) - \left(\widehat{\nabla}_{e_i} e_i\right)(\gamma).$$

Example 3. Let $(\mathbb{R}^2, g)$ be a statistical manifold with Riemannian metric $g = dx^2 + dy^2$ and ∇ an affine connection defined by

$$\nabla_{e_1} e_1 = e_2, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = e_1$$

where $\left\{e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}\right\}$ is an orthonormal frame field. Then $(\mathbb{R}^2, \nabla, g)$ is a statistical manifold of constant curvature -1 and $S_g = -2$. We want to determine γ such that $S_{\overline{g}} = 0$. By Corollary 2.2, we deduce that $S_{\overline{g}}$ vanish if and only if

$$2(\widehat{\Delta}\gamma) - \alpha E(\gamma) - 4\alpha^2 |grad\gamma|^2 + 2 = 0.$$

To solve this equation, we will present two cases :

- (1) If we assume that γ depends only on the variable x , then $S_{\overline{g}}$ vanish if and only if.

$$\gamma'' - 2\alpha^2 (\gamma')^2 + 1 = 0.$$

Note that if $\alpha = 0$, the solution of this last equation is,

$$\gamma(x) = -\frac{1}{2}x^2 + ax + b.$$

In the case where $\alpha \neq 0$, a particular solution is given by $\gamma(x) = \frac{1}{\alpha\sqrt{2}}x + b$.

- (2) If the function γ depends only on the variable y , we conclude that $S_{\overline{g}} = 0$ if and only if,

$$\gamma'' + \alpha\gamma' - 2\alpha^2 (\gamma')^2 + 1 = 0.$$

Using the same method, if $\alpha = 0$, the solution obtained is,

$$\gamma(y) = -\frac{1}{2}y^2 + ay + b$$

and if we take $\alpha \neq 0$, a particular solution is $\gamma(y) = \frac{1}{\alpha}y + b$.

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