On the α-connections and the α-conformal equivalence on statistical manifolds

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Abstract

Purpose – In this paper, we give some properties of the α-connections on statistical manifolds and we study the α-conformal equivalence where we develop an expression of curvature $\overline{R}$ for $\overline{\nabla}$ in relation to those for ∇ and $\widehat{\nabla}$.

Design/methodology/approach – In the first section of this paper, we prove some results about the α-connections of a statistical manifold where we give some properties of the difference tensor K and we determine a relation between the curvature tensors; this relation is a generalization of the results obtained in [1]. In the second section, we introduce the notion of α-conformal equivalence of statistical manifolds treated in [1, 3], and we construct some examples.

Findings – We give some properties of the difference tensor K and we determine a relation between the curvature tensors; this relation is a generalization of the results obtained in [1]. In the second section, we introduce the notion of α-conformal equivalence of statistical manifolds, we give the relations between curvature tensors and we construct some examples.

Originality/value – We give some properties of the difference tensor K and we determine a relation between the curvature tensors; this relation is a generalization of the results obtained in [1]. In the second section, we introduce the notion of α-conformal equivalence of statistical manifolds, we give the relations between curvature tensors and we construct some examples.

Keywords Statistical manifold, α-connections, α-conformal equivalence

Paper type Research paper

1. Introduction

Let (M^m, g) be a Riemannian manifold and ∇ a torsion free linear connection on M. The triple (M^m, ∇, g) is called a statistical manifold if ∇g is symmetric and the pair (∇, g) is called a statistical structure. For a statistical manifold (M^m, ∇, g), let ∇^* be an affine connection on M such that,

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g\left(Y, \overset{*}{\nabla}_X Z\right),$$

for all X, Y and Z in $\Gamma(TM)$. The affine connection $\overset{*}{\nabla}$ is torsion free, and $\overset{*}{\nabla} g$ symmetric. Then $\overset{*}{\nabla}$ is called the dual connection of ∇, the triple $\left(M^m, \overset{*}{\nabla}, g\right)$ is called the dual statistical manifold of (M^m, ∇, g) and $\left(\nabla, \overset{*}{\nabla}, g\right)$ is the dualistic structure. Denoted by $\widehat{\nabla}$ the Levi-Civita

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connection associated with g, the difference tensor K is the $(1,2)$-tensor defined by (see [1]).

$$K(X,Y) = \overset{*}{\nabla}_X Y - \nabla_X Y.$$

The difference tensor K satisfies for any vector fields X, Y, Z and any smooth function f on M the following properties:

$$\nabla_X Y = \widehat{\nabla}_X Y - \frac{1}{2}K(X,Y), \qquad \overset{*}{\nabla}_X Y = \widehat{\nabla}_X Y + \frac{1}{2}K(X,Y),$$

$$K(X,Y) = K(Y,X),$$
$$K(X,Y+Z) = K(X,Y) + K(X,Z)$$

and

$$K(fX,Y) = K(X,fY) = fK(X,Y).$$

Moreover, we have,

$$g(K(X,Y),Z) = g(K(X,Z),Y).$$

A statistical structure is called trace-free if $\nabla v_g = 0$ where v_g is the volume form determined by g. This condition is equivalent to the condition $Tr_g K = 0$. A statistical structure (∇, g) is said to be of constant curvature $k \in \mathbb{R}$ if the curvature tensor field R of ∇ satisfies,

$$R(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\}.$$

If $k = 0$, (∇, g) is called a Hessian structure. The concept of α-conformally equivalence was treated in [1] where the author develops an expression of the curvature $R^{(\alpha)}$. In [2], the authors studied a 1-conformally flat statistical submanifold of a flat statistical manifold; they proved that a 1-conformally flat statistical manifold with a Riemannian metric can be locally realized as a submanifold of a flat statistical manifold. The author in [3] gives a procedure to realize a statistical manifold, which is α-conformally equivalent to a manifold with an α-transitively flat connection, as a statistical submanifold and in [4], he describe a method to obtain α-conformally equivalent connections from the relation between tensors and the symmetric cubic form. In [5], the authors studied the statistical hypersurfaces of some types of the statistical manifolds, which enable to construct a structure of a constant curvature. The divergence of 1-conformally flat statistical manifolds is studied in [6] where the authors prove that the generalized Pythagorean theorem holds if the statistical manifold has a constant curvature. In the first section of this paper, we prove some results about the α-connections of a statistical manifolds where we give some properties of the difference tensor K and we determine a relation between the curvature tensors $R^{(\alpha)}$ and $R^{(\beta)}$; this relation is a generalization of the results obtained in [1]. In the second section, we introduce the notion of α-conformal equivalence of statistical manifolds treated in [1, 3], and we give the relations between $\overline{R}$, R and $\widehat{R}$ and we construct some examples.

2. Some results on the α-connections of a statistical manifolds

Let (M^m, ∇, g) a statistical manifold with a dualistic structure $\left(\nabla, \overset{*}{\nabla}, g\right)$. For $\alpha \in \mathbb{R}$, we define a family of torsion-free connections $\nabla^{(\alpha)}$ by,

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\overset{*}{\nabla}.$$

$\nabla^{(\alpha)}$ is called an α-connection of (M^m, ∇, g). The triple $\left(M^m, \nabla^{(\alpha)}, g\right)$ is also a statistical manifold, and $\nabla^{(-\alpha)}$ is the dual connection of $\nabla^{(\alpha)}$. In particular,

$$\nabla^{(0)} = \widehat{\nabla}, \quad \nabla^{(1)} = \nabla, \quad \nabla^{(-1)} = \overset{*}{\nabla}.$$

Moreover, we have the following equality,

$$\nabla^{(\alpha)}_X Y = \widehat{\nabla}_X Y - \frac{\alpha}{2} K(X, Y).$$

In general, for any $\alpha, \beta \in \mathbb{R}$, it is easy to see that,

$$\nabla^{(\alpha)}_X Y = \nabla^{(\beta)}_X Y - \frac{\alpha - \beta}{2} K(X, Y) \tag{1}$$

Proposition 1. Let (M^m, ∇, g) a statistical manifold with a dualistic structure $\left(\nabla, \overset{*}{\nabla}, g\right)$. For all vector fields X, Y, Z on M, we have,

$$\begin{aligned} \left(\nabla^{(\alpha)}_X K\right)(Y,Z) &= \left(\nabla^{(\beta)}_X K\right)(Y,Z) - \frac{\alpha-\beta}{2} K(X, K(Y,Z)) \\ &\quad + \frac{\alpha-\beta}{2} K(K(X,Y),Z) + \frac{\alpha-\beta}{2} K(Y, K(X,Z)). \end{aligned} \tag{2}$$

Proof of Proposition 1. Let $X, Y, Z \in \Gamma(TM)$, by definition, we have,

$$\left(\nabla^{(\alpha)}_X K\right)(Y,Z) = \nabla^{(\alpha)}_X K(Y,Z) - K\left(\nabla^{(\alpha)}_X Y, Z\right) - K\left(Y, \nabla^{(\alpha)}_X Z\right).$$

The properties of the difference tensor K gives us,

$$\nabla^{(\alpha)}_X K(Y,Z) = \nabla^{(\beta)}_X K(Y,Z) - \frac{\alpha-\beta}{2} K(X, K(Y,Z)),$$

$$K\left(\nabla^{(\alpha)}_X Y, Z\right) = K\left(\nabla^{(\beta)}_X Y, Z\right) - \frac{\alpha-\beta}{2} K(K(X,Y),Z)$$

and

$$K\left(Y, \nabla^{(\alpha)}_X Z\right) = K\left(Y, \nabla^{(\beta)}_X Z\right) - \frac{\alpha-\beta}{2} K(Y, K(X,Z)),$$

then

$$\begin{aligned} \left(\nabla^{(\alpha)}_X K\right)(Y,Z) &= \nabla^{(\beta)}_X K(Y,Z) - \frac{\alpha-\beta}{2} K(X, K(Y,Z)) - K\left(\nabla^{(\beta)}_X Y, Z\right) \\ &\quad + \frac{\alpha-\beta}{2} K(K(X,Y),Z) - K\left(Y, \nabla^{(\beta)}_X Z\right) + K(Y, K(X,Z)) \end{aligned}$$

Finally, using the fact that,

$$\left(\nabla^{(\beta)}_X K\right)(Y,Z) = \nabla^{(\beta)}_X K(Y,Z) - K\left(\nabla^{(\beta)}_X Y, Z\right) - K\left(Y, \nabla^{(\beta)}_X Z\right),$$

we deduce that,

$$\begin{aligned} \left(\nabla^{(\alpha)}_X K\right)(Y,Z) &= \left(\nabla^{(\beta)}_X K\right)(Y,Z) - \frac{\alpha-\beta}{2} K(X, K(Y,Z)) + \frac{\alpha-\beta}{2} K(K(X,Y),Z) \\ &\quad + \frac{\alpha-\beta}{2} K(Y, K(X,Z)). \end{aligned}$$

Remark 1. As particular cases of Eqn (2), we have

$$\begin{aligned}\left(\nabla_X^{(\alpha)}K\right)(Y,Z) &= \left(\widehat{\nabla}_X K\right)(Y,Z) - \frac{\alpha}{2}K(X,K(Y,Z)) + \frac{\alpha}{2}K(K(X,Y),Z)\\ &\quad + \frac{\alpha}{2}K(Y,K(X,Z))\\ &= (\nabla_X K)(Y,Z) - \frac{\alpha-1}{2}K(X,K(Y,Z)) + \frac{\alpha-1}{2}K(K(X,Y),Z)\\ &\quad + \frac{\alpha-1}{2}K(Y,K(X,Z))\\ &= \left(\overset{*}{\nabla}_X K\right)(Y,Z) - \frac{\alpha+1}{2}K(X,K(Y,Z)) + \frac{\alpha+1}{2}K(K(X,Y),Z)\\ &\quad + \frac{\alpha+1}{2}K(Y,K(X,Z))\end{aligned}$$

For a statistical structure $\left(\nabla, \overset{*}{\nabla}, g\right)$, we denote R, R^*, $\widehat{R}$ the curvature tensors for ∇, $\overset{*}{\nabla}$, $\widehat{\nabla}$, respectively, and $R^{(\alpha)}$ the curvature tensor for $\nabla^{(\alpha)}$. In the first results, we give the relation between $R^{(\alpha)}$ and $R^{(\beta)}$ for any $\alpha, \beta \in \mathbb{R}$.

Theorem 1. Let (M^m, ∇, g) a statistical manifold. The relation between $R^{(\alpha)}$ and $R^{(\beta)}$ is given by,

$$\begin{aligned}R^{(\alpha)}(X,Y)Z &= R^{(\beta)}(X,Y)Z + \frac{\beta-\alpha}{2}\left(\nabla_X^{(\beta)}K\right)(Y,Z) - \frac{\beta-\alpha}{2}\left(\nabla_Y^{(\beta)}K\right)(X,Z)\\ &\quad + \frac{(\beta-\alpha)^2}{4}K(X,K(Y,Z)) - \frac{(\beta-\alpha)^2}{4}K(Y,K(X,Z)),\end{aligned} \tag{3}$$

for all $X, Y, Z \in \Gamma(TM)$.

Proof of Theorem 1. Let $X, Y, Z \in \Gamma(TM)$, By definition we have,

$$R^{(\alpha)}(X,Y)Z = \nabla_X^{(\alpha)}\nabla_Y^{(\alpha)}Z - \nabla_X^{(\alpha)}\nabla_Y^{(\alpha)}Z - \nabla_{[X,Y]}^{(\alpha)}Z. \tag{4}$$

For the first term $\nabla_X^{(\alpha)}\nabla_Y^{(\alpha)}Z$, we have,

$$\nabla_Y^{(\alpha)}Z = \nabla_Y^{(\beta)}Z + \frac{\beta-\alpha}{2}K(Y,Z),$$

then

$$\nabla_X^{(\alpha)}\nabla_Y^{(\alpha)}Z = \nabla_X^{(\alpha)}\widehat{\nabla}_Y Z - \frac{\alpha}{2}\nabla_X^{(\alpha)}K(Y,Z).$$

It is simple to see that,

$$\nabla_X^{(\alpha)}\nabla_Y^{(\beta)}Z = \nabla_X^{(\beta)}\nabla_Y^{(\beta)}Z + \frac{\beta-\alpha}{2}K\left(X, \nabla_Y^{(\beta)}Z\right)$$

and

$$\nabla_X^{(\alpha)}K(Y,Z) = \nabla_X^{(\beta)}K(Y,Z) + \frac{\beta-\alpha}{2}K(X,K(Y,Z)),$$

which gives us

$$\begin{aligned}\nabla_X^{(\alpha)}\nabla_Y^{(\alpha)}Z &= \nabla_X^{(\beta)}\nabla_Y^{(\beta)}Z + \frac{\beta-\alpha}{2}K\left(X,\nabla_Y^{(\beta)}Z\right)\\ &+\frac{\beta-\alpha}{2}\nabla_X^{(\beta)}K(Y,Z)+\frac{(\beta-\alpha)^2}{4}K(X,K(Y,Z)).\end{aligned} \tag{5}$$

A similar calculation gives,

$$\begin{aligned}\nabla_Y^{(\alpha)}\nabla_X^{(\alpha)}Z &= \nabla_Y^{(\beta)}\nabla_X^{(\beta)}Z + \frac{\beta-\alpha}{2}K\left(Y,\nabla_X^{(\beta)}Z\right)\\ &+\frac{\beta-\alpha}{2}\nabla_Y^{(\beta)}K(X,Z)+\frac{(\beta-\alpha)^2}{4}K(Y,K(X,Z)).\end{aligned} \tag{6}$$

Finally, we have,

$$\nabla_{[X,Y]}^{(\alpha)}Z = \nabla_{[X,Y]}^{(\beta)}Z + \frac{\beta-\alpha}{2}K([X,Y],Z) \tag{7}$$

If we replace (5), (6) and (7) in (4), we deduce that,

$$\begin{aligned}R^{(\alpha)}(X,Y)Z &= R^{(\beta)}(X,Y)Z + \frac{\beta-\alpha}{2}\nabla_X^{(\beta)}K(Y,Z) - \frac{\beta-\alpha}{2}\nabla_Y^{(\beta)}K(X,Z)\\ &+\frac{\beta-\alpha}{2}K\left(X,\nabla_Y^{(\beta)}Z\right)-\frac{\beta-\alpha}{2}K\left(Y,\nabla_X^{(\beta)}Z\right)-\frac{\beta-\alpha}{2}K([X,Y],Z)\\ &+\frac{(\beta-\alpha)^2}{4}K(X,K(Y,Z))-\frac{(\beta-\alpha)^2}{4}K(Y,K(X,Z))\end{aligned}$$

Using the fact that,

$$\nabla_X^{(\beta)}K(Y,Z) = \left(\nabla_X^{(\beta)}K\right)(Y,Z) + K\left(\nabla_X^{(\beta)}Y,Z\right) + K\left(Y,\nabla_X^{(\beta)}Z\right),$$

$$\nabla_Y^{(\beta)}K(X,Z) = \left(\nabla_Y^{(\beta)}K\right)(X,Z) + K\left(\nabla_Y^{(\beta)}X,Z\right) + K\left(X,\nabla_Y^{(\beta)}Z\right)$$

and

$$K([X,Y],Z) = K\left(\nabla_X^{(\beta)}Y,Z\right) - K\left(\nabla_Y^{(\beta)}X,Z\right),$$

we get

$$\begin{aligned}R^{(\alpha)}(X,Y)Z &= R^{(\beta)}(X,Y)Z + \frac{\beta-\alpha}{2}\left(\nabla_X^{(\beta)}K\right)(Y,Z) - \frac{\beta-\alpha}{2}\left(\nabla_Y^{(\beta)}K\right)(X,Z)\\ &+\frac{(\beta-\alpha)^2}{4}K(X,K(Y,Z))-\frac{(\beta-\alpha)^2}{4}K(Y,K(X,Z)).\end{aligned}$$

As particular cases of Theorem 1, we get the following Corollary:

Corollary 1. Let (M^m,∇,g) a statistical manifold with a dualistic structure $\left(\nabla,\overset{*}{\nabla},g\right)$. The relations between $R^{(\alpha)}$, $\widehat{R}$, R and R^* are given by

$$R^{(\alpha)}(X,Y)Z = \widehat{R}(X,Y)Z - \frac{\alpha}{2}\left(\widehat{\nabla}_X K\right)(Y,Z) + \frac{\alpha}{2}\left(\widehat{\nabla}_Y K\right)(X,Z) + \frac{\alpha^2}{4}K(X,K(Y,Z)) - \frac{\alpha^2}{4}K(Y,K(X,Z)), \tag{8}$$

$$R^{(\alpha)}(X,Y)Z = R(X,Y)Z + \frac{1-\alpha}{2}(\nabla_X K)(Y,Z) - \frac{1-\alpha}{2}(\nabla_Y K)(X,Z) + \frac{(1-\alpha)^2}{4}K(X,K(Y,Z)) - \frac{(1-\alpha)^2}{4}K(Y,K(X,Z)) \tag{9}$$

and

$$R^{(\alpha)}(X,Y)Z = R^*(X,Y)Z - \frac{1+\alpha}{2}\left(\overset{*}{\nabla}_X K\right)(Y,Z) + \frac{1+\alpha}{2}\left(\overset{*}{\nabla}_Y K\right)(X,Z) + \frac{(1+\alpha)^2}{4}K(X,K(Y,Z)) - \frac{(1+\alpha)^2}{4}K(Y,K(X,Z)), \tag{10}$$

for all $X,Y,Z \in \Gamma(TM)$.

Remark 2. From Theorem 1, we can give other relations:

(1) The relation between $R^{(\alpha)}$ and $R^{(-\alpha)}$ is given by (see [1]).

$$R^{(\alpha)}(X,Y)Z = R^{(-\alpha)}(X,Y)Z + \alpha(R(X,Y)Z - R^*(X,Y)Z).$$

(2) The relation between $R^{(\alpha)}$, R and R^* is given by (see [1]).

$$R^{(\alpha)}(X,Y)Z = \frac{1+\alpha}{2}R(X,Y)Z + \frac{1-\alpha}{2}R^*(X,Y)Z - \frac{1-\alpha^2}{4}K(X,K(Y,Z)) + \frac{1-\alpha^2}{4}K(Y,K(X,Z))$$

Corollary 2. Let $\{e_i\}_{i=1}^m$ be a local orthonormal frame field on (M^m,g), for a statistical structure $\left(\nabla, \overset{*}{\nabla}, g\right)$, if we denote

$$Ricci^{(\alpha)}(X) = Tr_g R^{(\alpha)}(X,\cdot)\cdot = R^{(\alpha)}(X,e_i)e_i$$

and

$$Ricci^{(\beta)}(X) = Tr_g R^{(\beta)}(X,\cdot)\cdot = R^{(\beta)}(X,e_i)e_i,$$

for any $X \in \Gamma(TM)$, then the relation between $Ricci^{(\alpha)}(X)$ and $Ricci^{(\beta)}(X)$ is given by the following formula :

$$Ricci^{(\alpha)}(X) = Ricci^{(\beta)}(X) + \frac{(\beta-\alpha)^2}{4}K(X,E) + \frac{\beta-\alpha}{2}Tr_g\left(\nabla_X^{(\beta)}K\right)(\cdot,\cdot) - \frac{\beta-\alpha}{2}Tr_g\left(\nabla_\cdot^{(\beta)}K\right)(X,\cdot) - \frac{(\beta-\alpha)^2}{4}Tr_g K(K(X,\cdot),\cdot),$$

where

$$Tr_g\left(\nabla_X^{(\beta)}K\right)(\cdot,\cdot) = \left(\nabla_X^{(\beta)}K\right)(e_i,e_i),$$

$$Tr_g\left(\nabla_\cdot^{(\beta)}K\right)(X,\cdot) = \left(\nabla_{e_i}^{(\beta)}K\right)(X,e_i),$$

$$Tr_g K(K(X,\cdot),\cdot) = K(K(X,e_i),e_i),$$

and

$$E = Tr_g K = K(e_i,e_i).$$

In particular for $\beta \in \{-1,0,1\}$, we obtain,

$$\begin{aligned} Ricci^{(\alpha)}(X) &= \widehat{Ricci}(X) + \frac{\alpha^2}{4}K(X,E) - \frac{\alpha}{2}Tr_g\left(\widehat{\nabla}_X K\right)(\cdot,\cdot) \\ &\quad + \frac{\alpha}{2}Tr_g\left(\widehat{\nabla}.K\right)(X,\cdot) - \frac{\alpha^2}{4}Tr_g K(K(X,\cdot),\cdot), \\ Ricci^{(\alpha)}(X) &= Ricci(X) + \frac{(1-\alpha)^2}{4}K(X,E) + \frac{1-\alpha}{2}Tr_g(\nabla_X K)(\cdot,\cdot) \\ &\quad - \frac{1-\alpha}{2}Tr_g(\nabla.K)(X,\cdot) - \frac{(1-\alpha)^2}{4}Tr_g K(K(X,\cdot),\cdot) \end{aligned}$$

and

$$\begin{aligned} Ricci^{(\alpha)}(X) &= Ricci^*(X) + \frac{(1+\alpha)^2}{4}K(X,E) - \frac{1+\alpha}{2}Tr_g\left(\overset{*}{\nabla}_X K\right)(\cdot,\cdot) \\ &\quad + \frac{1+\alpha}{2}Tr_g\left(\overset{*}{\nabla}\cdot K\right)(X,\cdot) - \frac{(1+\alpha)^2}{4}Tr_g K(K(X,\cdot),\cdot) \end{aligned}$$

Example 1. Let $(\mathbb{R}^2,g)$ be a statistical manifold with Riemannian metric $g = dx^2 + dy^2$ *and* ∇ an affine connection defined by

$$\nabla_{e_1}e_1 = e_2, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = e_1$$

where $\left\{e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}\right\}$ is an orthonormal frame field. A simple calculation gives,

$$\overset{*}{\nabla}_{e_1}e_1 = -e_2, \quad \overset{*}{\nabla}_{e_2}e_2 = 0, \quad \overset{*}{\nabla}_{e_1}e_2 = \overset{*}{\nabla}_{e_2}e_1 = -e_1.$$

We deduce that,

$$K(e_1,e_1) = -2e_2, \quad K(e_2,e_2) = 0, \quad K(e_1,e_2) = K(e_2,e_1) = -2e_1,$$

then,

$$E = Tr_g K = K(e_1,e_1) + K(e_2,e_2) = -2e_2.$$

In this case, we have,

$$\nabla^{(\alpha)}_{e_1}e_1 = \alpha e_2, \quad \nabla^{(\alpha)}_{e_2}e_2 = 0, \quad \nabla^{(\alpha)}_{e_1}e_2 = \nabla^{(\alpha)}_{e_2}e_1 = \alpha e_1,$$

$$R^{(\alpha)}(e_1,e_2)e_2 = -\alpha^2 e_1, \quad R^{(\alpha)}(e_2,e_1)e_1 = -\alpha^2 e_2.$$

and

$$Ricci^{(\alpha)}(X) = -\alpha^2 X, \quad Ric^{(\alpha)}(X,Y) = -\alpha^2 g(X,Y), \quad S^{(\alpha)}_g = -2\alpha^2.$$

Then $(\mathbb{R}^2,\nabla^{(\alpha)},g)$ is a statistical manifold of constant curvature $-\alpha^2$ and it is a Hessian structure if and only if $\alpha = 0$.

Example 2. Let $\left(H^2 = \{(x,y) \in R^2, y \succ 0\}, g\right)$ be a statistical manifold with Riemannian metric $g = \frac{1}{y^2}\left(dx^2 + dy^2\right)$ and ∇ an affine connection defined by

$$\nabla_{e_1}e_1 = \nabla_{e_2}e_2 = 2e_2, \qquad \nabla_{e_1}e_2 = 0, \qquad \nabla_{e_2}e_1 = e_1,$$

where $\left\{e_1 = y\frac{\partial}{\partial x}, e_2 = y\frac{\partial}{\partial y}\right\}$ is an orthonormal frame field. A simple calculation gives

$$\overset{*}{\nabla}_{e_1}e_1 = 0, \qquad \overset{*}{\nabla}_{e_2}e_2 = -2e_2, \qquad \overset{*}{\nabla}_{e_1}e_2 = -2e_1, \qquad \overset{*}{\nabla}_{e_2}e_1 = -e_1.$$

Then,

$$\nabla^{(\alpha)}_{e_1}e_1 = (1+\alpha)e_2, \qquad \nabla^{(\alpha)}_{e_2}e_2 = 2\alpha e_2, \qquad \nabla^{(\alpha)}_{e_1}e_2 = -(1-\alpha)e_1, \qquad \nabla^{(\alpha)}_{e_2}e_1 = \alpha e_1.$$

We deduce that,

$$R^{(\alpha)}(e_1,e_2)e_2 = \left(\alpha^2-1\right)e_1, \qquad R^{(\alpha)}(e_2,e_1)e_1 = \left(\alpha^2-1\right)e_2,$$

it follows that,

$$Ricci^{(\alpha)}(X) = \left(\alpha^2-1\right)X, \qquad Ric^{(\alpha)}(X,Y) = \left(\alpha^2-1\right)g(X,Y), \qquad S^{(\alpha)}_g = 2\left(\alpha^2-1\right).$$

In this case, $\left(H^2, \nabla^{(\alpha)}, g\right)$ is a statistical manifold of constant curvature $\alpha^2 - 1$ and it is a Hessian structure if and only if $\alpha = \pm 1$.

3. The α-conformal equivalence

For a real number α, statistical manifolds (M^m, ∇, g) and $\left(M^m, \overline{\nabla}, \overline{g}\right)$ are said to be α-conformally equivalent if there exists a function γ on M such that the Riemannian metrics $\overline{g}$ and g and h are related by the following relation,

$$\overline{g}(X,Y) = e^{2\gamma}g(X,Y) \tag{11}$$

and the connection $\overline{\nabla}$ is given by,

$$\overline{\nabla}_X Y = \nabla_X Y + (1-\alpha)Y(\gamma)X + (1-\alpha)X(\gamma)Y - (1+\alpha)g(X,Y)grad\gamma, \tag{12}$$

for all $X, Y, Z \in \Gamma(TM)$. Using the fact that $\nabla_X Y = \widehat{\nabla}_X Y - \frac{1}{2}K(X,Y)$, we obtain,

$$\overline{\nabla}_X Y = \widehat{\nabla}_X Y + (1-\alpha)Y(\gamma)X + (1-\alpha)X(\gamma)Y - (1+\alpha)g(X,Y)grad\gamma - \frac{1}{2}K(X,Y). \tag{13}$$

Theorem 2.

$$\begin{aligned}\overline{R}(X,Y)Z =& \widehat{R}(X,Y)Z - (1+\alpha)g(Y,Z)\widehat{\nabla}_X grad\gamma + (1+\alpha)g(X,Z)\widehat{\nabla}_Y grad\gamma \\ &+ (1+\alpha)^2 g(Y,Z)X(\gamma)grad\gamma - (1+\alpha)^2 g(X,Z)Y(\gamma)grad\gamma + (1-\alpha)^2 Y(\gamma)Z(\gamma)X \\ &- (1-\alpha^2)g(Y,Z)|grad\gamma|^2 X - \frac{1-\alpha}{2}K(Y,Z)(\gamma)X - (1-\alpha)g\left(\widehat{\nabla}_Y grad\gamma, Z\right)X \\ &- (1-\alpha)^2 X(\gamma)Z(\gamma)Y + (1-\alpha)g\left(\widehat{\nabla}_X grad\gamma, Z\right)Y + (1-\alpha^2)g(X,Z)|grad\gamma|^2 Y \\ &+ \frac{1-\alpha}{2}K(X,Z)(\gamma)Y - \frac{1}{2}\left(\widehat{\nabla}_X K\right)(Y,Z) + \frac{1}{2}\left(\widehat{\nabla}_Y K\right)(X,Z) \\ &+ \frac{1+\alpha}{2}g(Y,Z)K(X,grad\gamma) - \frac{1+\alpha}{2}g(X,Z)K(Y,grad\gamma) \\ &+ \frac{1}{4}K(X,K(Y,Z)) - \frac{1}{4}K(Y,K(X,Z)).\end{aligned} \tag{14}$$

Proof of Theorem 2. By definition, we have,

$$\overline{R}(X,Y)Z=\overline{\nabla}_X\overline{\nabla}_Y Z-\overline{\nabla}_Y\overline{\nabla}_X Z-\overline{\nabla}_{[X,Y]}Z. \tag{15}$$

We will study the right side of this equation term by term. By (13), we obtain,

$$\overline{\nabla}_Y Z=\widehat{\nabla}_Y Z+(1-\alpha)Z(\gamma)Y+(1-\alpha)Y(\gamma)Z-(1+\alpha)g(Y,Z)grad\gamma-\frac{1}{2}K(Y,Z),$$

which gives us,

$$\begin{aligned}\overline{\nabla}_X\overline{\nabla}_Y Z &=\overline{\nabla}_X\widehat{\nabla}_Y Z+(1-\alpha)\overline{\nabla}_X Z(\gamma)Y+(1-\alpha)\overline{\nabla}_X Y(\gamma)Z\\ &\quad-(1+\alpha)\overline{\nabla}_X g(Y,Z)grad\gamma-\frac{1}{2}\overline{\nabla}_X K(Y,Z).\end{aligned}$$

Using Eqn (13), we deduce that,

$$\begin{aligned}\overline{\nabla}_X\widehat{\nabla}_Y Z &=\widehat{\nabla}_X\widehat{\nabla}_Y Z+(1-\alpha)\left(\widehat{\nabla}_Y Z\right)(\gamma)X+(1-\alpha)X(\gamma)\widehat{\nabla}_Y Z\\ &\quad-(1+\alpha)g\left(X,\widehat{\nabla}_Y Z\right)grad\gamma-\frac{1}{2}K\left(X,\widehat{\nabla}_Y Z\right),\end{aligned}$$

$$\begin{aligned}\overline{\nabla}_X Z(\gamma)Y &=Z(\gamma)\widehat{\nabla}_X Y+(1-\alpha)Y(\gamma)Z(\gamma)X+(1-\alpha)X(\gamma)Z(\gamma)Y\\ &\quad-(1+\alpha)g(X,Y)Z(\gamma)grad\gamma-\frac{1}{2}Z(\gamma)K(X,Y)+g\left(\widehat{\nabla}_X grad\gamma,Z\right)Y\\ &\quad+g\left(grad\gamma,\widehat{\nabla}_X Z\right)Y,\end{aligned}$$

$$\begin{aligned}\overline{\nabla}_X Y(\gamma)Z &=Y(\gamma)\widehat{\nabla}_X Z+(1-\alpha)Y(\gamma)Z(\gamma)X+(1-\alpha)X(\gamma)Y(\gamma)Z\\ &\quad-(1+\alpha)g(X,Z)Y(\gamma)grad\gamma-\frac{1}{2}Y(\gamma)K(X,Z)+g\left(\widehat{\nabla}_X grad\gamma,Y\right)Z\\ &\quad+g\left(grad\gamma,\widehat{\nabla}_X Y\right)Z,\end{aligned}$$

$$\begin{aligned}\overline{\nabla}_X g(Y,Z)grad\gamma &=g(Y,Z)\widehat{\nabla}_X grad\gamma+(1-\alpha)g(Y,Z)|grad\gamma|^2X-2\alpha g(Y,Z)X(\gamma)grad\gamma\\ &\quad-\frac{1}{2}g(Y,Z)K(X,grad\gamma)+g\left(\widehat{\nabla}_X Y,Z\right)grad\gamma+g\left(Y,\widehat{\nabla}_X Z\right)grad\gamma\end{aligned}$$

and

$$\begin{aligned}\overline{\nabla}_X K(Y,Z) &=\widehat{\nabla}_X K(Y,Z)+(1-\alpha)K(Y,Z)(\gamma)X+(1-\alpha)X(\gamma)K(Y,Z)\\ &\quad-(1+\alpha)g(X,K(Y,Z))grad\gamma-\frac{1}{2}K(X,K(Y,Z)).\end{aligned}$$

It follows that,

$$
\begin{aligned}
\overline{\nabla}_X\overline{\nabla}_Y Z =& \widehat{\nabla}_X\widehat{\nabla}_Y Z-(1+\alpha)g(Y,Z)\widehat{\nabla}_X grad\gamma+\frac{1+\alpha}{2}g(X,K(Y,Z))grad\gamma\\
&-(1-\alpha^2)g(X,Y)Z(\gamma)grad\gamma-(1-\alpha^2)g(X,Z)Y(\gamma)grad\gamma\\
&+2\alpha(1+\alpha)g(Y,Z)X(\gamma)grad\gamma-(1+\alpha)g\left(X,\widehat{\nabla}_Y Z\right)grad\gamma\\
&-(1+\alpha)g\left(\widehat{\nabla}_X Y,Z\right)grad\gamma-(1+\alpha)g\left(Y,\widehat{\nabla}_X Z\right)grad\gamma\\
&-(1-\alpha^2)g(Y,Z)|grad\gamma|^2X+2(1-\alpha)^2Y(\gamma)Z(\gamma)X+(1-\alpha)\left(\widehat{\nabla}_Y Z\right)(\gamma)X\\
&+(1-\alpha)^2X(\gamma)Z(\gamma)Y+(1-\alpha)g\left(\widehat{\nabla}_X grad\gamma,Z\right)Y+(1-\alpha)g\left(grad\gamma,\widehat{\nabla}_X Z\right)Y\\
&+(1-\alpha)^2X(\gamma)Y(\gamma)Z+(1-\alpha)g\left(\widehat{\nabla}_X grad\gamma,Y\right)Z+(1-\alpha)g\left(grad\gamma,\widehat{\nabla}_X Y\right)Z\\
&+(1-\alpha)X(\gamma)\widehat{\nabla}_Y Z-\frac{1}{2}K\left(X,\widehat{\nabla}_Y Z\right)+(1-\alpha)Z(\gamma)\widehat{\nabla}_X Y+(1-\alpha)Y(\gamma)\widehat{\nabla}_X Z\\
&-\frac{1-\alpha}{2}Y(\gamma)K(X,Z)-\frac{1-\alpha}{2}Z(\gamma)K(X,Y)+\frac{1+\alpha}{2}g(Y,Z)K(X,grad\gamma)\\
&-\frac{1}{2}\widehat{\nabla}_X K(Y,Z)-\frac{1-\alpha}{2}K(Y,Z)(\gamma)X-\frac{1-\alpha}{2}X(\gamma)K(Y,Z)+\frac{1}{4}K(X,K(Y,Z)).
\end{aligned}
\tag{16}
$$

A similar calculation gives us,

$$
\begin{aligned}
\overline{\nabla}_Y\overline{\nabla}_X Z =& \widehat{\nabla}_Y\widehat{\nabla}_X Z-(1+\alpha)g(X,Z)\widehat{\nabla}_Y grad\gamma+\frac{1+\alpha}{2}g(Y,K(X,Z))grad\gamma\\
&-(1-\alpha^2)g(X,Y)Z(\gamma)grad\gamma-(1-\alpha^2)g(Y,Z)X(\gamma)grad\gamma\\
&+2\alpha(1+\alpha)g(X,Z)Y(\gamma)grad\gamma-(1+\alpha)g\left(Y,\widehat{\nabla}_X Z\right)grad\gamma\\
&-(1+\alpha)g\left(\widehat{\nabla}_Y X,Z\right)grad\gamma-(1+\alpha)g\left(X,\widehat{\nabla}_Y Z\right)grad\gamma\\
&-(1-\alpha^2)g(X,Z)|grad\gamma|^2Y+2(1-\alpha)^2X(\gamma)Z(\gamma)Y+(1-\alpha)\left(\widehat{\nabla}_X Z\right)(\gamma)Y\\
&+(1-\alpha)^2Y(\gamma)Z(\gamma)X+(1-\alpha)g\left(\widehat{\nabla}_Y grad\gamma,Z\right)X+(1-\alpha)g\left(grad\gamma,\widehat{\nabla}_Y Z\right)X\\
&+(1-\alpha)^2X(\gamma)Y(\gamma)Z+(1-\alpha)g\left(\widehat{\nabla}_X grad\gamma,Y\right)Z+(1-\alpha)g\left(grad\gamma,\widehat{\nabla}_Y X\right)Z\\
&+(1-\alpha)Y(\gamma)\widehat{\nabla}_X Z-\frac{1}{2}K\left(Y,\widehat{\nabla}_X Z\right)+(1-\alpha)Z(\gamma)\widehat{\nabla}_Y X+(1-\alpha)X(\gamma)\widehat{\nabla}_Y Z\\
&-\frac{1-\alpha}{2}X(\gamma)K(Y,Z)-\frac{1-\alpha}{2}Z(\gamma)K(X,Y)+\frac{1+\alpha}{2}g(X,Z)K(Y,grad\gamma)\\
&-\frac{1}{2}\widehat{\nabla}_Y K(X,Z)-\frac{1-\alpha}{2}K(X,Z)(\gamma)Y-\frac{1-\alpha}{2}Y(\gamma)K(X,Z)+\frac{1}{4}K(Y,K(X,Z)).
\end{aligned}
\tag{17}
$$

Finally, it is easy to see that,

$$\begin{aligned}\overline{\nabla}_{[X,Y]}Z &= \widehat{\nabla}_{[X,Y]}Z-(1+\alpha)g\left(\widehat{\nabla}_XY,Z\right)grad\gamma+(1+\alpha)g\left(\widehat{\nabla}_YX,Z\right)grad\gamma\\ &+(1-\alpha)\left(\widehat{\nabla}_XY\right)(\gamma)Z-(1-\alpha)\left(\widehat{\nabla}_YX\right)(\gamma)Z+(1-\alpha)Z(\gamma)\widehat{\nabla}_XY\\ &-(1-\alpha)Z(\gamma)\widehat{\nabla}_YX-\frac{1}{2}K\left(\widehat{\nabla}_XY,Z\right)+\frac{1}{2}K\left(\widehat{\nabla}_YX,Z\right).\end{aligned} \tag{18}$$

By replacing (16), (17) and (18) in (15), we conclude that,

$$\begin{aligned}\overline{R}(X,Y)Z=&\widehat{R}(X,Y)Z-(1+\alpha)g(Y,Z)\widehat{\nabla}_Xgrad\gamma+(1+\alpha)g(X,Z)\widehat{\nabla}_Ygrad\gamma\\ &+(1+\alpha)^2g(Y,Z)X(\gamma)grad\gamma-(1+\alpha)^2g(X,Z)Y(\gamma)grad\gamma+(1-\alpha)^2Y(\gamma)Z(\gamma)X\\ &-(1-\alpha^2)g(Y,Z)|grad\gamma|^2X-\frac{1-\alpha}{2}K(Y,Z)(\gamma)X-(1-\alpha)g\left(\widehat{\nabla}_Ygrad\gamma,Z\right)X\\ &-(1-\alpha)^2X(\gamma)Z(\gamma)Y+(1-\alpha)g\left(\widehat{\nabla}_Xgrad\gamma,Z\right)Y+(1-\alpha^2)g(X,Z)|grad\gamma|^2Y\\ &+\frac{1-\alpha}{2}K(X,Z)(\gamma)Y-\frac{1}{2}\left(\widehat{\nabla}_XK\right)(Y,Z)+\frac{1}{2}\left(\widehat{\nabla}_YK\right)(X,Z)\\ &+\frac{1+\alpha}{2}g(Y,Z)K(X,grad\gamma)-\frac{1+\alpha}{2}g(X,Z)K(Y,grad\gamma)+\frac{1}{4}K(X,K(Y,Z))\\ &-\frac{1}{4}K(Y,K(X,Z)).\end{aligned}$$

The same method of calculation used in Theorem 2 and the following equations,

$$\widehat{\nabla}_XY=\nabla_XY+\frac{1}{2}K(X,Y),$$

$$\begin{aligned}\widehat{R}(X,Y)Z &= R(X,Y)Z+\frac{1}{2}(\nabla_XK)(Y,Z)-\frac{1}{2}(\nabla_YK)(X,Z)\\ &+\frac{1}{4}K(X,K(Y,Z))-\frac{1}{4}K(Y,K(X,Z))\end{aligned}$$

gives us the following theorem

Theorem 3.

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z-(1+\alpha)g(Y,Z)\nabla_Xgrad\gamma+(1+\alpha)g(X,Z)\nabla_Ygrad\gamma\\ &-(1+\alpha)^2g(X,Z)Y(\gamma)grad\gamma+(1+\alpha)^2g(Y,Z)X(\gamma)grad\gamma\\ &-(1-\alpha^2)g(Y,Z)|grad\gamma|^2X+(1-\alpha^2)g(X,Z)|grad\gamma|^2Y\\ &-(1-\alpha)g(\nabla_Ygrad\gamma,Z)X+(1-\alpha)g(\nabla_Xgrad\gamma,Z)Y\\ &+(1-\alpha)^2Y(\gamma)Z(\gamma)X-(1-\alpha)^2X(\gamma)Z(\gamma)Y\\ &-(1-\alpha)g(K(Y,Z),grad\gamma)X+(1-\alpha)g(K(X,Z),grad\gamma)Y\end{aligned} \tag{19}$$

Corollary 3. Let us choose $\{e_i\}_{1\le i\le m}$ to be an orthonormal frame on (M^m,∇,g), an orthonormal frame on $\left(M^m,\overline{\nabla},\overline{g}=e^{2\gamma}g\right)$ is given by $\{\overline{e}_i=e^{-\gamma}e_i\}_{1\le i\le m}$. For any $X,Y\in\Gamma(TM)$, we define

$$Ricci(X) = Tr_g R(X,\cdot)\cdot = R(X,e_i)e_i, \quad \overline{Ricci}(X) = Tr_{\overline{g}}\overline{R}(X,\cdot)\cdot = \overline{R}(X,\overline{e_i})\overline{e_i},$$
$$Ric(X,Y) = g(Ricci(X),Y), \quad \overline{Ric}(X,Y) = \overline{g}\big(\overline{Ricci}(X),Y\big)$$

and

$$S_g = Tr_g Ric = Ric(e_i,e_i), \quad S_{\overline{g}} = Tr_{\overline{g}}\overline{Ric} = \overline{Ric}(\overline{e_i},\overline{e_i}).$$

Using Theorem 3, we obtain the following relations,

$$\begin{aligned}\overline{Ricci}(X) &= e^{-2\gamma}Ricci(X) + ((m-2)\alpha^2 + 2m\alpha + m - 2)e^{-2\gamma}X(\gamma)grad\gamma \\ &\quad - (m\alpha + m - 2)e^{-2\gamma}\nabla_X grad\gamma + (m\alpha^2 - 2\alpha - m + 2)e^{-2\gamma}|grad\gamma|^2 X \\ &\quad - (1-\alpha)e^{-2\gamma}\left(\widehat{\Delta}\gamma\right)X - \frac{(1-\alpha)}{2}e^{-2\gamma}E(\gamma)X + (1-\alpha)e^{-2\gamma}K(X,grad\gamma),\end{aligned}$$

$$\begin{aligned}\overline{Ric}(X,Y) &= Ric(X,Y) + ((m-2)\alpha^2 + 2m\alpha + m - 2)X(\gamma)Y(\gamma) \\ &\quad + (m\alpha^2 - 2\alpha - m + 2)|grad\gamma|^2 g(X,Y) - (1-\alpha)\left(\widehat{\Delta}\gamma\right)g(X,Y) \\ &\quad - (m\alpha + m - 2)g(\nabla_X grad\gamma, Y) - \frac{(1-\alpha)}{2}E(\gamma)g(X,Y) \\ &\quad + (1-\alpha)g(K(X,Y),grad\gamma)\end{aligned}$$

and

$$\begin{aligned}S_{\overline{g}} &= e^{-2\gamma}S_g + (m-1)((m+2)\alpha^2 - m + 2)e^{-2\gamma}|grad\gamma|^2 \\ &\quad - 2(m-1)e^{-2\gamma}\left(\widehat{\Delta}\gamma\right) + (m-1)\alpha e^{-2\gamma}E(\gamma)\end{aligned}$$

Corollary 4. Theorem 3 and Corollary 3 gives us two particular cases:

(1) If $\alpha = 1$, we obtain,

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z - 2g(Y,Z)\nabla_X grad\gamma + 2g(X,Z)\nabla_Y grad\gamma \\ &\quad - 4g(X,Z)Y(\gamma)grad\gamma + 4g(Y,Z)X(\gamma)grad\gamma,\end{aligned}$$

$$\overline{Ricci}(X) = e^{-2\gamma}(Ricci(X) + 4(m-1)X(\gamma)grad\gamma - 2(m-1)\nabla_X grad\gamma),$$

$$\overline{Ric}(X,Y) = Ric(X,Y) + 4(m-1)X(\gamma)Y(\gamma) - 2(m-1)g(\nabla_X grad\gamma, Y)$$

and

$$S_{\overline{g}} = e^{-2\gamma}\left(S_g + 4(m-1)|grad\gamma|^2 - 2(m-1)\left(\widehat{\Delta}\gamma\right) + (m-1)E(\gamma)\right).$$

(2) If $\alpha = -1$, we obtain,

$$\begin{aligned}\overline{R}(X,Y)Z &= R(X,Y)Z - 2g(\nabla_Y grad\gamma, Z)X + 2g(\nabla_X grad\gamma, Z)Y \\ &\quad + 4Y(\gamma)Z(\gamma)X - 4X(\gamma)Z(\gamma)Y - 2g(K(Y,Z),grad\gamma)X \\ &\quad + 2g(K(X,Z),grad\gamma)Y,\end{aligned}$$

$$\begin{aligned}\overline{Ricci}(X) &= e^{-2\gamma}Ricci(X) - 4e^{-2\gamma}X(\gamma)grad\gamma + 2e^{-2\gamma}\nabla_X grad\gamma \\ &\quad + 4e^{-2\gamma}|grad\gamma|^2 X - 2e^{-2\gamma}\left(\widehat{\Delta}\gamma\right)X - e^{-2\gamma}E(\gamma)X \\ &\quad + 2e^{-2\gamma}K(X,grad\gamma),\end{aligned}$$

$$\begin{aligned}\overline{Ric}(X,Y) &= Ric(X,Y) - 4X(\gamma)Y(\gamma) + 2g(\nabla_X grad\gamma, Y)\\ &\quad + 4|grad\gamma|^2 g(X,Y) - 2\left(\widehat{\Delta}\gamma\right)g(X,Y)\\ &\quad - E(\gamma)g(X,Y) + 2g(K(X,Y), grad\gamma)\end{aligned}$$

and

$$S_{\overline{g}} = e^{-2\gamma}S_g + (m-1)e^{-2\gamma}\left(4|grad\gamma|^2 - 2\left(\widehat{\Delta}\gamma\right) + E(\gamma)\right),$$

where

$$\widehat{\Delta}\gamma = g\left(\widehat{\nabla}_{e_i}grad\gamma, e_i\right) = e_i(e_i(\gamma)) - \left(\widehat{\nabla}_{e_i}e_i\right)(\gamma).$$

Example 3. Let $(\mathbb{R}^2, g)$ be a statistical manifold with Riemannian metric $g = dx^2 + dy^2$ *and* ∇ an affine connection defined by

$$\nabla_{e_1}e_1 = e_2, \qquad \nabla_{e_2}e_2 = 0, \qquad \nabla_{e_1}e_2 = \nabla_{e_2}e_1 = e_1$$

where $\left\{e_1 = \frac{\partial}{\partial x}, e_2 = \frac{\partial}{\partial y}\right\}$ is an orthonormal frame field. Then $(\mathbb{R}^2, \nabla, g)$ is a statistical manifold of constant curvature -1 and $S_g = -2$. We want to determine γ such that $S_{\overline{g}} = 0$. By Corollary\enleadertwodots , we deduce that $S_{\overline{g}}$ vanish if and only if

$$2\left(\widehat{\Delta}\gamma\right) - \alpha E(\gamma) - 4\alpha^2|grad\gamma|^2 + 2 = 0.$$

To solve this equation, we will present two cases :

(1) If we assume that γ depends only on the variable x, then $S_{\overline{g}}$ vanish if and only if.

$$\gamma'' - 2\alpha^2\left(\gamma'\right)^2 + 1 = 0.$$

Note that if $\alpha = 0$, the solution of this last equation is,

$$\gamma(x) = -\frac{1}{2}x^2 + ax + b.$$

In the case where $\alpha \neq 0$, a particular solution is given by $\gamma(x) = \frac{1}{\alpha\sqrt{2}}x + b.$

(2) If the function γ depends only on the variable y, we conclude that $S_{\overline{g}} = 0$ if and only if,

$$\gamma'' + \alpha\gamma' - 2\alpha^2\left(\gamma'\right)^2 + 1 = 0.$$

Using the same method, if $\alpha = 0$, the solution obtained is,

$$\gamma(y) = -\frac{1}{2}y^2 + ay + b$$

and if we take $\alpha \neq 0$, a particular solution is $\gamma(y) = \frac{1}{\alpha}y + b.$

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