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On the geometry of the tangent bundle with gradient Sasaki metric

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Abstract

Purpose – Let (M, g) be a n-dimensional smooth Riemannian manifold. In the present paper, the authors introduce a new class of natural metrics denoted by g^f and called gradient Sasaki metric on the tangent bundle TM. The authors calculate its Levi-Civita connection and Riemannian curvature tensor. The authors study the geometry of (TM, g^f) and several important results are obtained on curvature, scalar and sectional curvatures.

Design/methodology/approach – In this paper the authors introduce a new class of natural metrics called gradient Sasaki metric on tangent bundle.

Findings – The authors calculate its Levi-Civita connection and Riemannian curvature tensor. The authors study the geometry of (TM, g^f) and several important results are obtained on curvature scalar and sectional curvatures.

Originality/value – The authors calculate its Levi-Civita connection and Riemannian curvature tensor. The authors study the geometry of (TM, g^f) and several important results are obtained on curvature scalar and sectional curvatures.

Keywords Horizontal lift, Vertical lift, Gradient Sasaki metric, Sectional curvatures

Paper type Research paper

1. Introduction

We recall some basic facts about the geometry of the tangent bundle. In the present paper, we denote by $\Gamma(\mathrm{TM})$ the space of all vector fields of a Riemannian manifold (M, g). Let (M, g) be an n-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle.

A local chart $(U, x^i)_{i=1\ldots n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1\ldots n}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by

$$\mathcal{V}_{(x,u)} = \ker(d\pi_{(x,u)}) = \left\{ a^i \frac{\partial}{\partial y^i} |(x, u);\ a^i \in \mathbb{R} \right\}$$

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$$\mathcal{H}_{(x,u)} = \left\{ a^i \frac{\partial}{\partial x^i}_{|(x,u)} - a^i y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}_{|(x,u)}; a^i \in \mathbb{R} \right\}$$

where $(x, u) \in \mathrm{TM}$, such that $T_{(x,u)}\mathrm{TM} = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \left\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \right\} \tag{1.1}$$

For consequences, we have $\left(\frac{\partial}{\partial x^i}\right)^H = \frac{\delta}{\delta x^i}$ and $\left(\frac{\partial}{\partial x^i}\right)^V = \frac{\partial}{\partial y^i}$, then $\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right)_{i=1..n}$ is a local adapted frame in TTM.

The geometry of tangent bundle of a Riemannian manifold (M, g) is very important in many areas of mathematics and physics. In recent years, a lot of studies about their local or global geometric properties have been published in the literature. When the authors studied this topic, they used different metrics which are called natural metrics on the tangent bundle. First, the geometry of a tangent bundle has been studied by using a new metric g^s, which is called Sasaki metric, with the aid of a Riemannian metric g on a differential manifold M in 1958 by Sasaki [1]. It is uniquely determined by

$$\begin{aligned} g^s\left(X^H, Y^H\right) &= g(X, Y) \circ \pi \\ g^s\left(X^H, Y^V\right) &= 0 \\ g^s\left(X^V, Y^V\right) &= g(X, Y) \circ \pi \end{aligned} \tag{1.2}$$

for all vector fields X and Y on M. More intuitively, the metric g^s is constructed in such a way that the vertical and horizontal subbundles are orthogonal and the bundle map $\pi : (\mathrm{TM}, g^s) \to (M, g)$ is a Riemannian submersion.

After that, the tangent bundle could be split to its horizontal and vertical subbundles with the aid of Levi-Civita connection ∇ on (M, g). Later, the Lie bracket of the tangent bundle TM, the Levi-Civita connection ∇^s on TM and its Riemannian curvature tensor R^s have been obtained in Refs. [2, 3]. Furthermore, the explicit formulas of another natural metric g_{CG}, which is called Cheeger-Gromoll metric, on the tangent bundle TM of a Riemannian manifold (M, g). It is uniquely determined by

$$\begin{aligned} g_{CG}\left(X^H, Y^H\right) &= g(X, Y) \circ \pi \\ g_{CG}\left(X^H, Y^V\right) &= 0 \\ g_{CG}\left(X^V, Y^V\right) &= \frac{1}{\alpha}\{g(X, Y) + g(X, u)g(Y, u)\} \circ \pi \end{aligned} \tag{1.3}$$

where $X, Y \in \Gamma(\mathrm{TM})$, $(x, u) \in \mathrm{TM}$, $\alpha = 1 + g_x(u, u)$. This metric has been given by Musso and Tricerri in Ref. [4], using Cheeger and Gromoll's study [5]. The Levi-Civita connection ∇^{CG} and the Riemannian curvature tensor R^{CG} of $(\mathrm{TM}, g_{\mathrm{CG}})$ have been obtained in Refs. [6, 7], respectively. The sectional curvatures and the scalar curvature of this metric have been obtained in Refs. [8–16]. These results are completed in 2002 by S. Gudmundson and E. Kappos in Ref. [6]. They have also shown that the scalar curvature of the Cheeger-Gromoll metric is never constant if the metric on the base manifold has constant sectional curvature. Furthermore, in Ref. [17] M.T.K. Abbassi, M. Sarih have proved that TM with the Cheeger-Gromoll metric is never a space of constant sectional curvature. A more general metric is given by M. Anastasiei in Ref. [18] which generalizes both of the two metrics mentioned above in the following sense: it preserves the orthogonality of the two distributions, on the horizontal distribution it is the same as on the base manifold, and finally the Sasaki and the Cheeger-Gromoll metric can be obtained as particular cases of this metric. A compatible almost complex structure is also introduced and hence TM becomes a locally conformal almost Käherian manifold. V.Oproiu and his collaborators constructed a family of Riemannian metrics on the tangent bundles of Riemannian manifolds which possess interesting geometric properties (see Refs. [19, 20]). In particular, the scalar curvature of TM can be constant also for a non-flat base manifold with constant sectional curvature. Then M.T.K. Abbassi and M. Sarih proved in Ref. [21] that the considered metrics by Oproiu form a particular subclass of the so-called g-natural metrics on the tangent bundle. Recently, the geometry of the tangent bundles with Cheeger-Gromoll metric has been studied by many mathematicians (see Refs. [17, 22, 23] and etc).

Zayatuev in [24] introduced a Riemannian metric on TM given by

$$\begin{aligned} g_f^s\left(X^H, Y^H\right) &= f(p)g_p(X, Y) \\ g_f^s\left(X^H, Y^V\right) &= 0 \\ g_f^s\left(X^V, Y^V\right) &= g_p(X, Y) \end{aligned} \tag{1.4}$$

for all vector fields X and Y on (M, g), where f is strictly positive smooth function on (M, g). In Ref. [25] J. Wang, Y. Wang called g_f^s the rescaled Sasaki metric and studied the geometry of TM endowed with g_f^s.

H. M. Dida, F. Hathout in Ref. [26], we define a new class of naturally metric on TM given by

$$\begin{aligned} G^f_{(p,u)}\left(X^H, Y^H\right) &= g_p(X, Y) \\ G^f_{(p,u)}\left(X^H, Y^V\right) &= 0 \\ G^f_{(p,u)}\left(X^V, Y^V\right) &= f(p)g_p(X, Y) \end{aligned} \tag{1.5}$$

for some strictly positive smooth function f in (M, g) and any vector fields X and Y on M. We call G^f vertical rescaled metric.

L. Belarbi, H. El Hendi in Ref. [27], we define a new class of naturally metric on TM given by

$$\begin{aligned} G^{f,h}_{(p,u)}\left(X^H, Y^H\right) &= f(p)g_p(X, Y) \\ G^{f,h}_{(p,u)}\left(X^V, Y^H\right) &= 0 \\ G^{f,h}_{(p,u)}\left(X^V, Y^V\right) &= h(p)g_p(X, Y) \end{aligned} \tag{1.6}$$

where f, h be strictly positive smooth functions on M and any vector fields X and Y on M. For $h = 1$ the metric $G^{f,h}$ is exactly the rescaled Sasaki metric. If $f = 1$, the metric $G^{f,h}$ is exactly the vertical rescaled metric. We call $G^{f,h}$ the twisted Sasaki metric.

Motivated by the above studies, we define a new class of naturally metric on TM given by

$$\begin{aligned} g^f\left(X^H, Y^H\right)_{(x,u)} &= g_x(X, Y) \\ g^f\left(X^V, Y^H\right)_{(x,u)} &= 0 \\ g^f\left(X^V, Y^V\right)_{(x,u)} &= g_x(X, Y) + X_x(f)Y_x(f) \end{aligned} \tag{1.7}$$

where f be strictly positive smooth functions on M and any vector fields X and Y on M. If f is constant the metric g^f is exactly the Sasaki metric.

In this paper, we introduce the gradient Sasaki metric on the tangent bundle TM as a new natural metric non-rigid on TM. First we investigate the geometry of the gradient Sasaki metric and we characterize the sectional curvature (Proposition 2.1) and the scalar curvature (Proposition 2.2).

2. Gradient Sasaki metric

Definition 2.1. Let (M, g) be a Riemannian manifold and $f : M \to [0, +\infty]$. then the gradient Sasaki metric g^f on the tangent bundle TM of M is given by

$$g^f\left(X^H, Y^H\right)_{(x,u)} = g_x(X, Y)$$

$$g^f\left(X^V, Y^H\right)_{(x,u)} = 0$$

$$g^f\left(X^V, Y^V\right)_{(x,u)} = g_x(X, Y) + X_x(f)Y_x(f)$$

for all vector fields $X, Y \in \Gamma(\mathrm{TM})$, $(x, u) \in \mathrm{TM}$.

Remark 2.1.

(1) If f is constant, then g^f is the Sasaki metric.

(2) $g^f(X^H, (\mathrm{gradf})^H) = g(X, \mathrm{gradf}) = X(f)$

(3) $g^f(X^V, (\mathrm{grad}f)^V) = (1 + \|\mathrm{grad}f\|^2)X(f) = \alpha X(f)$, where $\alpha = 1 + \|\mathrm{grad}f\|^2$.

(4) $g^f(X^V, Y^V) - g^f(X^H, Y^H) = X(f)Y(f)$, where $X, Y \in \Gamma(\mathrm{TM})$.

2.1 Levi-Civita connection of g^f

Lemma 2.1. Let (M, g) be a Riemannian manifold and ∇ (resp ∇^f) denote the Levi-Civita connection of (M, g) (resp(TM, g^f)), then we have:

(1) $g^f(\nabla^f_{X^H} Y^H, Z^H) = g^f((\nabla_X Y)^H, Z^H)$

(2) $g^f(\nabla^f_{X^H} Y^H, Z^V) = -\frac{1}{2} g^f((R(X, Y)u)^V, Z^V)$

(3) $g^f(\nabla^f_{X^H} Y^V, Z^H) = \frac{1}{2} g^f((R(u, Y)X)^H + Y(f)(R(u, \mathrm{grad}f)X)^H, Z^H)$

(4) $g^f(\nabla^f_{X^H} Y^V, Z^V) = g^f((\nabla_X Y)^V, Z^V) + \frac{1}{2} Y(f) g^f((\nabla_X \mathrm{grad}f)^V, Z^V)$
$+ \frac{1}{2\alpha}[g(Y, \nabla_X \mathrm{grad}f) - \frac{1}{2} X(\alpha) Y(f)] g^f((\mathrm{grad}f)^V, Z^V)$

(5) $g^f(\nabla^f_{X^V} Y^H, Z^H) = \frac{1}{2} g^f((R(u, X)Y)^H + X(f)(R(u, \mathrm{grad}f)Y)^H, Z^H)$

(6) $g^f(\nabla^f_{X^V} Y^H, Z^V) = \frac{1}{2} X(f) g^f((\nabla_Y \mathrm{grad}f)^V, Z^V) + \frac{1}{2\alpha}[g(X, \nabla_Y \mathrm{grad}f)$
$- \frac{1}{2} Y(\alpha) X(f)] \; g^f((\mathrm{grad}f)^V, Z^V)$

(7) $g^f(\nabla^f_{X^V} Y^V, Z^H) = -\frac{1}{2} g^f(X(f)(\nabla_Y \mathrm{grad}f)^H + Y(f)(\nabla_X \mathrm{grad}f)^H, Z^H)$

(8) $g^f(\nabla^f_{X^V} Y^V, Z^V) = 0$

Using Lemma 2.1, we have the theorem

Theorem 2.1. Let (M, g) be a Riemannian manifold and ∇^f be the Levi-Civita connection of the tangent bundle (TM, g^f). Then, we have

$$\left(\nabla^f_{X^H}Y^H\right)_p = (\nabla_X Y)^H_p - \frac{1}{2}(R(X,\, Y)u)^V_p$$

$$\left(\nabla^f_{X^H}Y^V\right)_p = \frac{1}{2}(R(u,\, Y)X)^h_p + \frac{1}{2}Y(f)(R(u,\, \mathrm{grad}f)X)^H + \frac{1}{2}Y(f)(\nabla_X \mathrm{grad}f)^V + (\nabla_X Y)^V$$
$$+ \frac{1}{2\alpha}\left[g(Y,\, \nabla_X \mathrm{grad}f) - \frac{1}{2}X(\alpha)Y(f)\right](\mathrm{grad}f)^V$$

$$\left(\nabla^f_{X^V}Y^H\right)_p = \frac{1}{2}(R(u,\, X)Y)^h_p + \frac{1}{2}X(f)(R(u,\, \mathrm{grad}f)Y)^H + \frac{1}{2}X(f)(\nabla_Y \mathrm{grad}f)^V$$
$$+ \frac{1}{2\alpha}\left[g(X,\, \nabla_Y \mathrm{grad}f) - \frac{1}{2}Y(\alpha)X(f)\right](\mathrm{grad}f)^V$$

$$\left(\nabla^f_{X^V}Y^V\right)_p = -\frac{1}{2}X(f)(\nabla_Y \mathrm{grad}f)^H - \frac{1}{2}Y(f)(\nabla_X \mathrm{grad}f)^H$$

for all vector fields $X,\, Y \in \Gamma(TM)$, $p = (x,\, u) \in TM$.

2.2 Curvature tensor of gradient Sasaki metric

Using Theorem 2.1 and the formula of curvature, we have

Theorem 2.2. Let $(M,\, g)$ be a Riemannian manifold and $(TM,\, g^f)$ its tangent bundle equipped with the gradient Sasaki metric. If R (resp R^f) denote the Riemann curvature tensor of M (resp TM), then we have the following formulas

(1)
$$\begin{aligned}
R^f_p(X^H,\, Y^H)Z^H &= (R(X,\, Y)Z^H_p + \frac{1}{2}(R(u,\, R(X,\, Y)u)Z)^H_p \\
&+ \frac{1}{4}(R(u,\, R(X,\, Z)u)Y^H_p - \frac{1}{4}(R(u,\, R(Y,\, Z)u)X)^H_p \\
&- \frac{1}{4}g_x(R(Y,\, Z)u,\, \mathrm{grad}f)(R(u,\, \mathrm{grad}f)X)^H_p \\
&+ \frac{1}{4}g_x(R(X,\, Z)u,\, \mathrm{grad}f)(R(u,\, \mathrm{grad}f)Y)^H_p \\
&+ \frac{1}{2}g_x(R(X,\, Y)u,\, \mathrm{grad}f)(R(u,\, \mathrm{grad}f)Z)^H_p \\
&+ \frac{1}{2}((\nabla_Z R)(X,\, Y)u)^V_p + \frac{1}{2}g_x(R(X,\, Y)u,\, \mathrm{grad}f)(\nabla_Z \mathrm{grad}f)^V_p \\
&+ \frac{1}{4}g_x(R(X,\, Z)u,\, \mathrm{grad}f)(\nabla_Y \mathrm{grad}f)^V_p \\
&- \frac{1}{4}g_x(R(Y,\, Z)u,\, \mathrm{grad}f)(\nabla_X \mathrm{grad}f)^V_p \\
&+ \frac{1}{4\alpha}[g_x(R(X,\, Z)u,\, \nabla_Y \mathrm{grad}f) - g_x(R(Y,\, Z)u,\, \nabla_X \mathrm{grad}f) \\
&+ 2g_x(R(X,\, Y)u,\, \nabla_Z \mathrm{grad}f)](\mathrm{grad}f)^V_p \\
&+ \frac{1}{8\alpha}[X_x(\alpha)g_x(R(Y,\, Z)u,\, \mathrm{grad}f) - Y_x(\alpha)g_x(R(X,\, Z)u,\, \mathrm{grad}f) \\
&- 2Z_x(\alpha)g_x(R(X,\, Y)u,\, \mathrm{grad}f)](\mathrm{grad}f)^V_p
\end{aligned}$$

(2)
$$R_p^f(X^H, Y^V)Z^V = -\frac{1}{2}Y_x(f)(\nabla_X\nabla_Z\mathrm{grad}f)_p^H - \frac{1}{4}Y_x(f)(R(u, \mathrm{grad}f)R(u, Z)X)_p^H$$
$$-\frac{1}{2}Z_x(f)(\nabla_X\nabla_Y\mathrm{grad}f)_p^H - \frac{1}{4}(R(u, Y)R(u, Z)X)_p^H$$
$$-\frac{1}{2}Z_x(f)(R(Y, \mathrm{grad}f)X)_p^H - \frac{1}{4}Z_x(f)(R(u, Y)R(u, \mathrm{grad}f)X)_p^H$$
$$-\frac{1}{4}Y_x(f)Z_x(f)(R(u, \mathrm{grad}f)(R(u, \mathrm{grad}f)X)_p^H - \frac{1}{2}(R(Y, Z)X)_p^H$$
$$+\frac{1}{4}Y_x(f)Z_x(f)\left(\nabla_{(\nabla_X\mathrm{grad}f)}\mathrm{grad}f\right)_p^H + \frac{1}{2}Z_x(f)\left(\nabla_{(\nabla_X Y)}\mathrm{grad}f\right)_p^H$$
$$-\frac{1}{2}g_x(Y, \nabla_X\mathrm{grad}f)(\nabla_Z\mathrm{grad}f)_p^H + \frac{1}{2}Y_x(f)\left(\nabla_{(\nabla_X Z)}\mathrm{grad}f\right)_p^H$$
$$+Y_x(f)\left[\frac{1}{8\alpha}g_x(Z, \nabla_X\mathrm{grad}f) - \frac{1}{16\alpha}X_x(\alpha)Z_x(f)\right](\mathrm{grad}\alpha)_p^H$$
$$+\left[\frac{1}{8\alpha}X_x(\alpha)Z_x(f) - \frac{1+\alpha}{4\alpha}g_x(Z, \nabla_X\mathrm{grad}f)\right](\nabla_Y\mathrm{grad}f)_p^H$$
$$+\frac{1}{4}Y_x(f)(R(X, \nabla_Z\mathrm{grad}f)u)_p^V + \frac{1}{4}Z_x(f)(R(X, \nabla_Y\mathrm{grad}f)u)_p^V$$
$$-\frac{1}{4}Y_x(f)(\nabla_{(R(u,Z)X)}\mathrm{grad}f)_p^V - \frac{1}{4}Y_x(f)Z_x(f)(\nabla_{(R(u,\mathrm{grad}f)X)}\mathrm{grad}f)_p^V$$
$$+\left[\frac{1}{8\alpha}Y_x(f)g_x(R(u, Z)X, \mathrm{grad}\alpha) - \frac{1}{4\alpha}g_x(R(u, Z)X, \nabla_Y\mathrm{grad}f)\right.$$
$$\left.+\frac{1}{8\alpha}Y_x(f)Z_x(f)g_x(R(u, \mathrm{grad}f)X, \mathrm{grad}\alpha) - \frac{1}{4\alpha}Z_x(f)g_x\Big(R(u, \mathrm{grad}f)X, \nabla_Y\mathrm{grad}f\Big)\right](\mathrm{grad}f)_p^V$$

(3)
$$R_p^f(X^V, Y^V)Z^H = \frac{1}{4}Y_x(f)(R(u, X)R(u, \mathrm{grad}f)Z)_p^H + \frac{1}{2}Y_x(f)(R(X, \mathrm{grad}f)Z)_p^H$$
$$-\frac{1}{4}X_x(f)(R(u, Y)R(u, \mathrm{grad}f)Z)_p^H - \frac{1}{2}X_x(f)(R(Y, \mathrm{grad}f)Z)_p^H$$
$$+\frac{1}{4}(R(u, X)R(u, Y)Z)_p^H - \frac{1}{4}(R(u, Y)R(u, X)Z)_p^H$$
$$+(R(X, Y)Z)_p^H - \frac{1}{4}X_x(f)(R(u, \mathrm{grad}f)(R(u, Y)Z)_p^H$$
$$-\frac{1}{4}Y_x(f)(R(u, \mathrm{grad}f)(R(u, X)Z)_p^H$$
$$-\left[\frac{1}{8\alpha}Y_x(f)Z_x(\alpha) + \frac{\alpha-1}{4\alpha}g_x(Y, \nabla_Z\mathrm{grad}f)\right](\nabla_X\mathrm{grad}f)_p^H$$
$$+\left[\frac{1}{8\alpha}X_x(f)Z_x(\alpha) + \frac{\alpha-1}{4\alpha}g_x(X, \nabla_Z\mathrm{grad}f)\right](\nabla_Y\mathrm{grad}f)_p^H$$
$$+\frac{1}{8\alpha}[Y_x(f)g_x(X, \nabla_Z\mathrm{grad}f) - X_x(f)g_x(Y, \nabla_Z\mathrm{grad}f)](\mathrm{grad}\alpha)_p^H$$
$$+\frac{1}{4}X_x(f)(\nabla_{(R(u,Y)Z)}\mathrm{grad}f)_p^V - \frac{1}{4}Y_x(f)(\nabla_{(R(u,X)Z)}\mathrm{grad}f)_p^V$$
$$+\left[\frac{1}{8\alpha}Y_x(f)g_x(R(u, X)Z, \mathrm{grad}\alpha) - \frac{1}{8\alpha}X_x(f)g_x(R(u, Y)Z, \mathrm{grad}\alpha)\right.$$
$$+\frac{1}{4\alpha}g_x(R(u, Y)Z, \nabla_X\mathrm{grad}f) - \frac{1}{4\alpha}g_x(R(u, X)Z, \nabla_Y\mathrm{grad}f)$$
$$\left.+\frac{1}{4\alpha}Y_x(f)g_x(R(u, \mathrm{grad}f)Z, \nabla_X\mathrm{grad}f) - \frac{1}{4\alpha}X_x(f)g_x\Big(R(u, \mathrm{grad}f)Z, \nabla_Y\mathrm{grad}f\Big)\right](\mathrm{grad}f)_p^V$$

$$(4)\quad R^f_p(X^H, Y^V)Z^H = \frac{1}{2}((\nabla_X R)(u, Y)Z)^H_p + \frac{1}{2} Y_x(f)((\nabla_X R)(u, \mathrm{grad}f)Z^H_p$$
$$+\frac{1}{2} Y_x(f)(R(u, \nabla_X \mathrm{grad}f)Z)^H_p + \frac{1}{2} g_x(Y, \nabla_X \mathrm{grad}f)(R(u, \mathrm{grad}f)Z)^H_p$$
$$+\frac{1}{4} Y_x(f)(R(u, \nabla_Z \mathrm{grad}f)X)^H_p + \frac{1}{4} g_x(Y, \nabla_Z \mathrm{grad}f)(R(u, \mathrm{grad}f)X)^H_p$$
$$-\frac{1}{4} Y_x(f)(\nabla_{(R(X,Z)u)} \mathrm{grad}f)^H_p - \frac{1}{4} g_x(R(X, Z)u, \mathrm{grad}f)(\nabla_Y \mathrm{grad}f)^H_p$$
$$+\frac{1}{2} (R(X, Z)Y)^V_p + \frac{1}{2} g_x(Y, \nabla_X \mathrm{grad}f)(\nabla_Z \mathrm{grad}f)^V_p$$
$$+\frac{1}{2} Y_x(f)(\nabla_X \nabla_Z \mathrm{grad}f)^V_p - \frac{1}{2} Y_x(f)\big(\nabla_{(\nabla_X Z)} \mathrm{grad}f)^V_p$$
$$-\frac{1}{4} (R(X, R(u, Y)Z)u)^V_p - \frac{1}{4} Y_x(f)(R(X, R(u, \mathrm{grad}f)Z)u)^V_p$$
$$+\left[\frac{\alpha+1}{4\alpha} g_x(Y, \nabla_Z \mathrm{grad}f) - \frac{1}{8\alpha} Y_x(f)Z_x(\alpha)\right](\nabla_X \mathrm{grad}f)^V_p$$
$$+\Bigg[\frac{1}{4\alpha} Y_x(f)g_x(\nabla_Z \mathrm{grad}f, \nabla_X \mathrm{grad}f) - \frac{1}{2\alpha} g_x(\nabla_X Z, \nabla_Y \mathrm{grad}f)$$
$$+\frac{1}{2\alpha} g_x(\nabla_X \nabla_Z \mathrm{grad}f, Y) - \frac{1}{4\alpha} Y_x(f)g_x(Z, \nabla_X \mathrm{grad}\alpha)$$
$$-\frac{1}{8\alpha^2} X_x(\alpha)Y_x(f)Z_x(\alpha) - \frac{1}{4\alpha} Z_x(\alpha)g_x(Y, \nabla_X \mathrm{grad}f)$$
$$-\frac{\alpha+2}{8\alpha^2} X_x(\alpha)g_x(Y, \nabla_Z \mathrm{grad}f)\Bigg](\mathrm{grad}f)^V_p$$

$$(5)\quad R^f_p(X^H, Y^H)Z^V = \frac{1}{2}((\nabla_X R)(u, Z)Y)^H_p + \frac{1}{4} g_x(Z, \nabla_X \mathrm{grad}f)(R(u, \mathrm{grad}f)Y)^H_p$$
$$-\frac{1}{2}((\nabla_Y R)(u, Z)X)^H_p - \frac{1}{4} g_x(Z, \nabla_Y \mathrm{grad}f)(R(u, \mathrm{grad}f)X)^H_p$$
$$+\frac{1}{2} Z_x(f)((\nabla_X R)(u, \mathrm{grad}f)Y)^H_p - \frac{1}{2} Z_x(f)((\nabla_Y R)(u, \mathrm{grad}f)X)^H_p$$
$$+\frac{1}{4} Z_x(f)(R(u, \nabla_X \mathrm{grad}f)Y)^H_p - \frac{1}{4} Z_x(f)(R(u, \nabla_Y \mathrm{grad}f)X)^H_p$$
$$-\frac{1}{2} g_x(R(X, Y)u, \mathrm{grad}f)(\nabla_Z \mathrm{grad}f)^H_p - \frac{1}{2} Z_x(f)(\nabla_{(R(X,Y)u)} \mathrm{grad}f)^H_p$$
$$+(R(X, Y)Z)^V_p + \frac{1}{2} Z_x(f)(R(X, Y)\mathrm{grad}f)^V_p$$
$$-\frac{1}{4}(R(X, R(u, Z)Y)u)^V_p - \frac{1}{4} Z_x(f)(R(X, R(u, \mathrm{grad}f)Y)u)^V_p$$
$$+\frac{1}{4}(R(Y, R(u, Z)X)u)^V_p + \frac{1}{4} Z_x(f)(R(Y, R(u, \mathrm{grad}f)X)u)^V_p$$
$$-\left[\frac{1}{8\alpha} Z_x(f)Y_x(\alpha) + \frac{\alpha-1}{4\alpha} g_x(Z, \nabla_Y \mathrm{grad}f)\right](\nabla_X \mathrm{grad}f)^V_p$$
$$+\left[\frac{1}{8\alpha} Z_x(f)X_x(\alpha) + \frac{\alpha-1}{4\alpha} g_x(Z, \nabla_X \mathrm{grad}f)\right](\nabla_Y \mathrm{grad}f)^V_p$$
$$+\left[\frac{\alpha-2}{8\alpha^2} X_x(\alpha)g_x(\nabla_Y \mathrm{grad}f, Z) + \frac{1}{2\alpha} g_x(R(X, Y)\mathrm{grad}f, Z) - \frac{\alpha-2}{8\alpha^2} Y_x(\alpha)g_x(\nabla_X \mathrm{grad}f, Z)\right](\mathrm{grad}f)^V_p$$

(6) $$R_p^f(X^V, Y^V)Z^V = \frac{1}{4}X_x(f)(R(u, Y)(\nabla_Z \mathrm{grad} f))_p^H - \frac{1}{4}Y_x(f)(R(u, X)(\nabla_Z \mathrm{grad} f))_p^H$$

$$+\frac{1}{4}Z_x(f)(R(u, Y)(\nabla_X \mathrm{grad} f))_p^H - \frac{1}{4}Z_x(f)(R(u, X)(\nabla_Y \mathrm{grad} f))_p^H$$

$$+\frac{1}{4}Y_x(f)Z_x(f)(R(u, \mathrm{grad} f)(\nabla_X \mathrm{grad} f))_p^H$$

$$-\frac{1}{4}X_x(f)Z_x(f)(R(u, \mathrm{grad} f)(\nabla_Y \mathrm{grad} f))_p^H$$

$$-\frac{1}{8\alpha}Y_x(f)Z_x(f)g_x(\nabla_X \mathrm{grad} f, \mathrm{grad}\alpha)(\mathrm{grad} f)_p^V$$

$$+\frac{1}{8\alpha}X_x(f)Z_x(f)g_x(\nabla_Y \mathrm{grad} f, \mathrm{grad}\alpha)(\mathrm{grad} f)_p^V$$

$$-\frac{1}{4\alpha}Y_x(f)g_x(\nabla_X \mathrm{grad} f, \nabla_Z \mathrm{grad} f)(\mathrm{grad} f)_p^V$$

$$+\frac{1}{4\alpha}X_x(f)g_x(\nabla_Y \mathrm{grad} f, \nabla_Z \mathrm{grad} f)(\mathrm{grad} f)_p^V$$

$$+\frac{1}{4}Y_x(f)Z_x(f)\left(\nabla_{(\nabla_X \mathrm{grad} f)} \mathrm{grad} f\right)_p^V$$

$$-\frac{1}{4}X_x(f)Z_x(f)\left(\nabla_{(\nabla_Y \mathrm{grad} f)} \mathrm{grad} f\right)_p^V$$

for all $p = (x, u) \in \mathrm{TM}$ and $X, Y, Z \in \Gamma(\mathrm{TM})$.

2.3 Sectional curvature of the gradient Sasaki metric

Let V and W be two orthonormal tangent vectors $V, W \in T_{(x,u)}\mathrm{TM}$. The sectional curvatures of the tangent bundle (TM, g^f) is given by

$$K^f(V, W) = \frac{G^f(V, W)}{Q^f(V, W)} \tag{2.1}$$

where

$$Q^f(V, W) = g^f(V, V)g^f(W, W) - |g^f(V, W)|^2 \text{ and } G^f(V, W) = g^f(R^f(V, W)W, V)$$

Lemma 2.2. Let (M, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with the gradient Sasaki metric, then for any orthonormal vectors fields $X, Y \in \Gamma(\mathrm{TM})$, we have

(1) $Q^f(X^H, Y^H) = 1$

(2) $Q^f(X^H, Y^V) = 1 + |Y(f)|^2$

(3) $Q^f(X^V, Y^V) = 1 + |X(f)|^2 + |Y(f)|^2$

(4) $G^f(X^H, Y^H) = g(R(X, Y)Y, X) - \frac{3}{4}\|R(X, Y)u\|^2 - \frac{3}{4}|g(R(X, Y)u), \mathrm{grad} f|^2$

(5)
$$G^f(X^H, Y^V) = Y(f)[g(\nabla_X \mathrm{grad} f, \nabla_X Y) - g(\nabla_X \nabla_Y \mathrm{grad} f, X) + \frac{1}{4\alpha} X(\alpha) g(\nabla_X \mathrm{grad} f, Y) + \frac{1}{2} g(R(u, Y)X, R(u, \mathrm{grad} f)X)] + |Y(f)|^2 \left[\frac{1}{4}\|R(u, \mathrm{grad} f)X\|^2 + \frac{1}{4}\|\nabla_X \mathrm{grad} f\|^2 - \frac{1}{16\alpha}|X(\alpha)|^2 \right. \frac{1}{4}\|R(u, Y)X\|^2 - \frac{3\alpha+1}{4\alpha}|g(\nabla_X \mathrm{grad} f, Y)|^2$$

(6)
$$G^f(X^V, Y^V) = \frac{1}{4}|Y(f)|^2 \|\nabla_X \mathrm{grad} f\|^2 + \frac{1}{4}|X(f)|^2 \nabla_Y \mathrm{grad} f^2 - \frac{1}{2} X(f) Y(f) g(\nabla_X \mathrm{grad} f, \nabla_Y \mathrm{grad} f)$$

Proposition 2.1. Let (M, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with the gradient Sasaki metric. If K, (resp K^f) denotes the sectional curvature of (M, g) (resp., (TM, g^f)), then for any orthonormal vectors fields $X, Y \in \Gamma(TM)$, we have

(1) $K^f(X^H, Y^H) = K(X, Y) - \frac{3}{4}\|R(X, Y)u\|^2 - \frac{3}{4}|g(R(X, Y)u, \mathrm{grad} f|^2$

(2)
$$K^f(X^H, Y^V) = \frac{Y(f)}{1+|Y(f)|^2}[g(\nabla_X \mathrm{grad} f, \nabla_X Y) - g(\nabla_X \nabla_Y \mathrm{grad} f, X) + \frac{1}{4\alpha} X(\alpha) g(\nabla_X \mathrm{grad} f, Y) + \frac{1}{2} g(R(u, Y)X, R(u, \mathrm{grad} f)X)] + \frac{|Y(f)|^2}{1+|Y(f)|^2}\left[\frac{1}{4}\|R(u, \mathrm{grad} f)X\|^2 + \frac{1}{4}\|\nabla_X \mathrm{grad} f\|^2 - \frac{1}{16\alpha}|X(\alpha)|^2\right] + \frac{1}{1+|Y(f)|^2}\left[\frac{1}{4}\|R(u, Y)X\|^2 - \frac{3\alpha+1}{4\alpha}|g(\nabla_X \mathrm{grad} f, Y)|^2\right]$$

(3)
$$K^f(X^V, Y^V) = \frac{1}{1+|X(f)|^2+|Y(f)|^2}\left[-\frac{1}{2} X(f) Y(f) g(\nabla_X \mathrm{grad} f, \nabla_Y \mathrm{grad} f) + \frac{1}{4}|Y(f)|^2 \|\nabla_X \mathrm{grad} f\|^2 + \frac{1}{4}|X(f)|^2 \|\nabla_Y \mathrm{grad} f\|^2\right]$$

Proof. The proof of Proposition 2.1 is deduced from equation (2.1) and Lemma 2.2.

Lemma 2.3. Let (M, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with the gradient Sasaki metric. If $(E_1, \ldots, E_m)$ be a local orthonormal frame on M

such that $E_1 = \frac{\text{grad} f}{\text{grad} f}$. Then $(F_1, \ldots, F_{2m})$ is a local orthonormal on (TM, g^f). Where $F_i = E_i^H$, $F_{m+1} = \frac{1}{\sqrt{\alpha}} E_1^V$ and $F_{m+j} = E_j^V$, $i = \overline{1, m}$, $\quad j = \overline{2, m}$.

Lemma 2.4. Let (M, g) be a Riemannian manifold and (TM, g^f) its tangent bundle equipped with the gradient Sasaki metric. If $(E_1, \ldots, E_m)$ (resp $(F_1, \ldots, F_{2m})$) are local orthonormal on M (resp., TM), then for all $i, j = \overline{1, m}$ et $k, l = \overline{2, m}$, we have

(1) $K^f(F_i, F_j) = K(E_i, E_j) - \frac{3}{4}\|R(E_i, E_j)u\|^2 - \frac{3}{4}\|g(R(E_i, E_j)u, \text{grad} f)\|^2$

(2)
$$K^f(F_i, F_{m+1}) = \frac{\alpha+3}{4\alpha}\|\nabla_{E_i}\text{grad} f\|^2 - \frac{\alpha^2-\alpha+4}{16\alpha^2(\alpha-1)}|E_i(\alpha)|^2 + \frac{\alpha}{4(\alpha-1)}||R(u, \text{grad} f)E_i||^2 - \frac{1}{2\alpha}g\left(\nabla_{E_i}\text{grad}\alpha, E_i\right)$$

(3) $K^f(F_i, F_{m+l}) = \frac{1}{4}\|R(u, E_l)E_i\|^2 - \frac{3\alpha+1}{4\alpha}|g(\nabla_{E_i}\text{grad} f, E_l)|^2$

(4) $K^f(F_{m+k}, F_{m+1}) = \frac{\alpha-1}{4\alpha}\|\nabla_{E_k}\text{grad} f\|^2$

(5) $K^f(F_{m+k}, F_{m+l}) = 0$

Proof. Using proposition 2.1, we have

(1) direct application

(2)
$$K^f(F_i, F_{m+1}) = G^f\left(E_i^H, \frac{1}{\sqrt{\alpha(\alpha-1)}}(\text{grad} f)^V\right)$$
$$= \frac{1}{\alpha(\alpha-1)}G^f\left(E_i^H, (\text{grad} f)^V\right)$$
$$= \frac{1}{\alpha(\alpha-1)}\left[(\alpha-1)\left[g\left(\nabla_{E_i}\text{grad} f, \nabla_{E_i}\text{grad} f\right) - g\left(\nabla_{E_i}\nabla_{\text{grad} f}\text{grad} f, E_i\right)\right.\right.$$
$$\left. = + \frac{1}{4\alpha}E_i(\alpha)g\left(\nabla_{E_i}\text{grad} f, \text{grad} f\right) + \frac{1}{2}g(R(u, \text{grad} f)E_i, R(u, \text{grad} f)E_i)\right]$$
$$+(\alpha-1)^2\left[\frac{1}{4}\|R(u, \text{grad} f)E_i\|^2 + \frac{1}{4}\|\nabla_{E_i}\text{grad} f\|^2 - \frac{1}{16\alpha}|E_i(\alpha)|^2\right.$$
$$\left.\left. + \frac{1}{4}\|R(u, \text{grad} f)E_i^2 \frac{3\alpha+1}{4\alpha}g\left(\nabla_{E_i}\text{grad} f, \text{grad} f\right)\right]\right]$$
$$= \frac{1}{\alpha}\|\nabla_{E_i}\text{grad} f^2 - \frac{1}{2\alpha}g\left(\nabla_{E_i}\text{grad}\alpha, E_i\right) + \frac{1}{8\alpha^2}|E_i(\alpha)|^2$$
$$+ \frac{1}{2\alpha}\|R(u, \text{grad} f)E_i\|^2 + \frac{\alpha-1}{4\alpha}R(u, \text{grad} f)E_i\|^2 + \frac{\alpha-1}{4\alpha}\nabla_{E_i}\text{grad} f^2$$

$$-\frac{\alpha-1}{16\alpha^2}|E_i(\alpha)|^2+\frac{1}{4\alpha(\alpha-1)}\|R(u,\,\mathrm{grad}f)E_i\|^2-\frac{3\alpha+1}{16\alpha^2(\alpha-1)}|E_i(\alpha)|^2$$
$$=\frac{3+\alpha}{4\alpha}\,\nabla_{E_i}\mathrm{grad}f\|^2-\frac{\alpha^2-\alpha+4}{16\alpha^2(\alpha-1)}|E_i(\alpha)|^2$$
$$+\frac{\alpha}{4(\alpha-1)}\|R(u,\,\mathrm{grad}f)E_i\|^2-\frac{1}{2\alpha}g\left(\nabla_{E_i}\mathrm{grad}\alpha,\,E_i\right)$$

(3) $$K^f(F_i,\,F_{m+l})=K^f\left(E_i^H,\,E_l^V\right)=\frac{1}{4}\|R(u,\,E_l)E_i\|^2-\frac{3\alpha+1}{4\alpha}\left|g\left(\nabla_{E_i}\mathrm{grad}f,\,E_l\right)\right|^2$$

(4) $$K^f(F_{m+k},\,F_{m+1})=G^f\left(E_k^V,\,\frac{1}{\sqrt{\alpha-1}}(\mathrm{grad}f)^V\right)$$
$$=\frac{1}{\alpha-1}G^f\left(E_k^H,\,(\mathrm{grad}f)^V\right)$$
$$=\frac{1}{\alpha(\alpha-1)}\,\frac{(\alpha-1)^2}{4}\|\nabla_{E_k}\mathrm{grad}f\|^2$$
$$=\frac{\alpha-1}{4\alpha}\|\nabla_{E_k}\mathrm{grad}f\|^2$$

(5) direct application □

Lemma 2.5. [18] Let $(E_1,\ \ldots,\ E_m$ be local orthonormal frame on M, then for all $i,\,j=\overline{1,\,m}$, we have

$$\sum_{i,j=1}^{m}\|R(u,\,E_i)E_j\|^2=\sum_{i,j=1}^{m}\|R(E_i,E_j)u\|^2$$

Proposition 2.2. Let $(M,\,g)$ be a Riemannian manifold and $(\mathrm{TM},\,g^f)$ its tangent bundle equipped with the metric of the gradient Sasaki metric. If σ (resp., σ^f denote the scalar curvature of $(M,\,g)$ (resp, $(\mathrm{TM},\,g^f)$), then for any local orthonormal frame $(E_1,\ \ldots,\ E_m)$ on M, we have

$$\sigma^f=\sigma-\frac{1}{4}\sum_{i,j=1}^{m}\|R(E_i,\,E_j)u\|^2-\frac{3}{4}\sum_{i,j=1}^{m}|g(R(E_i,\,E_j)u,\,\mathrm{grad}f)|^2$$
$$-\frac{3\alpha+1}{2\alpha}\sum_{i,j=1}^{m}\left|g\left(\nabla_{E_i}\mathrm{grad}f,\,E_j\right|^2-\frac{\alpha^2-\alpha+4}{8\alpha^2(\alpha-1)}\|\mathrm{grad}\alpha\|^2\right.$$
$$+\frac{\alpha+1}{\alpha}\|\nabla\mathrm{grad}f\|^2-\frac{1}{2}\mathrm{trace}_g\left((R(u,\mathrm{grad}f)^*)^2-\frac{1}{\alpha}\nabla_*\mathrm{grad}\alpha\right)$$

Proof. Using Lemma 2.3, we have

$$\sigma^f = \sum_{s,t=1}^{2m} K^f(F_s, F_t)$$

$$= \sum_{i,j=1,i\neq j}^{m} K^f(F_i, F_j) + 2\sum_{i,j=1}^{m} K^f(F_i, F_{m+j}) + \sum_{i,j=1,i\neq j}^{m} K^f(F_{m+i}, F_{m+j})$$

$$= \sum_{i,j=1,i\neq j}^{m} K^f(F_i, F_j) + 2\sum_{i=1}^{m} K^f(F_i, F_{m+1}) + 2\sum_{i=1,j=2}^{m} K^f(F_i, F_{m+j})$$

$$+ 2\sum_{i=1}^{m} K^f(F_{m+i}, F_{m+1}) + \sum_{i,j=2,i\neq j}^{m} K^f(F_{m+i}, F_{m+j})$$

$$\sigma^f = \sum_{i,j=1,i\neq j}^{m} [K(E_i, E_j) - \frac{3}{4}\|R(E_i, E_j)u\|^2 - \frac{3}{4}|g(R(E_i, E_j)u, gradf|^2]$$

$$+\sum_{i=1}^{m}\left[\frac{\alpha+3}{4\alpha}\|\nabla_{E_i}\mathrm{grad}f\|^2 - \frac{\alpha^2-\alpha+4}{16\alpha^2(\alpha-1)}|E_i(\alpha)|^2\right.$$

$$\left.+\frac{\alpha}{4(\alpha-1)}\|R(u, \mathrm{grad}f)E_i\|^2 - \frac{1}{2\alpha}g(\nabla_{E_i}\mathrm{grad}\alpha, E_i)\right]$$

$$+ 2\sum_{i=1,j=2}^{m}\left[\frac{1}{4}\|R(u, E_j)E_i\|^2 - \frac{3\alpha+1}{4\alpha}|g(\nabla_{E_i}\mathrm{grad}f, E_j)|^2\right]$$

$$+\sum_{i=2}^{m}\frac{\alpha-1}{4\alpha}\|\nabla_{E_i}\mathrm{grad}f\|^2$$

$$= \sigma - \frac{3}{4}\sum_{i,j=1,i\neq j}^{m}\|R(E_i, E_j)u\|^2 - \frac{3}{4}\sum_{i,j=1,i\neq j}^{m}|g(R(E_i, E_j)u, \mathrm{grad}f)|^2$$

$$+\frac{\alpha+3}{2\alpha}\sum_{i=1}^{m}\|\nabla_{E_i}\mathrm{grad}f\|^2 - \frac{\alpha^2-\alpha+4}{8\alpha^2(\alpha-1)}\sum_{i=1}^{m}|E_i(\alpha)|^2$$

$$+\frac{\alpha}{2(\alpha-1)}\sum_{i=1}^{m}\|R(u, \mathrm{grad}f)E_i\|^2 - \frac{1}{\alpha}\sum_{i=1}^{m}g(\nabla_{E_i}\mathrm{grad}\alpha, E_i)$$

$$+\frac{1}{2}\sum_{i=1,j=2}^{m}\|R(u, E_j)E_i\|^2 - \frac{3\alpha+1}{2\alpha}\sum_{i,j=2,i\neq j}^{m}|g(\nabla_{E_i}\mathrm{grad}f, E_j)|^2 + \frac{\alpha-1}{2\alpha}\sum_{i=1}^{m}\nabla_{E_i}\mathrm{grad}f^2$$

$$= \sigma - \frac{1}{4}\sum_{i,j=1,i\neq j}^{m}\|E(E_i, E_j)u\|^2 - \frac{3}{4}\sum_{i,j=1,i\neq j}^{m}|g(R(E_i, E_j)u, \mathrm{grad}f)|^2$$

$$+\frac{\alpha+1}{\alpha}\sum_{i=1}^{m}\|\nabla_{E_i}\mathrm{grad}f\|^2 - \frac{\alpha^2-\alpha+4}{8\alpha^2(\alpha-1)}\sum_{i=1}^{m}|E_i(\alpha)|^2$$

$$+\frac{\alpha}{2(\alpha-1)}\sum_{i=1}^{m}\|R(u, \mathrm{grad}f)E_i\|^2 - \frac{1}{2}\sum_{i=1}^{m}\|R(u, E_1)E_i\|^2$$

$$-\frac{1}{\alpha}\sum_{i=1}^{m}g(\nabla_{E_i}\mathrm{grad}\alpha, E_i) - \frac{3\alpha+1}{2\alpha}\sum_{i,j=1,i\neq j}^{m}|g(\nabla_{E_i}\mathrm{grad}f, E_j)|^2$$

$$\sigma^f = \sigma - \frac{1}{4}\sum_{i,j=1,i\neq j}^{m}\|R(E_i, E_j)u\|^2 - \frac{3}{4}\sum_{i,j=1,i\neq j}^{m}|g(R(E_i, E_j)u, \mathrm{grad}f)|^2$$

$$-\frac{3\alpha+1}{2\alpha}\sum_{i,j=1,i\neq j}^{m}|g(\nabla_{E_i}\mathrm{grad}f, E_j|^2 - \frac{\alpha^2-\alpha+4}{8\alpha^2(\alpha-1)}\|\mathrm{grad}\alpha\|^2$$

$$+\frac{\alpha+1}{\alpha}\|\nabla\mathrm{grad}f\|^2 - \frac{1}{2}\mathrm{trace}_g((R(u, \mathrm{grad}f) *)^2 - \frac{1}{\alpha}\nabla_*\mathrm{grad}\alpha)$$

□

Corollary 2.1. Let (M, g) be a Riemannian manifold of constant sectional curvature λ and (TM, g^f) its tangent bundle equipped with the gradient Sasaki metric. If σ^f denote the scalar curvature of TM, then for any local orthonormal frame $(E_1, \ldots, E_m)$ on M, we have

$$\sigma^f = m(m-1)\lambda + \lambda^2\left[\frac{2\alpha - m - 1}{2}\|u\|^2 - |g(u, \operatorname{grad} f)|^2\right]$$
$$-\frac{3\alpha+1}{2\alpha}\sum_{i,j=1}^{m}\left|g\left(\nabla_{E_i}\operatorname{grad} f, E_j\right)\right|^2 - \frac{\alpha^2-\alpha+4}{8\alpha^2(\alpha-1)}\|\operatorname{grad}\alpha\|^2$$
$$+\frac{\alpha+1}{\alpha}\nabla \operatorname{grad} f^2 - \frac{1}{\alpha}\operatorname{trace}_g(\nabla_* \operatorname{grad}\alpha)$$

Proof. Taking account that $\sigma = m(m-1)\lambda$ and for any vector fields $X, Y, Z \in \Gamma(TM)$

$$R(X, Y)Z = \lambda(g(Z, Y)X - g(X, Z)Y)$$

then we obtain

$$\begin{aligned}\sum_{i,j=1}^{m}\|R(E_i, E_j)u\|^2 &= \lambda^2\sum_{i,j=1}^{m}\|g(u, E_j)E_i - g(E_i, u)E_j\|^2\\ &= \lambda^2\sum_{i,j=1}^{m}\left[|g(u, E_j)|^2 - 2g(u, E_j)g(E_i, u)\delta_{ij} + |g(E_i, u)|^2\right.\\ &= \lambda^2[m\|u\|^2 - 2\|u\|^2 + m\|u\|^2]\\ &= 2(m-1)\lambda^2\|u\|^2\end{aligned}$$

From Proposition 2.2, we deduce

$$\sum_{i,j=1}^{m}|g(R(E_i, E_j)u, \operatorname{grad} f)|^2 = \lambda^2\sum_{i,j=1}^{m}|g(g(u, E_j)E_i - g(E_i, u)E_j, \operatorname{grad} f)|^2$$
$$= \lambda^2\sum_{i,j=1}^{m}|g(E_i, \operatorname{grad} f)g(u, E_j) - g(E_j, \operatorname{grad} f)g(E_i, u)|^2 = 0$$

and

$$\begin{aligned}\operatorname{trace}_g g(R(u, \operatorname{grad} f))^2 &= \sum_{i=1}^{m} g(R(u, \operatorname{grad} f)R(u, \operatorname{grad} f)E_i, E_i)\\ &= -\sum_{i=1}^{m}\|R(u, \operatorname{grad} f)E_i\|^2)\\ &= -\lambda^2\sum_{i=1}^{m}\|g(E_i, \operatorname{grad} f)u - g(u, E_i)\operatorname{grad} f\|^2\\ &= -\lambda^2\left[(\alpha-1)\|u\|^2 - 2|g(u, \operatorname{grad} f)|^2 + (\alpha-1)\|u\|^2\right]\\ &= -2\lambda^2\left[(\alpha-1)\|u\|^2 - |g(u, \operatorname{grad} f)|^2\right]\end{aligned}$$

□

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