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Fixed point theorems for Geraghty-type mappings applied to solving nonlinear Volterra-Fredholm integral equations in modular *G*-metric spaces

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Abstract

Purpose – The authors prove the existence and uniqueness of fixed point of mappings satisfying Geraghty-type contractions in the setting of preordered modular *G*-metric spaces. The authors apply the results in solving nonlinear Volterra-Fredholm-type integral equations. The results extend generalize compliment and include several known results as special cases.

Design/methodology/approach – The results of this paper are theoretical and analytical in nature.

Findings – The authors prove the existence and uniqueness of fixed point of mappings satisfying Geraghty-type contractions in the setting of preordered modular *G*-metric spaces. apply the results in solving nonlinear Volterra-Fredholm-type integral equations. The results extend, generalize, compliment and include several known results as special cases.

Research limitations/implications – The results are theoretical and analytical.

Practical implications – The results were applied to solving nonlinear integral equations.

Social implications – The results has several social applications.

Originality/value – The results of this paper are new.

Keywords Fixed point, Preordered, Modular *G*-metric spaces, Contractive mapping, Existence and uniqueness, Nonlinear Volterra-Fredholm integral equations

Paper type Research paper

1. Introduction

In 1973, Geraghty [1] introduced an interesting generalization of Banach contraction mapping principle using the concept of class $\mathcal{S}$ of functions, that is $\alpha : \mathbb{R}_+ \to [0, 1)$ with the condition that $\alpha(t_n) \to 1 \Rightarrow t_n \to 0$ where $\mathbb{R}_+$ is the set of all nonnegative real numbers and $t \in \mathbb{R}_+$ for all $n \in \mathbb{N}$. In 2012, Gordji *et al.* [2] proved some fixed point theorems for generalized Geraghty contraction in partially ordered complete metric spaces. Bhaskar and

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Lakshmikantham [3] proved a fixed point theorem for a mixed monotone mapping in a metric space endowed with partial order, using a weak contractivity type of assumption. Yolacan [4] established some new fixed point theorems in 0-complete ordered partial metric spaces. He also remarked on coupled generalized Banach contraction mapping. Faraji *et al.* [5] extended some fixed point theorems for Geraghty contractive mappings in b-complete b-metric spaces.

Furthermore, Gupta *et al.* [6], established some fixed point theorems in an ordered complete metric space using distance function. Chaipunya *et al.* [7] proved some fixed point theorems of Geraghty-type contractions concerning the existence and uniqueness of fixed points under the setting of modular metric spaces which also generalized the results in Gordji *et al.* [2] under the influence of a modular metric space.

Geraghty-type contractive mappings in metric spaces was generalized to the concept of preordered *G*-metric spaces in [8] and the authors in [8] obtained unique fixed point results. Furthermore, other interesting fixed point results in *G*-metric spaces can be found in [9] and the references therein.

In 2010, an essential study by Chistyakov [10] introduced an aspect of metric called modular metric spaces or parameterized metric space with the time parameter λ (say) and his purpose was to define the notion of a modular on an arbitrary set, develop the theory of metric spaces generated by modulars, called modular metric spaces and, on the basis of it, defined new metric spaces of (multi-valued) functions of bounded generalized variation of a real variable with values in metric semigroups and abstract convex cones.

In the same year, Chistyakov [11], as an application presented an exhausting description of Lipschitz continuous and some other classes of superposition (or Nemytskii) operators, acting in these modular metric spaces. He developed the theory of metric spaces generated by modulars and extended the results given by Nakano [12], Musielak and Orlicz [13], Musielak [14] to modular metric spaces. Modular spaces are extensions of Lebesgue, Riesz and Orlicz spaces of integrable functions.

Modular theories on linear spaces can be found in Nakano [12, 15], where he developed a spectral theory in semi-ordered linear spaces (vector lattices) and established the integral representation for projections acting in this modular space.

Nakano [12] established some modulars on real linear spaces which are convex functionals. Non-convex modulars and the corresponding modular linear spaces were constructed by Musielak and Orlicz [13]. Orlicz spaces and modular linear spaces have already become classical tools in modern nonlinear functional analysis.

Furthermore, the development of theory of metric spaces generated by modulars, called modular metric spaces attracted the attention of several mathematicians (see, e.g. [16–19]).

Okeke *et al.* [20] established some convergence results for three multi-valued ρ-quasi-nonexpansive mappings using a three step iterative scheme. Moreover, these fixed point results are applicable to nonlinear integral and differential equations see [19, 21–26] and the references therein, while [7] deals with application to partial differential equation in modular metric spaces.

In 2013, Azadifar *et al.* [27] introduced the notion of modular *G*-metric space and proved some fixed point theorems for contractive mappings defined on modular *G*-metric spaces. Based on definitions given in [27], we intend to extend the fixed point theorems obtained in [7] to preordered modular *G*-metric spaces in this paper. Furthermore, we prove some fixed point theorems for Geraghty-type contraction mappings in the setting of preordered modular *G*-metric spaces. We apply our results in proving the existence of a unique solution for a system of nonlinear Volterra-Fredholm integral equations in modular *G*-metric spaces, X_{ω^G}.

2. Preliminaries

We begin this section with the following results and definitions which will be useful in this paper.

Theorem 2.1.
[28] If $\{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}}, \{c_n\}_{n\in\mathbb{N}}$ are three sequences in $\mathbb{R}$ such that

(1) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \ell$,

(2) for some positive integer N, $a_n \leq c_n \leq b_n$ for all $n \geq N$.
Then $\lim_{n\to\infty} c_n = \ell$.

Definition 2.1. [29] A preorder set X is a relation $\preceq$ that is both,

(1) transitive i.e; $x \preceq y$ and $y \preceq z$ implies $x \preceq z$ and,

(2) reflexive i.e; $x \preceq x$.

A preordered set is a pair $(X, \preceq)$ consisting of a set X and a preorder $\preceq$ on X.

Remark 2.1. If a preorder $\preceq$ is antisymmetric i.e; $x \preceq y$ and $y \preceq x$ implies $x = y$, then $\preceq$ is called a partial order.

Definition 2.2. [1] Let $\mathcal{S}$ be the family of all Geraghty functions, that is functions $\alpha : [0, \infty) \to [0, 1)$ satisfying the condition $\{\alpha(t_n)\} \to 1 \Rightarrow \{t_n\} \to 0$.

For the rest of this paper, we denote the the class of all Geraghty functions by $\mathcal{S}_{Ger}$. Such Geraghty class was discussed in [7].

Definition 2.3. [7] Let $\mathcal{S}$ be the family of all Geraghty functions, that is functions $\beta_i : \mathbb{R}_+ \cup \{\infty\} \to [0, 1)$ satisfying the condition $\beta_i(t_k) \to \frac{1}{n} \Rightarrow \{t_k\} \to 0$ for all i.

Definition 2.4. [7] Let Ψ be the class of functions $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that the following conditions hold;

(1) ψ is decreasing,

(2) ψ is continuous,

(3) $\psi(t) = 0$ if and only if $t = 0$.

Extension of Definition 2.2 above is as follows:

Definition 2.5. [7] Let $\overline{\Psi}$ be the class of functions $\psi : \mathbb{R}_+ \cup \{\infty\} \to \mathbb{R}_+ \cup \{\infty\}$ such that the following conditions hold;

(1) ψ is subadditive,

(2) $\psi(t)$ is finite for $0 < t < \infty$,

(3) $\psi|_{\mathbb{R}_+} \in \overline{\Psi}$.

Definition 2.6. [27] Let X be a nonempty set, and let $\omega^G : (0, \infty) \times X \times X \times X \to [0, \infty]$ be a function satisfying;

(1) $\omega^G_\lambda(x, y, z) = 0$ for all $x, y \in X$ and $\lambda > 0$ if $x = y = z$,

(2) $\omega^G_\lambda(x, x, y) > 0$ for all $x, y \in X$ and $\lambda > 0$ with $x \neq y$,

(3) $\omega^G_\lambda(x, x, y) \leq \omega^G_\lambda(x, y, z)$ for all $x, y, z \in X$ and $\lambda > 0$ with $z \neq y$,

(4) $\omega^G_\lambda(x, y, z) = \omega^G_\lambda(x, z, y) = \omega^G_\lambda(y, z, x) = \ldots$ for all $\lambda > 0$ (symmetry in all three variables),

(5) $\omega^G_{\lambda+\mu}(x, y, z) \le \omega^G_\lambda(x, a, a) + \omega^G_\mu(a, y, z)$, for all $x, y, z, a \in X$ and $\lambda, \nu > 0$,

then the function ω^G_λ is called a modular G-metric on X.

Remarks 2.1.

(1) The pair (X, ω^G) is called a modular G-metric space, and without any confusion we will take X_{ω^G} as a modular G-metric space.
From condition (5), if ω^G is convex, then we have a strong form as,

(2) $\omega^G_{\lambda+\mu}(x, y, z) \le \omega^G_{\frac{\lambda}{\lambda+\mu}}(x, a, a) + \omega^G_{\frac{\mu}{\lambda+\mu}}(a, y, z)$,

(3) If $x = a$, then (5) above becomes $\omega^G_{\lambda+\mu}(a, y, z) \le \omega^G_\mu(a, y, z)$,

(4) Condition (5) is called rectangle inequality.

Definition 2.7. [27] Let (X, ω^G) be a modular G-metric space. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is ω^G-convergent to x, if it converges to x in the topology $\tau(\omega^G_\lambda)$.
A function $T : X_{\omega^G} \to X_{\omega^G}$ at $x \in X_{\omega^G}$ is called ω^G-continuous if $\omega^G_\lambda(x_n, x, x) \to 0$ then $\omega^G_\lambda(Tx_n, Tx, Tx) \to 0$, for all $\lambda > 0$.

Remark 2.2. $\{x_n\}_{n\in\mathbb{N}}$ modular G-converges to x as $n \to \infty$, if $\lim_{n\to\infty} \omega^G_\lambda(x_n, x_m, x) = 0$. That is for all $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\omega^G_\lambda(x_n, x_m, x) < \epsilon$ for all $n, m \ge n_0$. Here we say that x is modular G-limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 2.8. [27] Let (X, ω^G) be a modular G-metric space, then $\{x_n\}_{n\in\mathbb{N}} \subseteq X_{\omega^G}$ is said to be ω^G-Cauchy if for every $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\omega^G_\lambda(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \ge n_\epsilon$ and $\lambda > 0$.
A modular G-metric space X_{ω^G} is said to be ω^G-complete if every ω^G-Cauchy sequence in X_{ω^G} is ω^G-convergent in X_{ω^G}.

Proposition 2.2. [27] Let (X, ω^G) be a modular G-metric space, for any $x, y, x, a \in X$, it follows that:

(1) If $\omega^G_\lambda(x, y, z) = 0$ for all $\lambda > 0$, then $x = y = z$.

(2) $\omega^G_\lambda(x, y, z) \le \omega^G_{\lambda/2}(x, x, y) + \omega^G_{\lambda/2}(x, x, z)$ for all $\lambda > 0$.

(3) $\omega^G_\lambda(x, y, y) \le 2\omega^G_{\lambda/2}(x, x, y)$ for all $\lambda > 0$.

(4) $\omega^G_\lambda(x, y, z) \le \omega^G_{\lambda/2}(x, a, z) + \omega^G_{\lambda/2}(a, y, z)$ for all $\lambda > 0$.

(5) $\omega^G_\lambda(x, y, z) \le \frac{2}{3}\left(\omega^G_{\lambda/2}(x, y, a) + \omega^G_{\lambda/2}(x, a, z) + \omega^G_{\lambda/2}(a, y, z)\right)$ for all $\lambda > 0$.

(6) $\omega^G_\lambda(x, y, z) \le \omega^G_{\lambda/2}(x, a, a) + \omega^G_{\lambda/2}(y, a, a) + \omega^G_{\lambda/2}(z, a, a)$ for all $\lambda > 0$.

Proposition 2.3. [27] Let (X, ω^G) be a modular G-metric space and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_{ω^G}. Then the following are equivalent:

(1) $\{x_n\}_{n\in\mathbb{N}}$ is ω^G-convergent to x,

(2) $\omega^G_\lambda(x_n, x) \to 0$ as $n \to \infty$, i.e; $\{x_n\}_{n\in\mathbb{N}}$ converges to x relative to modular metric $\omega^G_\lambda(.)$,

(3) $\omega^G_\lambda(x_n, x_n, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$,

(4) $\omega^G_\lambda(x_n, x, x) \to 0$ as $n \to \infty$ for all $\lambda > 0$,

(5) $\omega^G_\lambda(x_m, x_n, x) \to 0$ as $m, n \to \infty$ for all $\lambda > 0$.

We give the following definition which will be useful in our results.

Definition 2.9. An ordered modular G-metric space is a triple $(X, \omega^G, \preceq)$ where (X, ω) is a modular metric space and $\preceq$ is a partial order on X_{ω^G}. If $\preceq$ is a preorder on X_{ω^G}, then $(X, \omega^G, \preceq)$ is a preordered modular G-metric space.

3. Main results

Theorem 3.1. Let (X, ω^G) be a complete modular G-metric space with a preorder, $\preceq$ and a nondecreasing self-mapping $T : X_{\omega^G} \to X_{\omega^G}$ on X_{ω^G} such that for each $\lambda > 0$, there is $\nu(\lambda) \in [0, \lambda)$ such that the following conditions hold:

(1)

$$\psi\left(\omega_\lambda^G(Tx, Ty, Ty)\right) \leq \alpha\left(\psi\left(\omega_\lambda^G(x, y, y)\right)\right)\psi\left(\omega_{\lambda+\nu(\lambda)}^G(x, y, y)\right) + \beta\left(\psi\left(\omega_\lambda^G(x, y, y)\right)\right)\psi\left(\omega_\lambda^G(x, Tx, Tx)\right) + \gamma\left(\psi\left(\omega_\lambda^G(x, y, y)\right)\right)\psi\left(\omega_\lambda^G(y, Ty, Ty)\right), \tag{3.1}$$

where $\psi \in \overline{\Psi}$ and $\{\alpha, \beta, \gamma\} \in \mathcal{S}_{Ger}$ with $\alpha(t) + 2\max\{\sup_{t\geq 0}\beta(t), \sup_{t\geq 0}\gamma(t)\} < 1$, and distinct $x, y \in X_{\omega^G}$. Assuming that if a nondecreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*, then $x_n \preceq x^*$ for each $n \in \mathbb{N}$,

(2) if ψ is subadditive and for any $x, y \in X_{\omega^G}$, there exists $z \in X_{\omega^G}$ with $z \preceq Tz$ and $\omega_\lambda^G(z, Tz, Tz)$ is finite for all $\lambda > 0$ such that z is comparable to both x and y. Then T has a fixed point $u \in X_{\omega^G}$ and the sequence define by $\{T^n x_0\}_{n\geq 1}$ converges to u. Moreover, the fixed point of T is unique.

Proof. Let $x_0 \in X_{\omega^G}$ be such that $x_0 \preceq Tx_0$ and let $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Regarding that T is nondecreasing mapping, we have that $x_0 \preceq Tx_0 = x_1$, which implies that $x_1 = Tx_0 \preceq Tx_1 = x_2$. Inductively, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \ldots \preceq x_{n-1} \preceq x_n \preceq x_{n+1} \preceq \ldots. \tag{3.2}$$

Assume that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$. Since $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is the fixed point of T. Now suppose that $x_n \precneqq x_{n+1}$ for all $n \in \mathbb{N}$, thus by inequality (3.2), we have that

$$x_0 \prec x_1 \prec x_2 \prec \ldots \prec x_{n-1} \prec x_n \prec x_{n+1} \prec \ldots. \tag{3.3}$$

Now for each $\lambda > 0$, and $x_0 \prec Tx_0$ for all $n \in \mathbb{N}$ implies that $\omega_\lambda^G(x_0, Tx_0, Tx_0) > 0$. Again, let $x_0 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, Tx_0, Tx_0) < \infty\ \forall\ \lambda > 0$.

First, we show that for all $n \in \mathbb{N}$, the sequence $\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) = 0$ for all $\lambda > 0$, as $n \to \infty$. Assume that, for each $n \in \mathbb{N}$, there exists $\lambda_n > 0$ such that $\omega_{\lambda_n}^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) \neq 0$. Otherwise the proof is complete. Suppose not, for each $n \geq 1$, if $0 < \lambda < \lambda_n$, then we have that $\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) \neq 0$. Since $T^n x_0 \preceq T^{n+1}x_0$, from inequality (3.1) we can see that $\psi\left(\omega_{\lambda_n}^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)\right) \leq \psi\left(\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)\right) = \psi\left(\omega_\lambda^G(TT^{n-1}x_0, TT^n x_0, TT^n x_0)\right)$. Take $x = T^{n-1}x_0$ and $y = T^n x_0$, then inequality (3.1) becomes;

$$\begin{aligned}
\psi\Big(\omega^G_{\lambda_n}(T^n x_0,\, T^{n+1}x_0,\, T^{n+1}x_0)\Big) &\le \psi\big(\omega^G_\lambda\big(T^n x_0,\, T^{n+1}x_0,\, T^{n+1}x_0\big)\big)\\
&\le \alpha\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\times\psi\big(\omega^G_{\lambda+\nu(\lambda)}\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\\
&\quad+\beta\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\times\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, TT^{n-1}x_0,\, TT^{n-1}x_0\big)\big)\\
&\quad+\gamma\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\times\psi\big(\omega^G_\lambda(T^n x_0,\, TT^n x_0,\, TT^n x_0)\big)\\
&= \alpha\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\times\psi\big(\omega^G_{\lambda+\nu(\lambda)}\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\\
&\quad+\beta\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\times\psi\big(\omega^G_\lambda\big(T^{n-1}x_0, T^n x_0, T^n x_0\big)\big)\\
&\quad+\gamma\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\times\psi\big(\omega^G_\lambda\big(T^n x_0,\, T^{n+1}x_0,\, T^{n+1}x_0\big)\big)\\
&\le \alpha\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\times\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\\
&\quad+\beta\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\times\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\\
&\quad+\gamma\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\times\psi\big(\omega^G_\lambda\big(T^n x_0,\, T^{n+1}x_0,\, T^{n+1}x_0\big)\big),
\end{aligned} \tag{3.4}$$

for which we have that

$$\begin{aligned}
\psi\big(\omega^G_\lambda\big(T^n x_0,\, T^{n+1}x_0,\, T^{n+1}x_0\big)\big) &\le \delta\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\\
&\le \psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\\
&\;\;\vdots\\
&\le \psi\big(\omega^G_\lambda(x_0,\, Tx_0,\, Tx_0)\big)\\
&< \infty,
\end{aligned} \tag{3.5}$$

where

$$\delta := \frac{\alpha\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)+\beta\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)}{1-\gamma\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)}. \tag{3.6}$$

Therefore, $\{\psi(\omega^G_\lambda(T^n x_0,\, T^{n+1}x_0,\, T^{n+1}x_0))\}_{n\ge1}$ is nonincreasing and bounded below, so the sequence $\{\psi(\omega^G_\lambda(T^n x_0,\, T^{n+1}x_0,\, T^{n+1}x_0))\}_{n\ge1}$ converges to some real number $k\ge0$. Assume $k>0$, we can see clearly that by using inequality 3, inequality 3 becomes

$$\begin{aligned}
\psi\big(\omega^G_\lambda\big(T^n x_0,\, T^{n+1}x_0,\, T^{n+1}x_0\big)\big) &\le \big(\alpha\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\\
&\quad+\beta\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)+\gamma\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\big)
\end{aligned} \tag{3.7}$$

as $n\to\infty$, we get

$$\begin{aligned}
1 &\le \liminf_{n\to\infty}\big(\alpha\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)+\beta\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\\
&\quad+\gamma\big(\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)\big)\big)
\end{aligned} \tag{3.8}$$

So, we have that

$$\psi\big(\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)\big)=0, \tag{3.9}$$

hence

$$\lim_{n\to\infty}\omega^G_\lambda\big(T^{n-1}x_0,\, T^n x_0,\, T^n x_0\big)=0 \tag{3.10}$$

for all $\lambda > 0$, which is a contradiction to our assumption. Therefore,

$$\lim_{n\to\infty}\psi\left(\omega_\lambda^G\left(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0\right)\right) = 0, \tag{3.11}$$

so,

$$\lim_{n\to\infty}\omega_\lambda^G\left(T^n x_0, T^{n+1}x_0, T^{n+1}x_0\right) = 0 \tag{3.12}$$

for all $\lambda > 0$. This shows that $\omega_\lambda^G(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0) = 0$ for all $\lambda > 0$, $n \geq 1$.

Next, we show that $\{T^n x_0\}_{n\geq 1}$ is a modular G-Cauchy sequence. Suppose, if possible that $\{T^n x_0\}_{n\geq 1}$ not a modular G-Cauchy sequence , then there exists real numbers, $\lambda_0 > 0$, $\epsilon > 0$ and also there exists two subsequences $\{T^{n_k}x_0\}_{k\geq 1}$ and $\{T^{m_k}x_0\}_{k\geq 1}$ of the sequence $\{T^n x_0\}_{n\geq 1}$ such that, for $n_k > m_k > k$, we have that $\omega_{\lambda_0}^G(T^{m_k}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0) \geq \epsilon$, but $\omega_{\lambda_0}^G(T^{m_k}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0) < \epsilon$. Now, since $T^{m_k}x_0 \leq T^{n_k}x_0$, we have that $\epsilon \leq \omega_{\lambda_0}^G(T^{m_k}x_0,$ $T^{n_k}x_0,\ T^{n_k}x_0)$ which implies that $\psi(\epsilon) \leq \psi(\omega_{\lambda_0}^G(T^{m_k}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0)) = \psi\ (\omega_{\lambda_0}^G(TT^{m_k-1}x_0,$ $TT^{n_k-1}x_0,\ TT^{n_k-1}x_0))$. Set $x = T^{m_k-1}x_0$ and $y = T^{n_k-1}x_0$ into inequality (3.1), then we have

$$\begin{aligned}
&\psi(\epsilon) \leq \psi\left(\omega_{\lambda_0}^G(T^{m_k}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0)\right) \leq \alpha\left(\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega_{\lambda_0+\nu(\lambda_0)}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right) + \beta\left(\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ TT^{m_k-1}x_0,\ TT^{m_k-1}x_0\right)\right) + \gamma\left(\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega_{\lambda_0}^G\left(T^{n_k-1}x_0,\ TT^{n_k-1}x_0,\ TT^{n_k-1}x_0\right)\right) = \alpha\left(\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega_{\lambda_0+\nu(\lambda_0)}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right) + \beta\left(\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right) + \gamma\left(\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega_{\lambda_0}^G\left(T^{n_k-1}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0\right)\right) \leq \alpha\left(\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&+\psi\left(\omega_{\lambda_0}^G\left(T^{m_k}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\Big) + \beta\left(\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right) + \gamma\left(\psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega_{\lambda_0}^G\left(T^{n_k-1}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0\right)\right) \leq \psi\left(\omega_{\nu(\lambda_0)}^G\left(T^{m_k-1}x_0, T^{m_k}x_0, T^{m_k}x_0\right)\right)\\
&+\psi\left(\omega_{\lambda_0}^G\left(T^{m_k}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right) + \psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)\\
&+\psi\left(\omega_{\lambda_0}^G\left(T^{n_k-1}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0\right)\right) \leq \psi\left(\omega_{\nu(\lambda_0)}^G\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)\\
&+\psi(\epsilon) + \psi\left(\omega_{\lambda_0}^G\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right) + \psi\left(\omega_{\lambda_0}^G\left(T^{n_k-1}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0\right)\right),
\end{aligned} \tag{3.13}$$

as $k \to \infty$, we obtain

$$\psi(\epsilon) \leq \lim_{k\to\infty}\psi\left(\omega_{\lambda_0}^G(T^{m_k}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0)\right) \leq \psi(\epsilon), \tag{3.14}$$

so that

$$\lim_{k\to\infty}\psi\left(\omega^G_{\lambda_0}(T^{m_k}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0)\right)=\psi(\epsilon). \tag{3.15}$$

Hence

$$\lim_{k\to\infty}\omega^G_{\lambda_0}(T^{m_k}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0)=\epsilon. \tag{3.16}$$

Again, using condition 5 of Definition 2.6, we get

$$\begin{aligned}
&\psi\left(\omega^G_{\lambda_0}(T^{m_k}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0)\right)\le\alpha\left(\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega^G_{\lambda_0+\nu(\lambda_0)}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)+\beta\left(\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)+\gamma\left(\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega^G_{\lambda_0}\left(T^{n_k-1}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0\right)\right)\le\psi\left(\omega^G_{\lambda_0+\nu(\lambda_0)}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\\
&+\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)+\psi\left(\omega^G_{\lambda_0}\left(T^{n_k-1}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0\right)\right)\\
&\le\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)+\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)\\
&+\psi\left(\omega^G_{\lambda_0}\left(T^{n_k-1}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0\right)\right)\le\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0\right)\right)\\
&+\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)+\psi\left(\omega^G_{\frac{\lambda_0}{2}}\left(T^{n_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)\\
&+\psi\left(\omega^G_{\frac{\lambda_0}{2}}(T^{m_k}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0)\right)\le\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\\
&+\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)+\psi\left(\omega^G_{\lambda_0}\left(T^{n_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)\\
&+\psi\left(\omega^G_{\lambda_0}(T^{m_k}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0)\right)\le\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\\
&+\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)+\psi\left(\omega^G_{\lambda_0}\left(T^{n_k-1}x_0,\ T^{m_k}x_0,\ T^{m_k}x_0\right)\right)\\
&+\psi\left(\omega^G_{\lambda_0}(T^{m_k}x_0,\ T^{n_k}x_0,\ T^{n_k}x_0)\right),
\end{aligned} \tag{3.17}$$

as $k\to\infty$, we have

$$\psi(\epsilon)\le\lim_{k\to\infty}\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)\le\psi(\epsilon), \tag{3.18}$$

so that

$$\lim_{k\to\infty}\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)\right)=\psi(\epsilon). \tag{3.19}$$

Hence

$$\lim_{k\to\infty}\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right)=\epsilon. \tag{3.20}$$

Thus, it follows that

$$1 \leq \liminf_{k\to\infty}\left(\alpha\left(\psi\left(\omega^G_{\lambda_0}\left(T^{m_k-1}x_0, T^{n_k-1}x_0, T^{n_k-1}x_0\right)\right)\right)\right). \tag{3.21}$$

Therefore, we conclude that

$$\lim_{k\to\infty}\omega^G_{\lambda_0}\left(T^{m_k-1}x_0,\ T^{n_k-1}x_0,\ T^{n_k-1}x_0\right) = 0 \ \ \forall \ \ \lambda > 0. \tag{3.22}$$

This is a contradiction. Therefore, it follows that $\{T^n x_0\}_{n\geq 1}$ is a modular G-Cauchy sequence in X_{ω^G}. Since X_{ω^G} is complete modular G-metric space, there exists $u \in X_{\omega^G}$ such that $T^n x_0 \to u \in X_{\omega^G}$. Now we show that u is a fixed point of T for any arbitrary $\lambda > 0$, using condition 5 of Definition 2.6 and inequality (3.1), we have that

$$\begin{aligned}
\psi\left(\omega^G_\lambda(T^n x_0,\ Tu,\ Tu)\right) &\leq \psi\left(\omega^G_{\frac{\lambda}{2}}\left(T^{n+1}x_0,\ Tu,\ Tu\right)\right)\\
&+ \psi\left(\omega^G_{\frac{\lambda}{2}}\left(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0\right)\right)\\
&= \psi\left(\omega^G_{\lambda/2}\left(T^{n+1}x_0,\ Tu,\ Tu\right)\right)\\
&+ \psi\left(\omega^G_{\lambda/2}\left(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0\right)\right)\\
&\leq \psi\left(\omega^G_\lambda\left(T^{n+1}x_0,\ Tu,\ Tu\right)\right)\\
&+ \psi\left(\omega^G_\lambda\left(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0\right)\right)\\
&\leq \alpha\left(\psi\left(\omega^G_\lambda(T^n x_0,\ u,\ u)\right)\right)\psi\left(\omega^G_{\lambda+\nu(\lambda)}(T^n x_0,\ u,\ u)\right)\\
&+ \beta\left(\psi\left(\omega^G_\lambda(T^n x_0,\ u,\ u)\right)\right)\psi\left(\omega^G_\lambda(T^n x_0,\ TT^n x_0,\ TT^n x_0)\right)\\
&+ \gamma\left(\psi\left(\omega^G_\lambda(T^n x_0,\ u,\ u)\right)\right)\psi\left(\omega^G_\lambda(u,\ Tu,\ Tu)\right)\\
&+ \psi\left(\omega^G_\lambda\left(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0\right)\right)\\
&= \alpha\left(\psi\left(\omega^G_\lambda(T^n x_0,\ u,\ u)\right)\right)\psi\left(\omega^G_{\lambda+\nu(\lambda)}(T^n x_0,\ u,\ u)\right)\\
&+ \beta\left(\psi\left(\omega^G_\lambda(T^n x_0,\ u,\ u)\right)\right)\psi\left(\omega^G_\lambda\left(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0\right)\right)\\
&+ \gamma\left(\psi\left(\omega^G_\lambda(T^n x_0,\ u,\ u)\right)\right)\psi\left(\omega^G_\lambda(u,\ Tu,\ Tu)\right)\\
&+ \psi\left(\omega^G_\lambda\left(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0\right)\right),
\end{aligned} \tag{3.23}$$

as $n \to\infty$, we have that

$$\psi\left(\omega^G_\lambda(u,\ Tu,\ Tu)\right) \leq \gamma(0)\psi\left(\omega^G_\lambda(u,\ Tu,\ Tu)\right) \tag{3.24}$$

for all $\lambda > 0$, which implies that

$$(1-\gamma(0))\omega^G_\lambda(u, Tu, Tu) \leq 0 \ \ \forall \ \ \lambda > 0. \tag{3.25}$$

Therefore,

$$\omega^G_\lambda(u, Tu, Tu) = 0 \ \ \forall \ \ \lambda > 0, \tag{3.26}$$

where $1-\gamma(0) < 1$. Hence, u is a fixed point of T for all $\lambda > 0$, i.e; $Tu = u$.

Finally, for the uniqueness, we can see from above that T has a fixed point $u \in X_{\omega^G}$. Suppose that there is another fixed point of T i.e; $Tv = v$, for $v \in X_{\omega^G}$, thus condition (2) of

Theorem 3.1 tells us that if $z \in X_{\omega^G}$ with $z \preceq Tz$ and it is comparable to both u and v and $T^n z$ is also comparable to u and v for each $n \in \mathbb{N}$. Now for $\lambda > 0$, then $\psi(\omega_\lambda^G(T^{n+1}z, u, u))$ and $\psi(\omega_\lambda^G(T^{n+1}z, v, v))$ are finite. Claim : $u = v$. Indeed, using inequality (3.1), we have by taking $x = T^n z$ and $y = u$. First consider $\psi(\omega_\lambda^G(T^{n+1}z, u, u)) < \infty$, so that we have the following inequality by using condition 6 of Proposition 2.2

$$
\begin{aligned}
&\psi\left(\omega_\lambda^G\left(T^{n+1}z, u, u\right)\right) = \psi\left(\omega_\lambda\left(T^{n+1}z, Tu, Tu\right)\right)\\
&= \psi\left(\omega_\lambda^G\left(TT^n z, Tu, Tu\right)\right)\\
&\leq \alpha\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_{\lambda+\nu(\lambda)}^G\left(T^n z, u, u\right)\right)\\
&+ \beta\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_\lambda^G\left(T^n z, TT^n z, TT^n z\right)\right)\\
&+ \gamma\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_\lambda^G\left(u, Tu, Tu\right)\right)\\
&= \alpha\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_{\lambda+\nu(\lambda)}^G\left(T^n z, u, u\right)\right)\\
&+ \beta\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_\lambda^G\left(T^n z, T^{n+1}z, T^{n+1}z\right)\right)\\
&\leq \alpha\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_{\lambda+\nu(\lambda)}^G\left(T^n z, u, u\right)\right)\\
&+ \psi\left(\omega_{\lambda/4}^G\left(T^{n+1}z, u, u\right)\right) + \psi\left(\omega_{\frac{\lambda}{4}}^G\left(T^{n+1}z, u, u\right)\right)\Big)\\
&= \alpha\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_{\lambda+\nu(\lambda)}^G\left(T^n z, u, u\right)\right)\\
&+ \beta\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\left(\psi\left(\omega_{\lambda/2}^G\left(T^n z, u, u\right)\right)\right.\\
&\left.+ 2\psi\left(\omega_{\lambda/4}^G\left(T^{n+1}z, u, u\right)\right)\right)\\
&\leq \alpha\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\Big)\\
&+ \beta\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right.\\
&\left.+ 2\psi\left(\omega_\lambda^G\left(T^{n+1}z, u, u\right)\right)\right)\\
&= \alpha\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\Big)\\
&+ \beta\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\\
&+ 2\beta\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)\psi\left(\omega_\lambda^G\left(T^{n+1}z, u, u\right)\right),
\end{aligned}
\tag{3.27}
$$

so, we have that

$$
\begin{aligned}
\psi\left(\omega_\lambda^G\left(T^{n+1}z, u, u\right)\right) &\leq \frac{\alpha\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right) + \beta\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)}{1 - 2\beta\left(\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\right)}\psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\\
&\leq \psi\left(\omega_\lambda^G\left(T^n z, u, u\right)\right)\\
&\leq \psi\left(\omega_\lambda^G\left(T^{n-1}z, u, u\right)\right)\\
&\;\;\vdots\\
&\leq \psi\left(\omega_\lambda^G\left(z, u, u\right)\right)\\
&< \infty.
\end{aligned}
\tag{3.28}
$$

Therefore, $\{\psi(\omega_\lambda^G(T^{n+1}z, u, u))\}_{n\geq 1}$ is nonincreasing sequence which is bounded below and converges to some real number $\ell \in [0, \infty)$ Assume that $\ell > 0$, using the fact that $\lim_{n\to\infty}\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) = 0$ for all $\lambda > 0$, from inequality 3, we have that

$$\begin{aligned}\psi\big(\omega_\lambda^G\big(T^{n+1}z, u, u\big)\big) &\leq \alpha\big(\psi\big(\omega_\lambda^G(T^n z, u, u)\big)\big)\psi\big(\omega_{\lambda+\nu(\lambda)}^G(T^n z, u, u)\big)\\ &+ \beta\big(\psi\big(\omega_\lambda^G(T^n z, u, u)\big)\big)\psi\big(\omega_\lambda^G\big(T^n z, T^{n+1}z, T^{n+1}z\big)\big)\\ &\leq \alpha\big(\psi\big(\omega_\lambda^G(T^n z, u, u)\big)\big)\psi\big(\omega_\lambda^G(T^n z, u, u)\big)\\ &+ \beta\big(\psi\big(\omega_\lambda^G(T^n z, u, u)\big)\big)\psi\big(\omega_\lambda^G\big(T^n z, T^{n+1}z, T^{n+1}z\big)\big).\end{aligned} \tag{3.29}$$

Using inequality 3 and letting $n\to\infty$, inequality 3 becomes

$$1 \leq \liminf_{n\to\infty} \alpha\big(\psi\big(\omega_\lambda^G(T^n z, u, u)\big)\big). \tag{3.30}$$

Thus, by condition 4 of Proposition 2.3 we have that

$$\lim_{n\to\infty} \omega_\lambda^G(T^n z, u, u) = 0 \tag{3.31}$$

for all $\lambda > 0$. Therefore, $T^n z \to u$ as $n\to\infty$.

Secondly consider $\psi(\omega_\lambda^G(T^{n+1}z, v, v)) < \infty$, from inequality (3.1), we have by taking $x = T^n z$ and $y = v$, so that we have the following inequality by using condition 6 of Proposition 2.2

$$\begin{aligned}\psi\big(\omega_\lambda^G\big(T^{n+1}z, v, v\big)\big) &= \psi\big(\omega_\lambda\big(T^{n+1}z, Tv, Tv\big)\big)\\ &= \psi\big(\omega_\lambda^G(TT^n z, Tv, Tv)\big)\\ &\leq \alpha\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_{\lambda+\nu(\lambda)}^G(T^n z, v, v)\big)\\ &+ \beta\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_\lambda^G(T^n z, TT^n z, TT^n z)\big)\\ &+ \gamma\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_\lambda^G(v, Tv, Tv)\big)\\ &= \alpha\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_{\lambda+\nu(\lambda)}^G(T^n z, v, v)\big)\\ &+ \beta\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_\lambda^G\big(T^n z, T^{n+1}z, T^{n+1}z\big)\big)\\ &\leq \alpha\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_{\lambda+\nu(\lambda)}^G(T^n z, v, v)\big)\\ &+ \beta\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\Big(\psi\Big(\omega_{\lambda/2}^G(T^n z, v, v)\Big)\Big)\\ &+ \psi\Big(\omega_{\lambda/4}^G\big(T^{n+1}z, v, v\big)\Big) + \psi\Big(\omega_{\lambda/4}^G\big(T^{n+1}z, v, v\big)\Big)\Big)\\ &= \alpha\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_{\lambda+\nu(\lambda)}^G(T^n z, v, v)\big)\\ &+ \beta\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\Big(\psi\Big(\omega_{\lambda/2}^G(T^n z, v, v)\Big)\\ &+ 2\psi\Big(\omega_{\lambda/4}^G\big(T^{n+1}z, v, v\big)\Big) \leq \alpha\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\\ &+ \beta\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\\ &+ 2\psi\big(\omega_\lambda^G\big(T^{n+1}z, v, v\big)\big)\big) = \alpha\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\\ &+ \beta\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\\ &+ 2\beta\big(\psi\big(\omega_\lambda^G(T^n z, v, v)\big)\big)\psi\big(\omega_\lambda^G\big(T^{n+1}z, v, v\big)\big).\end{aligned} \tag{3.32}$$

Therefore, we have

$$\begin{aligned}\psi\left(\omega_\lambda^G\left(T^{n+1}z, v, v\right)\right) &\leq \frac{\alpha\left(\psi\left(\omega_\lambda^G(T^n z, v, v)\right)\right)+\beta\left(\psi\left(\omega_\lambda^G(T^n z, v, v)\right)\right)}{1-2\beta\left(\psi\left(\omega_\lambda^G(T^n z, v, v)\right)\right)}\psi\left(\omega_\lambda^G(T^n z, v, v)\right)\\ &\leq \psi\left(\omega_\lambda^G(T^n z, v, v)\right)\\ &\leq \psi\left(\omega_\lambda^G\left(T^{n-1}z, v, v\right)\right)\\ &\ \ \vdots\\ &\leq \psi\left(\omega_\lambda^G(z, v, v)\right)\\ &< \infty.\end{aligned} \tag{3.33}$$

Hence, $\{\psi(\omega_\lambda^G(T^{n+1}z,\ v,\ v))\}_{n\geq 1}$ is nonincreasing sequence which is bounded below and converges to some real number $\ell_0 \in [0,\ \infty)$ Suppose that $\ell_0 > 0$, using the fact that $\lim_{n\to\infty}\omega_\lambda^G(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0) = 0$ for all $\lambda > 0$. From inequality 3, we have that

$$\begin{aligned}&\psi\left(\omega_\lambda^G\left(T^{n+1}z,\ v,\ v\right)\right) \leq \alpha\left(\psi\left(\omega_\lambda^G(T^n z,\ v,\ v)\right)\right)\psi\left(\omega_{\lambda+\nu(\lambda)}^G(T^n z,\ v,\ v)\right)\\ &+\beta\left(\psi\left(\omega_\lambda^G(T^n z,\ v,\ v)\right)\right)\psi\left(\omega_\lambda^G\left(T^n z,\ T^{n+1}z,\ T^{n+1}z\right)\right)\\ &\leq \alpha\left(\psi\left(\omega_\lambda^G(T^n z,\ v,\ v)\right)\right)\psi\left(\omega_\lambda^G(T^n z,\ v,\ v)\right)\\ &+\beta\left(\psi\left(\omega_\lambda^G(T^n z,\ v,\ v)\right)\right)\psi\left(\omega_\lambda^G\left(T^n z,\ T^{n+1}z,\ T^{n+1}z\right)\right).\end{aligned} \tag{3.34}$$

Using inequality 3 and letting $n\to\infty$, inequality 3 becomes

$$1 \leq \liminf_{n\to\infty} \alpha\left(\psi\left(\omega_\lambda^G(T^n z,\ v,\ v)\right)\right). \tag{3.35}$$

Thus, by condition 4 of Proposition 2.3 we have that

$$\lim_{n\to\infty}\omega_\lambda^G(T^n z,\ v,\ v) = 0 \tag{3.36}$$

for all $\lambda > 0$. Therefore, $T^n z \to v$ as $n\to\infty$.

Suppose, if possible, that $\lim_{n\to\infty} T^n z$ exists and not unique. Let $\lim_{n\to\infty} T^n z = u$ and $\lim_{n\to\infty} T^n z = v$ as we have seen above, where $u \neq v$. For each $\lambda > 0$, $u\neq v \Rightarrow \omega_\lambda^G(u,\ v,\ v) > 0$. If we take $\psi(\epsilon_1) = \frac{1}{3}\psi(\omega_\lambda(u,\ v,\ v))>0$, then for $\lambda > 0$, $\lim_{n\to\infty} T^n z = u \Rightarrow$ given $\epsilon_1 > 0, \exists\ m_1 \in \mathbb{N}$ such that $\psi(\omega_{\lambda/2}^G(u,\ T^n z,\ T^n z))<\psi(\epsilon_1)$ for $n > m_1$. Again, $\lim_{n\to\infty} T^n z = v \Rightarrow$ given $\epsilon_1 > 0$, $\exists\ m_2 \in \mathbb{N}$ such that $\psi(\omega_{\lambda/4}^G(v,\ T^n z,\ T^n z))<\psi(\epsilon_1)$ for $n > m_2$. Set $m = \max\{m_1,\ m_2\}$, then for $n\geq m$, by condition 6 of Proposition 2.2, we have

$$\begin{aligned}&\psi\left(\omega_\lambda^G(u,\ v,\ v)\right) \leq \psi\Big(\omega_{\lambda/2}^G(u,\ T^n z,\ T^n z)\\ &+2\omega_{\lambda/4}^G(v,\ T^n z,\ T^n z)\Big) \leq \psi\Big(\omega_{\lambda/2}^G(u,\ T^n z,\ T^n z)\Big)\\ &+2\psi\Big(\omega_{\lambda/4}^G(v,\ T^n z,\ T^n z)\Big)<\psi(\epsilon_1)+2\psi(\epsilon_1)\\ &= 3\psi(\epsilon_1),\end{aligned}$$

which shows that $\psi(\omega_\lambda^G(u, v, v)) < \psi(\omega_\lambda^G(u, v, v))$ for all $\lambda > 0$. This is a contradiction. Hence, $u = v$. Therefore the fixed point of T is unique and the proof is complete. □

We shall give an example to support Theorem 3.1 above.

Example 3.1. Let $X = \mathbb{R}$, define modular G-metric by $\omega_\lambda^G(x, y, y) = \infty$ if $\lambda \leq 2|x-y|$ and $\omega_\lambda^G(x, y, y) = 0$ if $\lambda > 2|x-y|$, and $\omega_\lambda^G(x, y, y) = \frac{G(x,y,y)}{\lambda}$, where $G(x, y, y) = 2|x-y|$ or $G(x, y, z) = |x-y| + |y-z| + |x-z|$ for $x, y, z \in \mathbb{R}$. We can see that $X_{\omega^G} = \mathbb{R}$. So it follows from Theorem 3.1 that $\mathbb{R}$ is a complete preordered modular G-metric space. Now define a map $T : \mathbb{R} \to \mathbb{R}$ by $Tx = \frac{x^3}{1+x^2}$. For $x, y \in \mathbb{R}$, then $\omega_\lambda^G(x, y, y) = \infty$ if $\lambda \leq 2\,|x-y|$, so inequality (3.1) is satisfied. Again, if $\lambda > 2\,|x-y|$ and $x, y \in \mathbb{R}$, then

$$G(Tx,\ Ty,\ Ty) = 2\left|\frac{x^3}{1+x^2} - \frac{y^3}{1+y^2}\right| \leq 4|x-y| < 2\lambda, \text{ therefore, } \omega_\lambda^G(Tx,\ Ty,\ Ty) = 2\omega_\lambda^G(x, y, y) \leq 0.$$

We can take $\psi(t) = t$, $\alpha(t) = \beta(t) = \gamma(t) = \frac{1}{2}$. But T has a fixed point at $x = 0$.

Corollary 3.2. Let (X, ω^G) be a complete modular G-metric space with a preorder, $\preceq$ and a nondecreasing self-mapping $T : X_{\omega^G} \to X_{\omega^G}$ on X_{ω^G} such that for each $\lambda > 0$, there is $\nu(\lambda) \in [0, \lambda)$ such that the following conditions hold:

(1)

$$\psi\left(\omega_\lambda^G(Tx,\ Ty,\ Ty)\right) \leq \alpha\left(\psi\left(\omega_\lambda^G(x, y, y)\right)\right)\psi\left(\omega_{\lambda+\nu(\lambda)}^G(x, y, y)\right), \tag{3.37}$$

where $\psi \in \overline{\Psi}$ and $\alpha \in \mathcal{S}_{Ger}$ and distinct $x, y \in X_{\omega^G}$. Assuming that if a nondecreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*, then $x_n \preceq x^*$ for each $n \in \mathbb{N}$,

(2) if ψ is subadditive and for any $x, y \in X_{\omega^G}$, there exists $z \in X_{\omega^G}$ with $z \preceq Tz$ and $\omega_\lambda^G(z,\ Tz,\ Tz)$ is finite for all $\lambda > 0$ such that z is comparable to both x and y. Then T has a fixed point $u \in X_{\omega^G}$ and the sequence define by $\{T^n x_0\}_{n\geq 1}$ converges to u. Moreover, the fixed point of T is unique.

Proof: Let $x_0 \in X_{\omega^G}$ be such that $x_0 \preceq Tx_0$ and let $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Regarding that T is nondecreasing mapping, we have that $x_0 \preceq Tx_0 = x_1$, which implies that $x_1 = Tx_0 \preceq Tx_1 = x_2$. Inductively, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{n-1} \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{3.38}$$

Assume that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$. Since $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is the fixed point of T. Now suppose that $x_n \precneqq x_{n+1}$ for all $n \in \mathbb{N}$, thus by inequality (3.38), we have that

$$x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n \prec x_{n+1} \prec \cdots. \tag{3.39}$$

Now for each $\lambda > 0$, and $x_0 \prec Tx_0$ for all $n \in \mathbb{N}$ implies that $\omega_\lambda^G(x_0,\ Tx_0,\ Tx_0) > 0$. Again, let $x_0 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0,\ Tx_0,\ Tx_0) < \infty\ \forall\ \lambda > 0$.

First, we show that for all $n \in \mathbb{N}$, the sequence $\omega_\lambda^G(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0) = 0$ for all $\lambda > 0$, as $n \to \infty$.

Assume that for each $n \in \mathbb{N}$, there exists $\lambda_n > 0$ such that $\omega_{\lambda_n}^G(T^{n+1}x_0,\ T^n x_0,\ T^n x_0) \neq 0$. Otherwise we are done. Suppose that for each $n \geq 1$, if $0 < \lambda < \lambda_n$, then we have $\omega_\lambda^G(T^{n+1}x_0,\ T^n x_0,\ T^n x_0) \neq 0$. Since $T^n x_0 \preceq T^{n+1}x_0$, we have from inequality (3.37) that $\psi(\omega_{\lambda_n}^G(T^{n+1}x_0,\ T^n x_0,\ T^n x_0)) \leq \psi(\omega_\lambda^G(T^{n+1}x_0,\ T^n x_0,\ T^n x_0)) = \psi(\omega_\lambda^G(TT^n x_0,\ TT^{n-1}x_0,\ TT^{n-1}x_0))$. Take $x = T^n x_0$ and $y = T^{n-1}x_0$, then inequality (3.37) becomes;

$$\begin{aligned}
&\psi\left(\omega_{\lambda}^{G}\left(T^{n+1}x_0,\ T^{n}x_0,\ T^{n}x_0\right)\right) \leq \alpha\left(\psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega_{\lambda+\nu(\lambda)}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right)\\
&\leq \alpha\left(\psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right)\right)\\
&\times\psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right)\\
&< \psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right).
\end{aligned} \tag{3.40}$$

Therefore, $\{\psi(\omega_{\lambda}^{G}(T^{n+1}x_0,\ T^{n}x_0,\ T^{n}x_0))\}_{n\geq 1}$ is nonincreasing and bounded below and converges to some real number $\tau \geq 0$. Assume that $\tau > 0$. In such a case,

$$\begin{aligned}
&\tau < \psi\left(\omega_{\lambda}^{G}\left(T^{n+1}x_0,\ T^{n}x_0,\ T^{n}x_0\right)\right)\\
&\leq \alpha\left(\psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right)\right)\psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right)\\
&< \psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right),
\end{aligned} \tag{3.41}$$

which implies that

$$\begin{aligned}
1 &< \frac{\psi\left(\omega_{\lambda}^{G}\left(T^{n+1}x_0,\ T^{n}x_0,\ T^{n}x_0\right)\right)}{\tau}\\
&\leq \frac{\alpha\left(\psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right)\right)\psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right)}{\tau}\\
&< \frac{\psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right)}{\tau}.
\end{aligned} \tag{3.42}$$

Letting, $n \to \infty$, Theorem 2.1 ensure that $\left\{\alpha(\psi(\omega_{\lambda}^{G}(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0)))\right\}_{n\geq 1} \to 1$ or from inequality 3,

$$1 \leq \liminf_{n\to\infty}\ \alpha\left(\psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right)\right). \tag{3.43}$$

As $\alpha \in \mathcal{S}_{Ger}$, then

$$\lim_{n\to\infty}\psi\left(\omega_{\lambda}^{G}\left(T^{n}x_0,\ T^{n-1}x_0,\ T^{n-1}x_0\right)\right) \to 0 \tag{3.44}$$

for all $\lambda > 0$, which contradicts the fact that $\tau > 0$. Thus $\tau = 0$, so that

$$\lim_{n\to\infty}\omega_{\lambda}^{G}\left(T^{n+1}x_0,\ T^{n}x_0,\ T^{n}x_0\right) = 0. \tag{3.45}$$

Hence $\omega_{\lambda}^{G}(T^{n+1}x_0,\ T^{n}x_0,\ T^{n}x_0) = 0$ for all $\lambda > 0$, $n \geq 1$. Following the proof of Theorem 3.1 above, we see that T has a unique fixed point in X_{ω^G}. □

Theorem 3.3. Let $(X,\ \omega^G)$ be a complete modular G-metric space with a preorder, $\preceq$, and a nondecreasing self-mapping $T : X_{\omega^G} \to X_{\omega^G}$ on X_{ω^G} such that for each $\lambda > 0$, there is $\nu(\lambda) \in [0, \lambda)$ such that the following conditions hold:

(1)

$$\psi\left(\omega_{\lambda}^{G}\left(T^{m}x,\ T^{m}y,\ T^{m}y\right)\right) \leq \alpha\left(\psi\left(\omega_{\lambda}^{G}(x,\ y, y)\right)\right)\psi\left(\omega_{\lambda+\nu(\lambda)}^{G}(x,\ y,\ y)\right), \tag{3.46}$$

where $\psi \in \overline{\Psi}$ and $\alpha \in \mathcal{S}_{Ger}$ and distinct $x,\ y \in X_{\omega^G}$. Assuming that if a nondecreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*, then $x_n \preceq x^*$ for each $n \in \mathbb{N}$,

(2) if ψ is subadditive and for any $x,\ y \in X_{\omega^G}$, there exists $z \in X_{\omega^G}$ with $z \preceq Tz$ and $\omega_{\lambda}^{G}(z,\ Tz,\ Tz)$ is finite for all $\lambda > 0$ such that z is comparable to both x and y.Then T

has a fixed point $u \in X_{\omega^G}$ for some positive integer m and the sequence define by $\{T^n x_0\}_{n\geq 1}$ converges to u. Moreover, the fixed point of T is unique.

Proof: By Corollary 3.2, T^m has a fixed point say $u \in X_{\omega^G}$ for some positive integer $m \geq 1$. Now $T^m(Tu) = T^{m+1}u = T(T^m u) = Tu$, so Tu is a fixed point of T^m. By the uniqueness of fixed point of T^m, we have $Tu = u$. Therefore, u is a fixed point of T. Since fixed point of T is also fixed point of T^m, hence T has a unique fixed point in X_{ω^G}. □

Theorem 3.4. Let (X, ω^G) be a complete modular G-metric space with a preorder, $\preceq$ and a nondecreasing self-mapping $T: X_{\omega^G} \to X_{\omega^G}$ on X_{ω^G} such that for each $\lambda > 0$, there is $\nu(\lambda) \in [0, \lambda)$ such that the following conditions hold:

(1)

$$\psi\left(\omega_\lambda^G(T^m x, T^m y, T^m y)\right) \leq \alpha\left(\psi\left(\omega_\lambda^G(x, y, y)\right)\right)\psi\left(\omega_{\lambda+\nu(\lambda)}^G(x, y, y)\right) + \beta\left(\psi\left(\omega_\lambda^G(x,y,y)\right)\right.$$
$$\left.\times\right)\psi\left(\omega_\lambda^G(x, T^m x, T^m x)\right) + \gamma\left(\psi\left(\omega_\lambda^G(x,y,y)\right)\right)\psi\left(\omega_\lambda^G(y, T^m y, T^m y)\right), \tag{3.47}$$

where $\psi \in \overline{\Psi}$ and $\{\alpha, \beta, \gamma\} \in \mathcal{S}_{Ger}$ with $\alpha(t) + 2\max\{\sup_{t\geq 0}\beta(t), \sup_{t\geq 0}\gamma(t)\} < 1$, and distinct $x, y \in X_{\omega^G}$. Assuming that if a nondecreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*, then $x_n \preceq x^*$ for each $n \in \mathbb{N}$,

(2) if ψ is subadditive and for any distinct $x, y \in X_{\omega^G}$, there exists $z \in X_{\omega^G}$ with $z \preceq Tz$ and $\omega_\lambda^G(z, Tz, Tz)$ is finite for all $\lambda > 0$ such that z is comparable to both x and y. Then T has a fixed point $u \in X_{\omega^G}$ for some positive integer $m \geq 1$ and the sequence define by $\{T^n x_0\}_{n\geq 1}$ converges to u. Moreover, the fixed point of T is unique.

Proof: By Theorem 3.1, T^m has a fixed point say $u_* \in X_{\omega^G}$ for some positive integer $m \geq 1$. Now $T^m(Tu_*) = T^{m+1}u_* = T(T^m u_*) = Tu_*$, so Tu_* is a fixed point of T^m. By the uniqueness of fixed point of T^m, we have $Tu_* = u_*$. Therefore, u_* is a fixed point of T. Since fixed point of T is also fixed point of T^m, hence T has a unique fixed point in X_{ω^G}. □

Theorem 3.5. Let (X, ω^G) be a complete modular G-metric space with a preorder, $\preceq$ and a nondecreasing self-mapping $T: X_{\omega^G} \to X_{\omega^G}$ on X_{ω^G} such that for each $\lambda > 0$, there is $\nu(\lambda) \in [0, \lambda)$ such that the following conditions hold:

(1)

$$\psi\left(\omega_\lambda^G(Tx, Ty, Tz)\right) \leq \alpha\left(\psi\left(\omega_\lambda^G(x,y,z)\right)\right)\psi\left(\omega_{\lambda+\nu(\lambda)}^G(x,y,z)\right) + \beta\left(\psi\left(\omega_\lambda^G(x,y,z)\right)\right.$$
$$\left.\times\right)\psi\left(\omega_\lambda^G(x, Tx, Tx)\right) + \gamma\left(\psi\left(\omega_\lambda^G(x,y,z)\right)\right)\psi\left(\omega_\lambda^G(y, Ty, Ty)\right) + \delta\left(\psi\left(\omega_\lambda^G(x,y,z)\right)\right.$$
$$\left.\times\right)\psi\left(\omega_\lambda^G(z, Tz, Tz)\right), \tag{3.48}$$

where $\psi \in \overline{\Psi}$ and $\{\alpha, \beta, \gamma, \delta\} \in \mathcal{S}_{Ger}$ with $\alpha(t) + 2\max\{\sup_{t\geq 0}\beta(t), \sup_{t\geq 0}\gamma(t), \sup_{t\geq 0}\delta(t)\} < 1$, and $x, y, z \in X_{\omega^G}$. Assuming that if a nondecreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x, then $x_n \preceq x$ for each $n \in \mathbb{N}$,

(2) if ψ is subadditive and for any $x, y, z \in X_{\omega^G}$, there exists $w \in X_{\omega^G}$ with $w \preceq Tw$ and $\omega_\lambda^G(w, Tw, Tw)$ is finite for all $\lambda > 0$ such that w is comparable to both x, y and z. Then T has a fixed point $u \in X_{\omega^G}$ and the sequence define by $\{T^n x_0\}_{n\geq 1}$ converges to u. Moreover, the fixed point of T is unique.

Proof. Let $x_0 \in X_{\omega^G}$ be such that $x_0 \preceq Tx_0$ and let $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}$. Regarding that T is nondecreasing mapping, we have that $x_0 \preceq Tx_0 = x_1$, which implies that $x_1 = Tx_0 \preceq Tx_1 = x_2$. Inductively, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_{n-1} \preceq x_n \preceq x_{n+1} \preceq \cdots. \tag{3.49}$$

Assume that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$. Since $x_{n_0} = x_{n_0+1} = Tx_{n_0}$, then x_{n_0} is the fixed point of T. Now suppose that $x_n \precneqq x_{n+1}$ for all $n \in \mathbb{N}$, thus by inequality (3.38), we have that

$$x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{n-1} \prec x_n \prec x_{n+1} \prec \cdots. \tag{3.50}$$

Now for each $\lambda > 0$, and $x_0 \prec Tx_0$ for all $n \in \mathbb{N}$ implies that $\omega_\lambda^G(x_0, Tx_0, Tx_0) > 0$. Again, let $x_0 \in X_{\omega^G}$ such that $\omega_\lambda^G(x_0, Tx_0, Tx_0) < \infty \ \forall \ \lambda > 0$.

First, we show that for all $n \in \mathbb{N}$, the sequence $\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) = 0$ for all $\lambda > 0$, as $n \to \infty$.

Assume that, for each $n \in \mathbb{N}$, there exists $\lambda_n > 0$ such that $\omega_{\lambda_n}^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0) \neq 0$. Otherwise there is nothing to prove. Suppose that for each $n \geq 1$, if $0 < \lambda < \lambda_n$, then we have that $\omega_\lambda^G(T^n x_0, T^n x_0, T^{n+1}x_0) \neq 0$. Since $T^n x_0 \preceq T^{n+1}x_0$, we have from inequality (3.5) that $\psi(\omega_{\lambda_n}^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)) \leq \psi(\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)) = \psi(\omega_\lambda^G(TT^{n-1}x_0, TT^n x_0, TT^n x_0))$. Take $x = T^{n-1}x_0$ and $y = T^n x_0 = z$, then inequality (3.5) becomes;

$$\begin{aligned}
&\psi\left(\omega_{\lambda_n}^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)\right) \leq \psi(\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)) \\
&\leq \alpha(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_{\lambda+\nu(\lambda)}^G(T^{n-1}x_0, T^n x_0, T^n x_0)) \\
&+ \beta(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_\lambda^G(T^{n-1}x_0, TT^{n-1}x_0, TT^{n-1}x_0)) \\
&+ \gamma(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_\lambda^G(T^n x_0, TT^n x_0, TT^n x_0)) \\
&+ \delta(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_\lambda^G(T^n x_0, TT^n x_0, TT^n x_0)) \\
&= \alpha(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_{\lambda+\nu(\lambda)}^G(T^{n-1}x_0, T^n x_0, T^n x_0)) \\
&+ \beta(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0)) \\
&+ \gamma(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)) \\
&+ \delta(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)) \\
&\leq \alpha(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0)) \\
&+ \beta(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0)) \\
&+ \gamma(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)) \\
&+ \delta(\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0))) \times \psi(\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)),
\end{aligned} \tag{3.51}$$

which implies that

$$\begin{aligned}
\psi(\omega_\lambda^G(T^n x_0, T^{n+1}x_0, T^{n+1}x_0)) &\leq \rho\psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0)) \\
&\leq \psi(\omega_\lambda^G(T^{n-1}x_0, T^n x_0, T^n x_0)) \\
&\ \ \vdots \\
&\leq \psi(\omega_\lambda^G(x_0, Tx_0, Tx_0)) < \infty,
\end{aligned} \tag{3.52}$$

where

$$\rho := \frac{\alpha\big(\psi\big(\omega_\lambda^G\big(T^{n-1}x_0,\ T^n x_0,\ T^n x_0\big)\big)\big) + \beta\big(\psi\big(\omega_\lambda^G\big(T^{n-1}x_0,\ T^n x_0,\ T^n x_0\big)\big)\big)}{1 - \big(\gamma\big(\psi\big(\omega_\lambda^G\big(T^{n-1}x_0,\ T^n x_0,\ T^n x_0\big)\big)\big) + \delta\big(\psi\big(\omega_\lambda^G\big(T^{n-1}x_0,\ T^n x_0,\ T^n x_0\big)\big)\big)\big)}. \tag{3.53}$$

Therefore, $\{\psi(\omega_\lambda^G(T^n x_0,\ T^{n+1}x_0,\ T^{n+1}x_0))\}_{n\geq 1}$ is nonincreasing and bounded below, hence converges to some real number $s \geq 0$. We can also see clearly that by taking $\theta(.) := \gamma(\psi(\omega_\lambda^G(T^{n-1}x_0,\ T^n x_0,\ T^n x_0))) + \delta(\psi(\omega_\lambda^G(T^{n-1}x_0,\ T^n x_0,\ T^n x_0)))$, as $y = T^n x_0 = z$, then following Theorem 3.1, T has a unique fixed point in X_{ω^G}. This complete the proof. □

Theorem 3.6. Let $(X,\ \omega^G)$ be a complete modular G-metric space with a preorder, $\preceq$ and a nondecreasing self-mapping $T: X_{\omega^G} \to X_{\omega^G}$ on X_{ω^G} such that for each $\lambda > 0$, there is $\nu(\lambda) \in [0,\ \lambda)$ such that the following conditions hold:

(1)

$$\begin{aligned}\psi\big(\omega_\lambda^G(T^m x,\ T^m y,\ T^m z)\big) \leq\ & \alpha\big(\psi\big(\omega_\lambda^G(x,\ y,\ z)\big)\big)\psi\big(\omega_{\lambda+\nu(\lambda)}^G(x,\ y,\ z)\big) \\ & + \beta\big(\psi\big(\omega_\lambda^G(x,\ y,\ z)\big)\big)\psi\big(\omega_\lambda^G(x,\ T^m x,\ T^m x)\big) + \gamma\big(\psi\big(\omega_\lambda^G(x,\ y,\ z)\big)\big)\psi\big(\omega_\lambda^G(y,\ T^m y,\ T^m y)\big) \\ & + \delta\big(\psi\big(\omega_\lambda^G(x,\ y,\ z)\big)\big)\psi\big(\omega_\lambda^G(z,\ T^m z,\ T^m z)\big),\end{aligned} \tag{3.54}$$

where $\psi \in \overline{\Psi}$ and $\{\alpha,\ \beta,\ \gamma,\ \delta\} \in \mathcal{S}_{Ger}$ with $\alpha(t) + 2\max\{\sup_{t\geq 0}\beta(t),\ \sup_{t\geq 0}\gamma(t),\ \sup_{t\geq 0}\delta(t)\} < 1$, and $x,\ y,\ z \in X_{\omega^G}$. Assuming that if a nondecreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x, then $x_n \preceq x$ for each $n \in \mathbb{N}$;

(2) if ψ is subadditive and for any $x,\ y,\ z \in X_{\omega^G}$, there exists $w \in X_{\omega^G}$ with $w \preceq Tw$ and $\omega_\lambda^G(w,\ Tw,\ Tw)$ is finite for all $\lambda > 0$ such that w is comparable to both x, y and z. Then T has a fixed point $u \in X_{\omega^G}$ for some positive integer $m \geq 1$ and the sequence define by $\{T^n x_0\}_{n\geq 1}$ converges to u. Moreover, the fixed point of T is unique.

Proof: Take $y = z$ and $\phi() = \gamma(\psi(\omega_\lambda^G(x,\ y,\ z))) + \delta(\psi(\omega_\lambda^G(x,\ y,\ z)))$, then Theorem 3.5 tells us that T^m has a fixed point say $u \in X_{\omega^G}$ for some positive integer $m \geq 1$. Therefore, Theorem 3.4 shows that T has a unique fixed point in X_{ω^G}. □

4. Applications to nonlinear Volterra-Fredholm-type integral equations

In this section, we construct a system of nonlinear integral equation that satisfies the conditions of Theorem 3.1. We consider the following general nonlinear Volterra-Fredholm-type integral equations.

$$u(t,\ x) = h(t,\ x) + \int_0^t \int_B F(t,\ x,\ s,\ y,\ u(s,\ y),\ (L^* u)(s,\ y))dyds, \tag{4.1}$$

and

$$v(t,\ x) = e(t,\ x) + \int_0^t \int_B G(t,\ x,\ s,\ y,\ v(s,\ y),\ (L^* v)(s,\ y))dsdy, \tag{4.2}$$

where

$$(L^*u)(t,x) = \int_0^t \int_B K(t,x,\tau,z,u(\tau,z))dzd\tau, \tag{4.3}$$

and

$$(L^*v)(t,x) = \int_0^t \int_B K(t,x,\tau,z,v(\tau,z))dzd\tau, \tag{4.4}$$

h, F, K and e, G, L are given functions and u, v are the unknown functions. We assume that $h, e \in C(\mathbb{R}_+ \times B, \mathbb{R}^n)$, $K \in C(\Omega \times \mathbb{R}^n, \mathbb{R}^n)$, $F \in C(\Omega \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $G \in C(\Omega \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and $\Omega = \{(t,x,s,y) : 0 \leq s \leq t < \infty;\ x, y \in B\}$, $B = \prod_{j=1}^n [a_j, b_j]$, $b_j > a_j$. Take $\sup_{(t,x)\in\mathbb{R}_+\times B} g(s,y) \leq \frac{1}{t\prod_{j=1}^n [a_j,b_j]}$, where $(t,x) \in \mathbb{R}_+ \times B$.

Let $\alpha, \beta, \gamma > 0$ with $\alpha(t) + 2\max_{t\in\Omega}\{\sup_{t\geq 0}\beta(t), \sup_{t\geq 0}\gamma(t)\}\langle 1$ such that

$$\|F(t,x,s,y,u(s,y),(L^*u)(s,y)) - G(t,x,s,y,v(s,y),(L^*v)(s,y))\|$$
$$\leq g(s,y)\{\alpha(\|u-v\|)\|u-v\| + \beta(\|u-v\|)m(u,L^*u) + \gamma(\|u-v\|)r(v,L^*v)\}$$

Let $F, G : C(\Omega \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n) \to \mathbb{R}^n$ be such that $F_u, G_v \in C(\Omega \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and let

$$F_u = \int_0^t \int_B F(t,x,s,y,u(s,y),(L^*u)(s,y))dyds, \tag{4.5}$$

and

$$G_v = \int_0^t \int_B G(t,x,s,y,v(s,y),(L^*v)(s,y))dsdy, \tag{4.6}$$

for $F \in C(\Omega \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $G \in C(\Omega \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$ and u, v are the unknown functions.

Now for any $\lambda > 0$, we define

$$\omega_\lambda^G(x,y,z) := \frac{1}{2(1+\lambda)} \sup_{(t,x)\in\mathbb{R}_+\times B}\{\|x(t)-y(t)\| + \|y(t)-z(t)\| + \|x(t)-z(t)\|\}, \tag{4.7}$$

so that

$$\omega_\lambda^G(x,y,y) := \frac{1}{(1+\lambda)} \sup_{(t,x)\in\mathbb{R}_+\times B}\{\|x(t)-y(t)\|\}. \tag{4.8}$$

In fact Eqns. (4.7) and (4.8) satisfies all the conditions in Definition 2.6 endowed with $X_{\omega^G} = (X, \omega^G) = C(\Omega \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$

Now, take $A = \frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\{\|F_u - G_v + h(t,x) - e(t,x)\|\}$, so that

$$\begin{aligned}
A &\le \frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\left\{\int_0^t\int_B \|F(t,x,s,y,u(s,y),(L^*u)(s,y))\right.\\
&\left.-G(t,x,s,y,v(s,y),(L^*v)(s,y))\|\times dyds\right\}\\
&\le \frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\left\{\int_0^t\int_B g(s,y)\{\alpha(\|u-v\|)\|u-v\|\right.\\
&\left.+\beta(\|u-v\|)m(u,L^*u)+\gamma(\|u-v\|)r(v,L^*v)\}dyds\right\}\\
&\le \frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\{\{\alpha(\|u-v\|)\|u-v\|+\beta(\|u-v\|)m(u,L^*u)\\
&+\gamma(\|u-v\|)\times r(v,L^*v)\}\}\\
&\le \frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\{\alpha(\|u-v\|)\|u-v\|\}\\
&+\frac{1}{1+\lambda}\sup_{t\in\Omega}\{\beta(\|u-v\|)m(u,L^*u)+\gamma(\|u-v\|)\times r(v,L^*v)\}\\
&\le \frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\{\alpha(\|u-v\|)\|u-v\|\}\\
&+\frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\{\beta(\|u-v\|)m(u,L^*u)\}\\
&+\frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\{\gamma(\|u-v\|)r(v,L^*v)\},
\end{aligned}\tag{4.9}$$

where $m, r \in C(\mathbb{R}_+\times B\times\mathbb{R}^n, \mathbb{R}^n)$

$$m(u,L^*u)(t,x) = \sup_{(t,x)\in\mathbb{R}_+\times B}\|F_u(t,x)+h(t,x)-u(t,x)\|,\tag{4.10}$$

$$r(v,L^*v)(t,x) = \sup_{(t,x)\in\mathbb{R}_+\times B}\|G_v(t,x)+e(t,x)-v(t,x)\|\tag{4.11}$$

Theorem 4.1. Let $X_{\omega^G} = C(\Omega\times\mathbb{R}^n\times\mathbb{R}^n, \mathbb{R}^n)$ be a complete modular G-metric space and $\omega^G : (0,\infty)\times X_{\omega^G}\times X_{\omega^G}\times X_{\omega^G}\to\mathbb{R}^n_+\cup\{\infty\}$ be defined by

$$\omega^G_\lambda(u,v,v) := \frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\text{\normu}(t,x) - v(t,x),\ \lambda > 0 \tag{4.12.}$$

and $u \le v \Leftrightarrow u(t,x) \le v(t,x)\ \ \forall\ \ (t,x)\in\mathbb{R}_+\times B$. Let $F_u, G_v : C(\Omega\times\mathbb{R}^n\times\mathbb{R}^n, \mathbb{R}^n)\to\mathbb{R}^n$ are such that $F_u, G_v \in X_{\omega^G}$ for each $u, v\in X_{\omega^G}$ and F_u, G_v satisfying Eqns. (4.5) and (4.6), respectively, for all $(t,x)\in\mathbb{R}_+\times B$. Suppose that there exists nonnegative reals $\alpha, \beta, \gamma > 0$ with $\alpha(t) + 2\max_{t\in\Omega}\{\sup_{t\ge 0}\beta(t), \sup_{t\ge 0}\gamma(t)\} < 1$ such that inequality 4 is satisfied for every $u, v\in X_{\omega^G}$. Moreover if ψ is subadditive and for any $u, v\in X_{\omega^G}$, there exists $w_0, w_1\in X_{\omega^G}$ with $w_0\le w_1$ and $\omega^G_\lambda(w_0, w_1, w_1)$ is finite for all $\lambda > 0$ such that w is comparable to both u and v. Then the system of integral Eqns (4.1) and (4.2) have a unique solution in X_{ω^G}.

Proof. Define the mapping $T: X_{\omega^G} \to X_{\omega^G}$ by $Tu = F_u + e$ and $Tv = G_v + h$. Then for $\lambda > 0$, $\omega_\lambda^G(Tu, Tv, Tv) = \frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\backslash normF_u(t, x) - G_v(t, x) + e(t, x) - h(t, x)$, $\omega_\lambda^G(u, Tu, Tu) = \frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\backslash normF_u(t, x) + e(t, x) - u(t, x)$ and $\omega_\lambda^G(v, Tv, Tv) = \frac{1}{1+\lambda}\sup_{(t,x)\in\mathbb{R}_+\times B}\backslash normG_v(t, x) + h(t, x) - v(t, x)$. So from inequality 4, we get by noticing that ψ is continuous and subadditive and there exists $\nu \in [0, \lambda)$ such that

$$\begin{aligned}\psi(\omega_\lambda^G(Tu, Tv, Tv)) &\le \alpha(\psi(\omega_\lambda^G(u, v, v)))\psi(\omega_{\lambda+\nu(\lambda)}^G(u, v, v)) \\ &+ \beta(\psi(\omega_\lambda^G(u, v, v)))\psi(\omega_\lambda^G(u, Tu, Tu)) + \gamma(\psi(\omega_\lambda^G(u,v,v)))\psi(\omega_\lambda^G(v, Tv, Tv)),\end{aligned} \tag{4.13}$$

where $\psi \in \overline{\Psi}$. By Theorem 3.1, we conclude that the system of nonlinear Volterra-Fredholm integral Eqns (4.1) and (4.2) have a unique solution in X_{ω^G}. □

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