Congruences modulo prime for fractional colour partition function

Congruences
for fractional
colour partition

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Abstract

Purpose – Let $p_{[1,r,t]}(n)$ be defined by $\sum_{n=0}^{\infty} p_{[1,r,t]}(n)q^n = (E_1 E_r)^t$, where t is a non-zero rational number, $r \geq 1$ is an integer and $E_r = \prod_{n=0}^{\infty} (1 - q^{r(n+1)})$ for $|q| < 1$. The function $p_{[1,r,t]}(n)$ is the generalisation of the two-colour partition function $p_{[1,r,-1]}(n)$. In this paper, the authors prove some new congruences modulo odd prime ℓ by taking $r = 5, 7, 11$ and 13 , and non-integral rational values of t .

Design/methodology/approach – Using q -series expansion/identities, the authors established general congruence modulo prime number for two-colour partition function.

Findings – In the paper, the authors study congruence properties of two-colour partition function for fractional values. The authors also give some particular cases as examples.

Originality/value – The partition functions for fractional value is studied in 2019 by Chan and Wang for Ramanujan's general partition function and then extended by Xia and Zhu in 2020. In 2021, Baruah and Das also proved some congruences related to fractional partition functions previously investigated by Chan and Wang. In this sequel, some congruences are proved for two-colour partitions in this paper. The results presented in the paper are original.

Keywords Integer partition, Fractional colour partition function, Partition congruence

Paper type Research paper

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1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers called parts, whose sum equals n . The number of partition of a non-negative integer n is usually denoted by $p(n)$ with $p(0) = 1$. The generating function of $p(n)$ (due to Euler) is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}}, \quad (1.1)$$

where for any complex number a ,

$$(a; q)_{\infty} = \prod_{n=0}^{\infty} (1 - aq^n), \quad |q| < 1. \quad (1.2)$$

For convenience, for any positive integer r , we write

$$E_r = (q^r; q^r)_{\infty}.$$

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Ramanujan [1, 2] found following three beautiful congruences modulo 5, 7 and 11 for the partition function $p(n)$:

$$p(5n + 4) \equiv 0 \pmod{5}, \quad (1.3)$$

$$p(7n + 5) \equiv 0 \pmod{7}, \quad (1.4)$$

and

$$p(11n + 6) \equiv 0 \pmod{11}. \quad (1.5)$$

In a letter to Hardy written from Fitzroy House late in 1918 [3, pp. 192–193], Ramanujan introduced the general partition function $p_t(n)$ for any non-negative integer n and non-zero integer t as

$$\sum_{n=0}^{\infty} p_t(n) q^n = \frac{1}{E_1^t}. \quad (1.6)$$

For $t = 1$, $p_1(n)$ is the usual unrestricted partition function $p(n)$ defined in (1.1). Ramanujan [3] further claimed that, if λ is a positive integer and $\bar{w}$ is a prime of the form $6\lambda - 1$, then

$$p_{-4}\left(n\bar{w} - \frac{(\bar{w} + 1)}{6}\right) \equiv 0 \pmod{\bar{w}}. \quad (1.7)$$

After Ramanujan, the congruence properties of the partition function $p_t(n)$ have been studied by many authors. For some recent works on $p_t(n)$, we refer to [4–13] and references therein.

Chan and Wang [14] studied the partition function $p_t(n)$ for non-integral rational values of t and found many congruences. This paper closely follows the techniques of Chan and Wang. Xia and Zhu [15] proved many of the congruences conjectured in [14]. Recently, Baruah and Das [16] proved some new families of congruences modulo powers of primes for $p_t(n)$ for non-integral rational values of t . Baruah and Das [16] also investigated another general partition function $p_{[1,r;t]}(n)$ which is defined by

$$\sum_{n=0}^{\infty} p_{[1,r;t]}(n) = (E_1 E_r)^t \quad (1.8)$$

for non-integral fractional values of t . The partition function $p_{[1,r;t]}(n)$ is the generalisation of the two-colour partition function $p_{[1,r;-1]}(n)$, one of the colours appears only in parts that are multiples of r . The partition function $p_{[1,r;-1]}(n)$ and other related functions have been studied by many authors. We refer [17–24] and references therein for details.

In [16], Baruah and Das established three congruences for $p_{[1,r;t]}(n)$ by taking $r = 2, 3$ and 4, and $t = -a/b$ with $\gcd(a, b) = 1$. For example, Baruah and Das [16, Theorem 1.13] proved that:

Theorem. (Baruah & Das) [16, Theorem 1.13] Suppose $a, b, d \in \mathbb{Z}$, $b \geq 1$ and $\gcd(a, b) = 1$. Let ℓ be an odd prime divisor of $a + db$ and $0 \leq r < \ell$. Suppose d, ℓ and r satisfy the following conditions: $d = 3$, $\ell \equiv 5$ or $11 \pmod{12}$ and $2r + 1 \equiv 0 \pmod{\ell}$. Then for all $n \geq 0$,

$$p_{[1,3;-a/b]}(\ell n + r) \equiv 0 \pmod{\ell}.$$

In this paper, we will prove four new congruences for $p_{[1,r;t]}(n)$ modulo an odd prime ℓ by taking $r = 5, 7, 11$ and 13 in (1.8) which are analogous to the above theorem. In order to verify some of our congruences, we list following q -series expansions:

$$(E_1 E_5)^{-1/4} = 1 + \frac{q}{4} + \frac{13}{32} q^2 + \frac{55}{128} q^3 + \frac{1235}{2048} q^4 + \frac{6663}{8192} q^5 + \frac{59897}{65536} q^6 + \frac{226967}{262144} q^7 + \frac{9878595}{8388608} q^8 + \frac{41862795}{33554432} q^9 + \frac{514093347}{268435456} q^{10} + O[q]^{11}, \quad (1.9)$$

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$$(E_1 E_7)^{-1/3} = 1 + \frac{q}{3} + \frac{5}{9} q^2 + \frac{50}{81} q^3 + \frac{215}{243} q^4 + \frac{646}{729} q^5 + \frac{8711}{6561} q^6 + \frac{32036}{19683} q^7 + \frac{112486}{59049} q^8 + \frac{3386515}{1594323} q^9 + \frac{12675397}{4782969} q^{10} + O[q]^{11}, \quad (1.10)$$

$$(E_1 E_{11})^{-1/2} = 1 + \frac{q}{2} + \frac{7}{8} q^2 + \frac{17}{16} q^3 + \frac{203}{128} q^4 + \frac{455}{256} q^5 + \frac{2723}{1024} q^6 + \frac{6001}{2048} q^7 + \frac{133107}{32768} q^8 + \frac{312011}{65536} q^9 + \frac{1613529}{262144} q^{10} + O[q]^{11}, \quad (1.11)$$

$$(E_1 E_{13})^{1/2} = 1 - \frac{q}{2} - \frac{5}{8} q^2 - \frac{5}{16} q^3 - \frac{45}{128} q^4 + \frac{33}{256} q^5 - \frac{209}{1024} q^6 + \frac{755}{2048} q^7 + \frac{1155}{32768} q^8 + \frac{15045}{65536} q^9 + \frac{45077}{262144} q^{10} + O[q]^{11} \quad (1.12)$$

and

$$(E_1 E_{13})^{-5/6} = 1 + \frac{5}{6} q + \frac{115}{72} q^2 + \frac{2915}{1296} q^3 + \frac{112585}{31104} q^4 + \frac{889249}{186624} q^5 + \frac{49269935}{6718464} q^6 + \frac{381199585}{40310784} q^7 + \frac{26238866555}{1934917632} q^8 + \frac{1853700203575}{104485552128} q^9 + \frac{30342040805197}{1253826625536} q^{10} + O[q]^{11}, \quad (1.13)$$

where $O[q]^{11}$ contains the terms involving q^{11} or higher powers of q .

2. New congruences for $p_{[1,r;\ell]}(n)$

In this section, we prove four new congruence modulo an odd prime ℓ . To prove our congruences, we employ the following q -series identity from [25]:

$$E_1^3 = \sum_{n=-\infty}^{\infty} (4n+1) q^{((4n+1)^2-1)/8} \quad (2.1)$$

We also require the following congruence which follows from binomial theorem (or see [26]): For prime ℓ and integer $k \geq 1$,

$$E_k^\ell \equiv E_{k\ell} \pmod{\ell}. \quad (2.2)$$

Theorem 2.1. *Suppose a and b are relatively prime integers with $b \geq 1$. Let ℓ be an odd prime divisor of $a + 3b$, and let r be an integer with $0 \leq r < \ell$. Suppose ℓ and r satisfy any of the following two conditions:*

- (1) $4r + 3 \equiv 0 \pmod{\ell}$, $\ell \equiv 2$ or $3 \pmod{5}$ and $\ell \equiv 1 \pmod{4}$,
- (2) $4r + 3 \equiv 0 \pmod{\ell}$, $\ell \equiv 1$ or $4 \pmod{5}$ and $\ell \equiv 3 \pmod{4}$.

Then for all $n \geq 0$,

$$p_{[1,5;-a/b]}(\ell n + r) \equiv 0 \pmod{\ell}. \quad (2.3)$$

Proof. Since $\ell|(a + db)$, so we can write $a + db = \ell m$ for some integer m . Also, since $\gcd(a, b) = 1$, it follows that $\gcd(b, \ell) = 1$. Setting $t = -a/b$ in (1.8), we find that

$$\sum_{n=0}^{\infty} p_{[1,5;-a/b]}(n)q^n = (E_1E_5)^{-a/b} = \frac{(E_1E_5)^d}{(E_1E_5)^{\ell m/b}}. \quad (2.4)$$

Employing (2.2) in (2.4), we obtain

$$\sum_{n=0}^{\infty} p_{[1,5;-a/b]}(n)q^n \equiv \frac{(E_1E_5)^d}{(E_1E_5)^{m/b}} \pmod{\ell}. \quad (2.5)$$

For $d = 3$, from (2.1), we find that

$$(E_1E_5)^3 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (4n+1)(4m+1)q^{((4n+1)^2+5(4m+1)^2-6)/8}. \quad (2.6)$$

We note that

$$N = ((4n+1)^2 + 5(4m+1)^2 - 6)/8$$

is equivalent to

$$8N + 6 = (4n+1)^2 + 5(4m+1)^2.$$

If

$$\ell \equiv 2 \text{ or } 3 \pmod{5} \quad \text{and} \quad \ell \equiv 1 \pmod{4},$$

or

$$\ell \equiv 1 \text{ or } 4 \pmod{5} \quad \text{and} \quad \ell \equiv 3 \pmod{4}.$$

then $\left(\frac{-5}{\ell}\right) = -1$, where here and throughout the paper $\left(\frac{\cdot}{\cdot}\right)$ denotes the Legendre symbol. So it follows that

$$8N + 6 \equiv 0 \pmod{\ell},$$

or

$$4N + 3 \equiv 0 \pmod{\ell},$$

if and only if $4n+1 \equiv 0 \pmod{\ell}$ and $4m+1 \equiv 0 \pmod{\ell}$. So the congruences (2.3) now follows by employing (2.6) and then comparing the coefficient of $q^{\ell n+r}$. $\square$

Corollary 2.2. *We have*

- (i) $p_{[1,5;-1/4]}(13n+9) \equiv 0 \pmod{13}$,
- (ii) $p_{[1,5;-2/3]}(11n+2) \equiv 0 \pmod{11}$,
- (iii) $p_{[1,5;-2/5]}(17n+12) \equiv 0 \pmod{17}$,
- (iv) $p_{[1,5;-7/8]}(31n+7) \equiv 0 \pmod{31}$.

Proof. (i) For $\ell \equiv 3 \pmod{5}$ and $\ell \equiv 1 \pmod{4}$, take $\ell = 13$. From [27, p. 176, Theorem 9.2(c)], we note that if p is an odd prime and a is a integer relatively prime to p , then

$$\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}. \quad (2.7)$$

From (2.7), it follows that

$$\left(\frac{-5}{13}\right) \equiv (-5)^{\frac{13-1}{2}} \pmod{13}.$$

Since $\frac{13-1}{2}$ is an even integer, we have

$$\left(\frac{-5}{13}\right) \equiv (5)^{\frac{13-1}{2}} \pmod{13}, \quad (2.8)$$

$$\equiv \left(\frac{5}{13}\right) \pmod{13}, \quad (2.9)$$

As Legendre symbol take values 1 and -1 only, (2.16) gives

$$\left(\frac{-5}{13}\right) = \left(\frac{5}{13}\right). \quad (2.10)$$

Using Quadratic Reciprocity Law for Legendre symbol [27, p. 186], we find that

$$\left(\frac{5}{13}\right) \left(\frac{13}{5}\right) = (-1)^{\frac{13-1}{2} \cdot \frac{5-1}{2}} = 1.$$

Since $13 \equiv 3 \pmod{5}$,

$$\left(\frac{13}{5}\right) = \left(\frac{3}{5}\right) \equiv 3^{\frac{5-1}{2}} \equiv 1 \pmod{5}.$$

So from (2.10), $\left(\frac{-5}{13}\right) = -1$. Now set $a = 1$ and $b = 4$ so that $a + 3b$ is divisible by 13. Also, if $r = 9$, then $4r + 3 \equiv 0 \pmod{13}$. Thus, (i) follows from Theorem 2.1.

Similarly, (ii)-(iv) follows from Theorem 2.1 with $\{a = 2, b = 3, \ell = 11, r = 2\}$, $\{a = 1, b = 4, \ell = 13, r = 9\}$, $\{a = 2, b = 5, \ell = 17, r = 12\}$ and $\{a = 7, b = 8, \ell = 31, r = 7\}$, respectively. $\square$

The congruence presented in Corollary 2.2(i) can be easily verified from the series expansion of $(E_1 E_5)^{-1/4}$ in (1.9).

Theorem 2.3. *Suppose a and b are relatively prime integers with $b \geq 1$. Let ℓ be an odd prime divisor of $a + 3b$ and let r be an integer with $0 \leq r < \ell$. Suppose ℓ and r satisfy any of the following two conditions:*

- (1) $r + 1 \equiv 0 \pmod{\ell}$, $\ell \equiv 3, 5, 6 \pmod{7}$ and $\ell \equiv 1 \pmod{4}$,
- (2) $r + 1 \equiv 0 \pmod{\ell}$, $\ell \equiv 3, 5, 6 \pmod{7}$ and $\ell \equiv 3 \pmod{4}$.

Then for all $n \geq 0$,

$$p_{[1,7,-a/b]}(\ell n + r) \equiv 0 \pmod{\ell}. \quad (2.11)$$

Proof. Since $\ell | (a + db)$, so we can write $a + db = \ell m$ for some integer m . Also, since $\gcd(a, b) = 1$, it follows that $\gcd(b, \ell) = 1$. Setting $t = -a/b$ in (1.8), we find that

$$\sum_{n=0}^{\infty} p_{[1,7,-a/b]}(n)q^n = (E_1E_7)^{-a/b} = \frac{(E_1E_7)^d}{(E_1E_7)^{dm/b}}. \quad (2.12)$$

Employing (2.2) in (2.12), we obtain

$$\sum_{n=0}^{\infty} p_{[1,7,-a/b]}(n)q^n \equiv \frac{(E_1E_7)^d}{(E_1E_7)^{dm/b}} \pmod{\ell}. \quad (2.13)$$

For $d = 3$, from (2.1), we find that

$$(E_1E_7)^3 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (4n+1)(4m+1)q^{((4n+1)^2+7(4m+1)^2-6)/8}. \quad (2.14)$$

We note that

$$N = \left((4n+1)^2 + 7(4m+1)^2 - 8 \right) / 8,$$

is equivalent to

$$8N + 8 = (4n+1)^2 + 7(4m+1)^2.$$

If

$$\ell \equiv 3, 5, 6 \pmod{7} \quad \text{and} \quad \ell \equiv 1 \pmod{4},$$

or

$$\ell \equiv 3, 5, 6 \pmod{7} \quad \text{and} \quad \ell \equiv 3 \pmod{4}.$$

then $\left(\frac{-7}{\ell}\right) = -1$. So it follows that

$$8N + 8 \equiv 0 \pmod{\ell},$$

or

$$N + 1 \equiv 0 \pmod{\ell},$$

if and only if $4n+1 \equiv 0 \pmod{\ell}$ and $4m+1 \equiv 0 \pmod{\ell}$. So the congruences (2.11) now follow by employing (2.14) and then comparing the coefficient of $q^{\ell n+r}$. $\square$

Corollary 2.4. *We have*

- (i) $p_{[1,7;-1/3]}(5n+4) \equiv 0 \pmod{5},$
- (ii) $p_{[1,7;-1/4]}(13n+12) \equiv 0 \pmod{13},$
- (iii) $p_{[1,7;-2/5]}(17n+16) \equiv 0 \pmod{17},$
- (iv) $p_{[1,7;-1/6]}(19n+18) \equiv 0 \pmod{19}.$

Proof. For $\ell \equiv 5 \pmod{7}$ and $\ell \equiv 1 \pmod{4}$, set $\ell = 5$. From (2.7), it follows that

$$\left(\frac{-7}{5}\right) \equiv (-7)^{\frac{5-1}{2}} \equiv (7)^{\frac{5-1}{2}} \equiv \left(\frac{7}{5}\right) \pmod{5}. \quad (2.15)$$

Using Quadratic Reciprocity Law, we have

$$\left(\frac{5}{7}\right) \left(\frac{7}{5}\right) = (-1)^{\frac{5-1}{2} \cdot \frac{7-1}{2}} = 1. \quad (2.16)$$

Now,

$$\left(\frac{5}{7}\right) \equiv 5^{\frac{7-1}{2}} \equiv -1 \pmod{7}. \quad (2.17)$$

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Therefore, by (2.16) $\left(\frac{-7}{5}\right) \equiv -1$. Take $a = 1$ and $b = 3$ so that $a + 3b$ is divisible by 5, and also if $r = 4$, then $r + 1 \equiv 0 \pmod{5}$. So (i) follows immediately from Theorem 2.3.

Similarly, to prove (ii)–(iv) we set $\{a = 1, b = 4, \ell = 13, r = 12\}$, $\{a = 2, b = 5, \ell = 17, r = 16\}$ and $\{a = 1, b = 6, \ell = 19, r = 18\}$ in Theorem 2.3, respectively. $\square$

The congruence presented in Corollary 2.4(i) can be easily verified from the series expansion of $(E_1 E_7)^{(-1/3)}$ in (1.10).

Theorem 2.5. *Suppose a and b are relatively prime integers with $b \geq 1$. Let ℓ be an odd prime divisor of $a + 3b$ and let r be an integer with $0 \leq r < \ell$. Suppose ℓ and r satisfy any of the following two conditions:*

- (1) $2r + 3 \equiv 0 \pmod{\ell}$, $\ell \equiv 2, 6, 7, 8, 10 \pmod{11}$ and $\ell \equiv 1 \pmod{4}$,
- (2) $2r + 3 \equiv 0 \pmod{\ell}$, $\ell \equiv 2, 6, 7, 8, 10 \pmod{11}$ and $\ell \equiv 3 \pmod{4}$.

Then for all $n \geq 0$,

$$p_{[1,11;-a/b]}(\ell n + r) \equiv 0 \pmod{\ell}. \quad (2.18)$$

Proof. Since $\ell | (a + db)$, so we can write $a + db = \ell m$ for some integer m . Also, since $\gcd(a, b) = 1$, it follows that $\gcd(b, \ell) = 1$. Setting $t = -a/b$ in (1.8), we find that

$$\sum_{n=0}^{\infty} p_{[1,11;-a/b]}(n) q^n = (E_1 E_{11})^{-a/b} = \frac{(E_1 E_{11})^d}{(E_1 E_{11})^{\ell m/b}}. \quad (2.19)$$

Employing (2.2) in (2.19), we obtain

$$\sum_{n=0}^{\infty} p_{[1,11;-a/b]}(n) q^n \equiv \frac{(E_1 E_{11})^d}{(E_{\ell} E_{11\ell})^{m/b}} \pmod{\ell}. \quad (2.20)$$

For $d = 3$, from (2.1), we find that

$$(E_1 E_{11})^3 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (4n+1)(4m+1) q^{((4n+1)^2 + 11(4m+1)^2 - 12)/8}. \quad (2.21)$$

We note that

$$N = \left((4n+1)^2 + 11(4m+1)^2 - 12 \right) / 8,$$

is equivalent to

$$8N + 12 = (4n+1)^2 + 11(4m+1)^2.$$

If

$$\ell \equiv 2, 6, 7, 8, 10 \pmod{11} \quad \text{and} \quad \ell \equiv 1 \pmod{4},$$

or

$$\ell \equiv 2, 6, 7, 8, 10 \pmod{11} \quad \text{and} \quad \ell \equiv 3 \pmod{4}.$$

then $\left(\frac{-11}{\ell}\right) = -1$. So it follows that

$$8N + 12 \equiv 0 \pmod{\ell},$$

or

$$2N + 3 \equiv 0 \pmod{\ell},$$

if and only if $4n + 1 \equiv 0 \pmod{\ell}$ and $4m + 1 \equiv 0 \pmod{\ell}$. So the congruences (2.18) now follow by employing (2.21) and then comparing the coefficient of $q^{\ell n + r}$. $\square$

Corollary 2.6. *We have*

- (i) $p_{[1,11;-1/2]}(7n + 2) \equiv 0 \pmod{7}$.
- (ii) $p_{[1,11;-2/5]}(17n + 7) \equiv 0 \pmod{17}$.
- (iii) $p_{[1,11;-5/8]}(29n + 13) \equiv 0 \pmod{29}$.
- (iv) $p_{[1,11;-8/11]}(41n + 19) \equiv 0 \pmod{41}$.

Proof. (i) For $\ell \equiv 7 \pmod{11}$ and $\ell \equiv 1 \pmod{4}$, take $\ell = 7$. From (2.7), it follows that

$$\left(\frac{-11}{7}\right) \equiv (-11)^{\frac{7-1}{2}} \equiv (11)^{\frac{7-1}{2}} \equiv \left(\frac{11}{7}\right) \pmod{7}.$$

Using Quadratic Reciprocity Law, we have

$$\left(\frac{11}{7}\right) \left(\frac{7}{11}\right) = (-1)^{\frac{11-1}{2} \cdot \frac{7-1}{2}} = 1. \quad (2.22)$$

Now,

$$\left(\frac{7}{11}\right) \equiv (7)^{\frac{11-1}{2}} \equiv -1 \pmod{5}.$$

Therefore, by (2.22) $\left(\frac{-11}{7}\right) \equiv -1$. Take $a = 1$ and $b = 2$ so that $a + 3b$ is divisible by 7, and also if $r = 2$, then $2r + 3 \equiv 0 \pmod{7}$. So (i) follows immediately from Theorem 2.5.

Similarly, we set $\{a = 2, b = 5, \ell = 17, r = 7\}$, $\{a = 5, b = 8, \ell = 29, r = 13\}$ and $\{a = 8, b = 11, \ell = 41, r = 19\}$ in Theorem 2.5 to arrive at (ii)–(iv), respectively. $\square$

The congruence in Corollary 2.6(i) can be easily verified from the series expansion of $(E_1 E_{11})^{-1/2}$ in (1.11).

Theorem 2.7. *Suppose a and b are relatively prime integers with $b \geq 1$. Let ℓ be an odd prime divisor of $a + 3b$ and let r be an integer with $0 \leq r < \ell$. Suppose ℓ and r satisfy any of the following two conditions:*

- (i) $4r + 7 \equiv 0 \pmod{\ell}$, $\ell \equiv 2, 5, 6, 7, 8, 11 \pmod{13}$ and $\ell \equiv 1 \pmod{4}$,
- (ii) $4r + 7 \equiv 0 \pmod{\ell}$, $\ell \equiv 1, 3, 4, 9, 10, 12 \pmod{13}$ and $\ell \equiv 3 \pmod{4}$.

Then for all $n \geq 0$,

$$p_{[1,13,-a/b]}(\ell n + r) \equiv 0 \pmod{\ell}. \quad (2.23)$$

Proof. Since $\ell | (a + db)$, so we can write $a + db = \ell m$ for some integer m . Also, since $\gcd(a, b) = 1$, it follows that $\gcd(b, \ell) = 1$. Setting $t = -a/b$ in (1.8), we find that

$$\sum_{n=0}^{\infty} p_{[1,13;-a/b]}(n)q^n = (E_1E_{13})^{-a/b} = \frac{(E_1E_{13})^d}{(E_1E_{13})^{\ell m/b}}. \quad (2.24)$$

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Employing (2.2) in (2.24), we obtain

$$\sum_{n=0}^{\infty} p_{[1,13;-a/b]}(n)q^n \equiv \frac{(E_1E_{13})^d}{(E_{\ell}E_{13\ell})^{m/b}} \pmod{\ell}. \quad (2.25)$$

For $d = 3$, from (2.1), we find that

$$(E_1E_{13})^3 = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (4n+1)(4m+1)q^{((4n+1)^2+13(4m+1)^2-14)/8}. \quad (2.26)$$

We note that

$$N = \left((4n+1)^2 + 13(4m+1)^2 - 14 \right) / 8,$$

is equivalent to

$$8N + 14 = (4n+1)^2 + 13(4m+1)^2.$$

Now if

$$\ell \equiv 2, 5, 6, 7, 8, 11 \pmod{13} \quad \text{and} \quad \ell \equiv 1 \pmod{4},$$

or

$$\ell \equiv 1, 3, 4, 9, 10, 12 \pmod{13} \quad \text{and} \quad \ell \equiv 3 \pmod{4},$$

then $\left(\frac{-13}{\ell}\right) = -1$. So it follows that

$$8N + 14 \equiv 0 \pmod{\ell},$$

or

$$4N + 7 \equiv 0 \pmod{\ell},$$

if and only if $4n+1 \equiv 0 \pmod{\ell}$ and $4m+1 \equiv 0 \pmod{\ell}$. So the congruences (2.23) now follows by employing (2.26) and then comparing the coefficient of $q^{\ell n+r}$. $\square$

Corollary 2.8. *We have*

- (i) $p_{[1,13;1/2]}(5n+2) \equiv 0 \pmod{5},$
- (ii) $p_{[1,13;-5/6]}(23n+4) \equiv 0 \pmod{23},$
- (iii) $p_{[1,13;-7/10]}(37n+26) \equiv 0 \pmod{37},$
- (iv) $p_{[1,13;-8/11]}(41n+29) \equiv 0 \pmod{41}.$

Proof. (i) For $\ell \equiv 5 \pmod{13}$ and $\ell \equiv 1 \pmod{4}$, set $\ell = 5$. From (2.7), it follows that

$$\left(\frac{-13}{5}\right) \equiv (-13)^{\frac{5-1}{2}} \equiv 13^{\frac{5-1}{2}} \equiv \left(\frac{13}{5}\right) \pmod{5}.$$

Using Quadratic Reciprocity Law, we have

$$\left(\frac{5}{13}\right) \left(\frac{13}{5}\right) = (-1)^{\frac{13-1}{2} \cdot \frac{5-1}{2}} = 1. \quad (2.27)$$

Now,

$$\left(\frac{5}{13}\right) \equiv 5^{\frac{13-1}{2}} \equiv -1 \pmod{13}.$$

Therefore, by (2.27) $\left(\frac{-13}{5}\right) \equiv -1$. Now set $a = -1$ and $b = 2$ so that $a + 3b$ is divisible by 5 and also if $r = 2$, then $4r + 7 \equiv 0 \pmod{5}$. So (i) follows immediately from Theorem 2.7.

(ii) For $\ell \equiv 10 \pmod{13}$ and $\ell \equiv 3 \pmod{4}$, take $\ell = 23$. From (2.7), it follows that

$$\left(\frac{-13}{23}\right) \equiv (-13)^{\frac{23-1}{2}} \equiv -(13)^{\frac{23-1}{2}} \equiv -\left(\frac{13}{23}\right) \pmod{23}.$$

Using Quadratic Reciprocity Law, we have

$$\left(\frac{23}{13}\right) \left(\frac{13}{23}\right) = (-1)^{\frac{13-1}{2} \cdot \frac{23-1}{2}} = 1. \quad (2.28)$$

Now,

$$\left(\frac{23}{13}\right) = \left(\frac{10}{13}\right) \equiv 10^{\frac{13-1}{2}} \equiv 1 \pmod{13}.$$

Therefore, (2.28) implies $\left(\frac{-13}{23}\right) \equiv -1$. Take $a = 5$ and $b = 6$ so that $a + 3b$ is divisible by 23, and also if $r = 4$, then $4r + 7 \equiv 0 \pmod{23}$. So (ii) follows immediately from Theorem 2.7.

Similarly, we set $\{a = 7, b = 10, \ell = 37, r = 26\}$ and $\{a = 8, b = 11, \ell = 41, r = 29\}$ in Theorem 2.7 to arrive at (iii)–(iv), respectively. $\square$

The congruences presented in Corollary 2.8(i) and (ii) can be verified from the series expansion of $(E_1 E_{13})^{1/2}$ and $(E_1 E_{13})^{-5/6}$ in (1.12) and (1.13), respectively.

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