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# Classical solutions for the generalized Kadomtsev–Petviashvili I equations

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## Abstract

**Purpose** – This paper is devoted to the generalized Kadomtsev–Petviashvili I equation. This study aims to propose a new approach for investigation for the existence of at least one global classical solution and the existence of at least two nonnegative global classical solutions. The main arguments in this paper are based on some recent theoretical results.

**Design/methodology/approach** – This paper is devoted to the generalized Kadomtsev–Petviashvili I equation. This study aims to propose a new approach for investigation for the existence of at least one global classical solution and the existence of at least two nonnegative global classical solutions. The main arguments in this paper are based on some recent theoretical results.

**Findings** – This paper is devoted to the generalized Kadomtsev–Petviashvili I equation. This study aims to propose a new approach for investigation for the existence of at least one global classical solution and the existence of at least two nonnegative global classical solutions. The main arguments in this paper are based on some recent theoretical results.

**Originality/value** – This article is devoted to the generalized Kadomtsev–Petviashvili I equation. This study aims to propose a new approach for investigation for the existence of at least one global classical solution and the existence of at least two nonnegative global classical solutions. The main arguments in this paper are based on some recent theoretical results.

**Keywords** Kadomtsev–Petviashvili equation, Existence, Classical solution

**Paper type** Research paper

## 1. Introduction

The Soviet physicists Boris Kadomtsev and Vladimir Petviashvili derived the equation that now bears their name, the Kadomtsev–Petviashvili equation (shortly the KP equation), as a model that describes the evolution of long ion-acoustic waves of small amplitude propagating in plasmas under the effect of long transverse perturbations. A particular case of this model is the Korteweg-de Vries (KdV) equation in the case of the absence of transverse dynamics. The KP equation is an extension of the classical KdV equation to two spatial dimensions, and it



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was used by Ablowitz and Segur for modeling of surface and internal water waves and for modeling in nonlinear optics, as well as in other physical settings.

The KP equation I can consider as a nonlinear partial differential equation in two spatial and one temporal coordinate. There are two distinct versions of the KP equation, which can be written in a normalized form in the following way:

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0,$$

where $u = u\,(x, y, t)$ is a scalar function, $x$ and $y$ are the longitudinal and transverse spatial coordinates, subscripts $x$, $y$, $t$ denote partial derivatives and $\sigma^2 = \pm 1$. When $\sigma = 1$, this equation is known as the KPII equation, and in this case, it models water waves with small surface tension. In the case when $\sigma = -1$, this equation is known as the KPI equation, and in this case, it models waves in thin films with high surface tension. In the references, the equation is often written with different coefficients in front of the various terms. Note that the particular values are inessential and they can be modified by appropriate rescaling of the dependent variables and of the independent variables.

This paper is devoted to the IVP for the generalized Kadomtsev–Petviashvilli I (gKP I) equation

$$\begin{aligned} \partial_x\left(\partial_t u + u\partial_x u + \mu^2 \partial_x^l u\right) + \nu \partial_{yy} u &= 0, \quad t > 0, \quad (x,y) \in \mathbb{R}^2, \\ u(0,x,y) &= u_0(x,y), \quad (x,y) \in \mathbb{R}^2, \end{aligned} \tag{1.1}$$

where

*H1.* $l \in \mathbb{N}, l \geq 5, u_0 \in \mathcal{C}^{l+1}(\mathbb{R}^2), |u_0(x,y)| \leq B, (x,y) \in \mathbb{R}^2, B > 0$ is a given constant.

*H2.* $\nu = \pm 1, \mu > 0$.

In the particular case, when $l = 5$, equation (1.1) is reduced to the fifth-order KP I equation and in the case $l = 3$, equation (1.1) is reduced to the KP equation.

In Ref. [1], when $l = 3$, the authors the local well-posedness for the Cauchy problem for the KP equation in certain Sobolev spaces. Generically, the solution of the KP equation develops a singularity in finite time $t$. It is discussed in Refs [2, 3] that this singularity develops at a point where the derivatives become divergent in all directions except one.

In Ref. [4], the authors established the local well-posedness of the Cauchy problem for the gKP I equation in anisotropic Sobolev spaces $H^{s_1,s_2}(\mathbb{R})$ when $s_1 > -\frac{\alpha-1}{4}$, $s_2 \geq 0$ and $\alpha \geq 4$, and global well-posedness in $H^{s_1,0}(\mathbb{R})$ when $s_1 > -\frac{(\alpha-1)(3\alpha-4)}{4(5\alpha+3)}$ and $4 \leq \alpha \leq 5$, as well as when $s_1 > -\frac{\alpha(3\alpha-4)}{4(5\alpha+4)}$ and $\alpha > 5$.

Mechanical systems with impact, heartbeats, blood flows, population dynamics, industrial robotics, biotechnology, economics, etc. are real-world and applied science phenomena which are abruptly changed in their states at some time instants due to short time perturbations whose duration is negligible in comparison with the duration of these phenomena. A natural framework for mathematical simulation of such phenomena is differential equations when more factors are taken into account, please see Refs [5–8].

This paper aims to investigate the IVP (1.1) for the existence of at least one and at least two global classical solutions. In addition of (*H*1) and (*H*2), suppose

*H3.* $g \in \mathcal{C}([0,\infty) \times \mathbb{R}^2)$ is a positive function on $[0,\infty) \times \mathbb{R}^2$ such that

$$g(0,x,y) = g(t,0,y) = g(t,x,0) = 0, \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2,$$

and

$$2^{2l+4}(l+1)!(1+t)^2\left(\sum_{r=0}^{l+1}|x|^r\right)\left(\sum_{r=0}^{l+1}|y|^r\right)$$

$$\int_0^t\left|\int_0^x\int_0^y g(t_1,x_1,y_1)dy_1dx_1\right|dt_1 \le A, \quad (t,x,y)\in[0,\infty)\times\mathbb{R}^2,$$

for some constant $A > 0$, and

*H4.* $\epsilon \in (0, 1)$, $A$ and $B$ satisfy the inequalities $\epsilon B_1(1 + A) < B$ and $AB_1 < 1$.

*H5.* Let $m > 0$ be large enough and $A$, $B$, $r$, $L$, $R_1$ be positive constants that satisfy the following conditions

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m}+1\right)L, \quad AB_1 < \frac{L}{5}.$$

In the last section, we will give an example of a function $g$ and constants $\epsilon, A, B, B_1, r, L, R_1$ and $m$ that satisfy (*H*3)–(*H*5).

With $X = \mathcal{C}^1([0,\infty)\times\mathcal{C}^{l+1}(\mathbb{R}^2))$, we denote the space of all continuous functions on $[0,\infty)\times\mathbb{R}^2$ so that $u_t$, $\partial_x^r u$, $\partial_{tx}u$, $\partial_y u$, $\partial_y^2 u$, $r = 1, \ldots, l+1$, exist and are continuous on $[0,\infty)\times\mathbb{R}^2$.

Our main result for existence of at least one global classical solution is as follows.

**Theorem 1.1.** *Suppose that* (*H*1)–(*H*4) *hold. Then the IVP* (1.1) *has at least one solution* $u \in X$.

Next theorem is our result for the existence of at least two global nonnegative classical solutions.

**Theorem 1.2.** *Suppose that* (*H*1)–(*H*3) *and* (*H*5) *hold. Then the IVP* (1.1) *has at least two nonnegative solutions* $u_1, u_2 \in X$.

The main idea for the proof of our main results is as follows. First, we find an integral representation of the solutions of the IVP (1.1). Then we construct a pair of operators so that any fixed point of their sum is a solution of the IVP (1.1). We find some *a-priori* estimates of the defined operators and using some fixed point theorems we conclude the existence of at least one global classical solution and the existence of at least two nonnegative classical solutions of the IVP (1.1).

The paper is organized as follows. In the next section, we give some auxiliary results. In Section 3, we give some preliminary results. In Section 4, we will prove our main results. In Section 5, we give an example to illustrate our main results.

## 2. Auxiliary results

In this section, as in Ref. [9], we will give some basic definitions and facts which will be used in this paper. Moreover, we will formulate the basic fixed-point theorems which we explore to prove our main results. For more details, we refer the reader to the papers [10–14] and references therein. To prove the existence of at least one global classical solution for the IVP (1.1), we will use the following fixed-point theorem.

**Theorem 2.1.** ([9, 12, 13]) *Suppose that the constants* $\epsilon$ *and* $B$ *are positive constants. Let* $E$ *be a Banach space and define the set* $X = \{x \in E\colon \|x\| \le B\}$ *and the operator* $Tx = -\epsilon x$, $x \in X$.

*Assume that the operator* $S : X \to E$ *is a continuous operator and the set* $(I-S)(X)$ *resides in a compact subset of E. Let also,*

$$\{x \in E : x = \lambda(I-S)x, \quad \|x\| = B\} = \varnothing \tag{2.1}$$

*for any* $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. *Then there exists* $x^* \in X$ *for which one has*

$$Tx^* + Sx^* = 0.$$

Below, assume that $X$ is a real Banach space. Now, we will recall the definition of a completely continuous operator in a Banach space.

**Definition 2.2.** [9] *A map* $K : X \to X$ *is called a completely continuous map if it is continuous and it maps any bounded set into a relatively compact set.*

For completeness, we will recall the definition of the Kuratowski measure of noncompactness, which will be used to be define $l$-set contraction mappings when $l \in \mathbb{N}_0$.

**Definition 2.3.** [9] *With* $\Omega_X$ *we will denote the class of all bounded sets of X. Then the Kuratowski measure of noncompactness* $\alpha : \Omega_X \to [0, \infty)$ *is defined by*

$$\alpha(Y) = \inf\left\{\delta > 0 : Y = \bigcup_{j=1}^{m} Y_j \quad \text{and} \quad \text{diam}(Y_j) \le \delta, \quad j \in \{1, \ldots, m\}\right\}.$$

*here, with diam* $(Y_j) = \sup\{\|x-y\|_X : x, y \in Y_j\}$ *we will denote the diameter of* $Y_j$, $j \in \{1, \ldots, m\}$.

For the main properties of the measure of noncompactness, we refer the reader to Ref. [10]. Now, we are ready to define an $l$-set contraction in a Banach space for any $l \in \mathbb{N}_0$.

**Definition 2.4.** [9] *A map* $K : X \to X$ *is called an l-set contraction if it is continuous, bounded and there exists a constant* $l \ge 0$ *for which one has the following inequality*

$$\alpha(K(Y)) \le l\alpha(Y)$$

*for any bounded set* $Y \subset X$. *The map K will be called a strict set contraction map if* $l < 1$.

Note that any completely continuous mapping $K : X \to X$ is a 0-set contraction (see Ref. [11], p. 264). Next, for our main results, we have a need for a definition of an expansive operator.

**Definition 2.5.** [9] *Let X and Y be real Banach spaces. A map* $K : X \to Y$ *is called expansive if there exists a constant* $h > 1$ *for which one has the following inequality*

$$\|Kx - Ky\|_Y \ge h\|x - y\|_X$$

*for any* $x, y \in X$.

Now, we will recall the definition of a cone in a Banach space.

**Definition 2.6.** [9] *A closed, convex set* $\mathcal{P}$ *in X is said to be cone if*

(1) $\alpha x \in \mathcal{P}$ *for any* $\alpha \ge 0$ *and for any* $x \in \mathcal{P}$,

(2) $x, -x \in \mathcal{P}$ *implies* $x = 0$.

Denote $\mathcal{P}^* = \mathcal{P} \backslash \{0\}$. The next result is a fixed-point theorem which we will use to prove the existence of at least two nonnegative global classical solutions of the IVP (1.1). For its proof, we refer the reader to the paper [14].

**Theorem 2.7.** *Let $\mathcal{P}$ be a cone of a Banach space E; $\Omega$ a subset of $\mathcal{P}$ and $U_1$, $U_2$ and $U_3$ three open bounded subsets of $\mathcal{P}$ such that $\overline{U}_1 \subset \overline{U}_2 \subset U_3$ and $0 \in U_1$. Assume that $T : \Omega \to \mathcal{P}$ is an expansive mapping with constant $h > 1$, $S : \overline{U}_3 \to E$ is a k-set contraction with $0 \le k < h - 1$ and $S(\overline{U}_3) \subset (I - T)(\Omega)$. Suppose that $(U_2 \backslash \overline{U}_1) \cap \Omega \neq \varnothing$, $(U_3 \backslash \overline{U}_2) \cap \Omega \neq \varnothing$, and there exists $u_0 \in \mathcal{P}^*$ such that the following conditions hold:*

(1) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_1 \cap (\Omega + \lambda u_0)$,*

(2) *There exists $\epsilon \ge 0$ such that $Sx \neq (I - T)(\lambda x)$, for all $\lambda \ge 1 + \epsilon$, $\quad x \in \partial U_2$ and $\lambda x \in \Omega$,*

(3) *$Sx \neq (I - T)(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial U_3 \cap (\Omega + \lambda u_0)$.*
*Then $T + S$ has at least two non-zero fixed points $x_1, x_2 \in \mathcal{P}$ such that*

$$x_1 \in \partial U_2 \cap \Omega \,\text{and}\, x_2 \in \left(\overline{U}_3 \backslash \overline{U}_2\right) \cap \Omega$$

or

$$x_1 \in (U_2 \backslash U_1) \cap \Omega \,\text{and}\, x_2 \in \left(\overline{U}_3 \backslash \overline{U}_2\right) \cap \Omega.$$

## 3. Preliminary results

In this section, we will define suitable operators and we will deduct some *a-priori* estimates which we will use to prove our main results. Let $X$ be endowed with the norm

$$\begin{aligned} \|u\| \quad = \quad \max\Bigg\{ & \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2} |u(t,x,y)|, \\ & \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2} |\partial_t u(t,x,y)|, \\ & \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2} |\partial_x^r u(t,x,y)|, \\ & \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2} |\partial_y^k u(t,x,y)|, \\ & r = 1, \ldots, l+1, \quad k = 1, 2, \qquad \sup_{(t,x,y)\in[0,\infty)\times\mathbb{R}^2} |\partial_{tx} u(t,x,y)| \Bigg\}, \end{aligned}$$

provided it exists. For $u \in X$, define the operator

$$\begin{aligned} S_1 u(t,x,y) \quad = \quad & u(t,x,y) - u_0(x,y) \\ & + \int_0^t \Big( -\partial_t u(t_1,x,y) + \partial_t \partial_x u(t_1,x,y) + (\partial_x u(t_1,x,y))^2 \\ & + u(t_1,x,y)\partial_{xx} u(t_1,x,y) \\ & + \mu^2 \partial_x^{l+1} u(t_1,x,y) + \nu \partial_{yy} u(t_1,x,y) \Big) dt_1, \\ & (t,x,y) \in [0,\infty) \times \mathbb{R}^2. \end{aligned}$$

In the next lemma, we will establish that any solution of an integral equation is a solution to the IVP (1.1).

**Lemma 3.1.** *Suppose that (H1) and (H2) hold. Let $u \in X$ be a solution of the equation*

$$S_1 u(t,x,y) = 0, \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2. \tag{3.1}$$

*Then it is a solution of the IVP (1.1).*

*Proof.* Let $u \in X$ be a solution of equation (3.1). Using the definition of $S_1$, we get the following integral equation

$$\begin{aligned} 0 &= u(t,x,y) - u_0(x,y) \\ &+ \int_0^t \Big( -\partial_t u(t_1,x,y) + \partial_t \partial_x u(t_1,x,y) + (\partial_x u(t_1,x,y))^2 \\ &+ u(t_1,x,y)\partial_{xx} u(t_1,x,y) \\ &+ \mu^2 \partial_x^{l+1} u(t_1,x,y) + \nu \partial_{yy} u(t_1,x,y) \Big) dt_1, \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2. \end{aligned}$$

For the last integral equation, we differentiate one time with respect to $t$ and we get the following integral equation

$$\begin{aligned} 0 &= \partial_t u(t,x,y) \\ &- \partial_t u(t,x,y) + \partial_t \partial_x u(t,x,y) + (\partial_x u(t,x,y))^2 \\ &+ u(t,x,y)\partial_{xx} u(t,x,y) \\ &+ \mu^2 \partial_x^{l+1} u(t,x,y) + \nu \partial_{yy} u(t,x,y), \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2. \end{aligned}$$

Therefore

$$\begin{aligned} 0 &= \partial_t \partial_x u(t,x,y) + (\partial_x u(t,x,y))^2 \\ &+ u(t,x,y)\partial_{xx} u(t,x,y) \\ &+ \mu^2 \partial_x^{l+1} u(t,x,y) + \nu \partial_{yy} u(t,x,y), \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2. \end{aligned}$$

Thus, $u$ satisfies the first equation of (1.1). Now, we put $t = 0$ and we arrive at the equality

$$u(0,x,y) = u_0(x,y), \quad (x,y) \in \mathbb{R}^2.$$

From here, we conclude that $u$ satisfies the second equation of (1.1). Consequently, $u$ is a solution to the IVP (1.1). This completes the proof. □

Now, we will give an *a-priori* estimate of the operator $S_1$. For this aim, we define the constant

$$B_1 = \left(2 + |\nu| + \mu^2\right) B + 2B^2.$$

**Lemma 3.2.** *Suppose that (H1) and (H2) hold. Let $u \in X$ be such that $\|u\| \le b$, for some constant $b > 1$. Then one has*

$$|S_1 u(t,x,y)| \le B_1 (1+t), \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2.$$

*Proof.* By the definition of the operator $S_1$, one gets

$$\begin{aligned}
|S_1u(t,x)| &= |u(t,x,y)-u_0(x,y)\\
&\quad+\int_0^t\Big(-\partial_t u(t_1,x,y)+\partial_t\partial_x u(t_1,x,y)+(\partial_x u(t_1,x,y))^2\\
&\quad+u(t_1,x,y)\partial_{xx}u(t_1,x,y)\\
&\quad+\mu^2\partial_x^{l+1}u(t_1,x,y)+\nu\partial_{yy}u(t_1,x,y)\Big)dt_1\Big|\\
&\le |u(t,x,y)|+|u_0(x,y)|\\
&\quad+\int_0^t\Big(|\partial_t u(t_1,x,y)|+|\partial_t\partial_x u(t_1,x,y)|+(\partial_x u(t_1,x,y))^2\\
&\quad+|u(t_1,x,y)||\partial_{xx}u(t_1,x,y)|\\
&\quad+\mu^2|\partial_x^{l+1}u(t_1,x,y)|+|\nu||\partial_{yy}u(t_1,x,y)|\Big)dt_1\\
&\le 2B+\int_0^t\big(B+B+B^2+B^2+\mu^2B+|\nu|B\big)dt_1\\
&= 2B+\big((2+|\nu|+\mu^2)B+2B^2\big)t\\
&\le B_1(1+t),\quad (t,x,y)\in[0,\infty)\times\mathbb{R}^2.
\end{aligned}$$

This completes the proof. □

For $u\in X$, we define the operator

$$S_2u(t,x,y)=\int_0^t\int_0^x\int_0^y (t-t_1)(x-x_1)^{l+1}(y-y_1)^{l+1}g(t_1,x_1)S_1u(t_1,x_1,y_1)dy_1dx_1dt_1,$$

$(t,x,y)\in[0,\infty)\times\mathbb{R}^2$. In the next lemma, we will give an estimate of the norm of the operator $S_2$.

**Lemma 3.3.** *Suppose that (H1)–(H3) hold. For $u\in X$, $\|u\|\le B$, one has the following estimate*

$$\|S_2u\|\le AB_1.$$

*Proof.* We will use the inequality $(v+w)^q\le 2^q(v^q+w^q)$, $q>0, v,w>0$, to find estimates for $S_2u$ and its derivatives. Then, we will deduct the desired estimate for the norm of $S_2u$. We have

$$\begin{aligned}
|S_2u(t,x,y)| &= \left|\int_0^t\int_0^x\int_0^y (t-t_1)(x-x_1)^{l+1}(y-y_1)^{l+1}g(t_1,x_1)S_1u(t_1,x_1,y_1)dy_1dx_1dt_1\right|\\
&\le \int_0^t\left|\int_0^x\int_0^y (t-t_1)|x-x_1|^{l+1}|y-y_1|^{l+1}g(t_1,x_1)|S_1u(t_1,x_1,y_1)|dy_1dx_1\right|dt_1\\
&\le B_1(1+t)\int_0^t\left|\int_0^x\int_0^y (t-t_1)|x-x_1|^{l+1}|y-y_1|^{l+1}g(t_1,x_1)dy_1dx_1\right|dt_1\\
&\le 2^{2l+4}B_1(1+t)^2|x|^{l+1}|y|^{l+1}\int_0^t\left|\int_0^x\int_0^y g(t_1,x_1,y_1)dy_1dx_1\right|dt_1\\
&\le (l+1)!2^{2l+4}B_1(1+t)^2\left(\sum_{r=0}^{l+1}|x|^r\right)\left(\sum_{r=0}^{l+1}|y|^r\right)\\
&\quad\times\int_0^t\left|\int_0^x\int_0^y g(t_1,x_1,y_1)dy_1dx_1\right|dt_1\\
&\le AB_1,\quad (t,x,y)\in[0,\infty)\times\mathbb{R}^2.
\end{aligned}$$

Now, we will estimate the first derivative with respect to $t$ of $S_2u$. For it, one has

$$
\begin{aligned}
|\partial_t S_2 u(t,x,y)| &= \left| \int_0^t \int_0^x \int_0^y (x-x_1)^{l+1}(y-y_1)^{l+1} g(t_1,x_1) S_1 u(t_1,x_1,y_1) dy_1 dx_1 dt_1 \right| \\
&\le \int_0^t \left| \int_0^x \int_0^y |x-x_1|^{l+1}|y-y_1|^{l+1} g(t_1,x_1)|S_1 u(t_1,x_1,y_1)| dy_1 dx_1 \right| dt_1 \\
&\le B_1 \int_0^t \left| \int_0^x \int_0^y |x-x_1|^{l+1}|y-y_1|^{l+1} g(t_1,x_1) dy_1 dx_1 \right| dt_1 \\
&\le 2^{2l+4} B_1 (1+t)^2 |x|^{2l+1}|y|^{2l+1} \int_0^t \left| \int_0^x \int_0^y g(t_1,x_1,y_1) dy_1 dx_1 \right| dt_1 \\
&\le (l+1)! 2^{2l+4} B_1 (1+t)^2 \left( \sum_{r=0}^{l+1} |x|^r \right) \left( \sum_{r=0}^{l+1} |y|^r \right) \\
&\quad \times \int_0^t \left| \int_0^x \int_0^y g(t_1,x_1,y_1) dy_1 dx_1 \right| dt_1 \\
&\le AB_1, \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2.
\end{aligned}
$$

For the derivatives of $S_2 u$ with respect to $x$, one deduct

$$
\begin{aligned}
|\partial_x^r S_2 u(t,x,y)| &= (l+1)\dots(l-r+2) \\
&\quad \times \left| \int_0^t \int_0^x \int_0^y (t-t_1)(x-x_1)^{l-r+1}(y-y_1)^{l+1} g(t_1,x_1) S_1 u(t_1,x_1,y_1) dy_1 dx_1 dt_1 \right| \\
&\le (l+1)\dots(l-r+2) \\
&\quad \times \int_0^t \left| \int_0^x \int_0^y (t-t_1)|x-x_1|^{l-r+1}|y-y_1|^{l+1} g(t_1,x_1)|S_1 u(t_1,x_1,y_1)| dy_1 dx_1 \right| dt_1 \\
&\le (l+1)! B_1 (1+t) \int_0^t \left| \int_0^x \int_0^y (t-t_1) \left| x-x_1 \right|^{l-r+1}|y-y_1|^{l+1} g(t_1,x_1) dy_1 dx_1 \right| dt_1 \\
&\le (l+1)! 2^{2l-r+4} B_1 (1+t)^2 |x|^{l-r+1}|y|^{l+1} \int_0^t \left| \int_0^x \int_0^y g(t_1,x_1,y_1) dy_1 dx_1 \right| dt_1 \\
&\le (l+1)! 2^{2l+4} B_1 (1+t)^2 \left( \sum_{r=0}^{l+1} |x|^r \right) \left( \sum_{r=0}^{l+1} |y|^r \right) \\
&\quad \times \int_0^t \left| \int_0^x \int_0^y g(t_1,x_1,y_1) dy_1 dx_1 \right| dt_1 \\
&\le AB_1, \ (t,x,y) \in [0,\infty) \times \mathbb{R}^2, \ r = 1,\dots,l+1.
\end{aligned}
$$

As above, one can get the following estimates

$$
|\partial_y^k S_2 u(t,x,y)| \le AB_1, \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2, \quad k = 1,2.
$$

Note that for the mixed derivative $\partial_{tx} S_2 u$, one has

$$\begin{aligned}
|\partial_{tx}S_2u(t,x,y)| &= (l+1)\left|\int_0^t\int_0^x\int_0^y (x-x_1)^l(y-y_1)^{l+1}g(t_1,x_1)S_1u(t_1,x_1,y_1)dy_1dx_1dt_1\right| \\
&\leq (l+1)\int_0^t\left|\int_0^x\int_0^y |x-x_1|^l|y-y_1|^{l+1}g(t_1,x_1)|S_1u(t_1,x_1,y_1)|dy_1dx_1\right|dt_1 \\
&\leq (l+1)B_1(1+t)\int_0^t\left|\int_0^x\int_0^y |x-x_1|^l|y-y_1|^{l+1}g(t_1,x_1)dy_1dx_1\right|dt_1 \\
&\leq (l+1)!B_1 2^{2l+3}B_1(1+t)^2|x|^l y^{l+1}\int_0^t\left|\int_0^x\int_0^y g(t_1,x_1,y_1)dy_1dx_1\right|dt_1 \\
&\leq (l+1)!2^{2l+4}B_1(1+t)^2\left(\sum_{r=0}^{l+1}|x|^r\right)\left(\sum_{r=0}^{l+1}|y|^r\right) \\
&\quad \times\int_0^t\left|\int_0^x\int_0^y g(t_1,x_1,y_1)dy_1dx_1\right|dt_1 \\
&\leq AB_1, \quad (t,x,y)\in[0,\infty)\times\mathbb{R}^2.
\end{aligned}$$

Thus,

$$\|S_2u\|\leq AB_1.$$

This completes the proof. □

In the next result, we will give other integral equations whose solutions are solutions to the IVP (1.1).

**Lemma 3.4.** *Suppose* ($H1$), ($H2$) *and let* $g\in\mathcal{C}([0,\infty)\times\mathbb{R})$ *be a positive function almost everywhere on* $[0,\infty)\times\mathbb{R}^2$. *If* $u\in X$ *satisfies the equation*

$$S_2u(t,x,y)=0, \quad (t,x,y)\in[0,\infty)\times\mathbb{R}^2, \tag{3.2}$$

*then* $u$ *is a solution to the IVP* (1.1).

*Proof.* We differentiate two times with respect to $t$, five times with respect to $x$ and five times with respect to $y$ equation (3.2) and we find

$$g(t,x,y)S_1u(t,x,y)=0, \quad (t,x,y)\in[0,\infty)\times\left(\mathbb{R}^2\backslash\{\{x=0\}\cup\{y=0\}\}\right),$$

whereupon

$$S_1u(t,x,y)=0, \quad (t,x,y)\in[0,\infty)\times\left(\mathbb{R}^2\backslash\{\{x=0\}\cup\{y=0\}\}\right).$$

since $S_1u(\cdot,\cdot,\cdot)$ is a continuous function on $[0,\infty)\times\mathbb{R}^2$, we get

$$\begin{aligned}
0 &= \lim_{t\to0}S_1u(t,0,0)=\lim_{x\to0}S_1u(0,x,0)=\lim_{y\to0}S_1u((0,0,y) \\
&= \lim_{t,x\to0}S_1u(t,x,0)=\lim_{t,y\to0}S_1u(t,0,y)=\lim_{x,y\to0}S_1u(0,x,y) \\
&= \lim_{t,x,y\to0}S_1u(t,x,y).
\end{aligned}$$

thus,

$$S_1u(t,x,y)=0, \quad (t,x,y)\in[0,\infty)\times\mathbb{R}^2.$$

hence and Lemma 3.1, we conclude that $u$ is a solution to the IVP (1.1). This completes the proof. □

## 4. Proof of the main results

*4.1 Proof of Theorem 1.1*

Let $\tilde{\tilde{\tilde{Y}}}$ denote the set of all equi-continuous families in $X$ with respect to the norm $\|\cdot\|$. Let also, $\tilde{\tilde{Y}} = \overline{\tilde{\tilde{\tilde{Y}}}}$ be the closure of $\tilde{\tilde{\tilde{Y}}}$, $\tilde{Y} = \tilde{\tilde{Y}} \cup \{u_0\}$,

$$Y = \left\{u \in \tilde{Y} : \|u\| \le B\right\}.$$

Note that $Y$ is a compact set in $X$. For $u \in X$, define the operators

$$\begin{aligned} Tu(t,x,y) &= -\epsilon u(t,x,y), \\ Su(t,x,y) &= u(t,x,y) + \epsilon u(t,x,y) + \epsilon S_2 u(t,x,y), \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2. \end{aligned}$$

For $u \in Y$, using Lemma 3.3, we have

$$\begin{aligned} \|(I-S)u\| &= \|\epsilon u - \epsilon S_2 u\| \\ &\le \epsilon\|u\| + \epsilon\|S_2 u\| \\ &\le \epsilon B_1 + \epsilon A B_1 \\ &= \epsilon B_1(1+A) \\ &< B \end{aligned}$$

Thus, $S : Y \to E$ is continuous and $(I-S)(Y)$ resides in a compact subset of $E$. Now, suppose that there is a $u \in E$ so that $\|u\| = B$ and

$$u = \lambda(I-S)u,$$

or

$$\frac{1}{\lambda}u = (I-S)u = -\epsilon u - \epsilon S_2 u,$$

or

$$\left(\frac{1}{\lambda} + \epsilon\right)u = -\epsilon S_2 u,$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Hence, $\|S_2 u\| \le AB_1 < B$,

$$\epsilon B < \left(\frac{1}{\lambda} + \epsilon\right)B = \left(\frac{1}{\lambda} + \epsilon\right)\|u\| = \epsilon\|S_2 u\| < \epsilon B,$$

which is a contradiction. Hence Theorem 2.1 follows that the operator $T + S$ has a fixed point $u^* \in Y$. Therefore,

$$\begin{aligned} u^*(t,x,y) &= Tu^*(t,x,y) + Su^*(t,x,y) \\ &= -\epsilon u^*(t,x,y) + u^*(t,x,y) + \epsilon u^*(t,x,y) + \epsilon S_2 u^*(t,x,y), \\ &\quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2, \end{aligned}$$

where

$$0 = S_2 u^*(t,x,y), \quad (t,x,y) \in [0,\infty) \times \mathbb{R}^2.$$

from here and from Lemma 3.4, it follows that $u$ is a solution to the IVP (1.1). This completes the proof.

*4.2 Proof of Theorem 1.2*

Let $X$ be the space used in the previous section. Let also,

$$\tilde{P} = \{u \in X : u \geq 0 \quad \text{on} \quad [0, \infty) \times \mathbb{R}^2\}.$$

with $\mathcal{P}$ we will denote the set of all equi-continuous families in $\tilde{P}$. For $v \in X$, define the operators

$$\begin{aligned} T_1 v(t,x,y) &= (1+m\epsilon)v(t,x,y) - \epsilon \frac{L}{10}, \\ S_3 v(t,x,y) &= -\epsilon S_2 v(t,x,y) - m\epsilon v(t,x,y) - \epsilon \frac{L}{10}, \end{aligned}$$

$t \in [0, \infty)$, $(x, y) \in \mathbb{R}^2$. Note that any fixed point $v \in X$ of the operator $T_1 + S_3$ is a solution to the IVP (1.1). Define

$$\begin{aligned} U_1 &= \mathcal{P}_r = \{v \in \mathcal{P} : \|v\| < r\}, \\ U_2 &= \mathcal{P}_L = \{v \in \mathcal{P} : \|v\| < L\}, \\ U_3 &= \mathcal{P}_{R_1} = \{v \in \mathcal{P} : \|v\| < R_1\}, \\ R_2 &= R_1 + \frac{A}{m} B_1 + \frac{L}{5m}, \\ \Omega &= \overline{\mathcal{P}_{R_2}} = \{v \in \mathcal{P} : \|v\| \leq R_2\}. \end{aligned}$$

(1) Let $v_1, v_2 \in \Omega$. Then, we get

$$\|T_1 v_1 - T_1 v_2\| = (1 + m\varepsilon)\|v_1 - v_2\|.$$

from the last equality, we conclude that the operator $T_1 : \Omega \to X$ is an expansive operator with a constant $h = 1 + m\varepsilon > 1$.

(2) Take $v \in \overline{\mathcal{P}}_{R_1}$ arbitrarily. Then

$$\begin{aligned} \|S_3 v\| &\leq \varepsilon \|S_2 v\| + m\varepsilon \|v\| + \varepsilon \frac{L}{10} \\ &\leq \varepsilon \left( AB_1 + mR_1 + \frac{L}{10} \right). \end{aligned}$$

from the last inequality, we conclude that the set $S_3(\overline{\mathcal{P}}_{R_1})$ is uniformly bounded. Because the operator $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is a continuous operator, we get that $S_3(\overline{\mathcal{P}}_{R_1})$ is equi-continuous. Therefore, the operator $S_3 : \overline{\mathcal{P}}_{R_1} \to X$ is a 0-set contraction.

(3) Take $v_1 \in \overline{\mathcal{P}}_{R_1}$ arbitrarily. Set

$$v_2 = v_1 + \frac{1}{m} S_2 v_1 + \frac{L}{5m}.$$

note that $S_2 v_1 + \frac{L}{5} \geq 0$ on $[0, \infty) \times \mathbb{R}^2$. Therefore, $v_2 \geq 0$ on $[0, \infty) \times \mathbb{R}^2$ and we have the following estimate

$$\begin{aligned}\|v_2\| &\leq \|v_1\| + \frac{1}{m}\|S_2 v_1\| + \frac{L}{5m}\\ &\leq R_1 + \frac{A}{m}B_1 + \frac{L}{5m}\\ &= R_2.\end{aligned}$$

consequently $v_2 \in \Omega$. Moreover,

$$-\varepsilon m v_2 = -\varepsilon m v_1 - \varepsilon S_2 v_1 - \varepsilon\frac{L}{10} - \varepsilon\frac{L}{10}$$

or

$$\begin{aligned}(I - T_1)v_2 &= -\varepsilon m v_2 + \varepsilon\frac{L}{10}\\ &= S_3 v_1.\end{aligned}$$

Therefore, $S_3(\overline{\mathcal{P}}_{R_1}) \subset (I - T_1)(\Omega)$.

(4) Suppose that for any $v_0 \in \mathcal{P}^*$, there exist $\lambda > 0$ and $z \in \partial\mathcal{P}_r \cap (\Omega + \lambda v_0)$ or $z \in \partial\mathcal{P}_{R_1} \cap (\Omega + \lambda v_0)$ such that

$$S_3 z = (I - T_1)(z - \lambda v_0).$$

hence,

$$-\epsilon S_2 z - m\epsilon z - \epsilon\frac{L}{10} = -m\epsilon(z - \lambda v_0) + \epsilon\frac{L}{10}$$

or

$$-S_2 z = \lambda m v_0 + \frac{L}{5}.$$

from the last equation, we arrive at

$$\|S_2 z\| = \left\|\lambda m v_0 + \frac{L}{5}\right\| > \frac{L}{5}.$$

this is a contradiction.

(5) Assume that for any $\epsilon_1 \geq 0$ small enough there exist a $x_1 \in \partial\mathcal{P}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$ and

$$S_3 x_1 = (I - T_1)(\lambda_1 x_1). \tag{4.1}$$

When $\epsilon_1 > \frac{2}{5m}$, one has $x_1 \in \partial\mathcal{P}_L$, $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, $\lambda_1 \geq 1 + \epsilon_1$ and (4.1) holds. Since $x_1 \in \partial\mathcal{P}_L$ and $\lambda_1 x_1 \in \overline{\mathcal{P}}_{R_1}$, it follows that

$$\left(\frac{2}{5m} + 1\right)L < \lambda_1 L = \lambda_1\|x_1\| \leq R_1.$$

in addition,

$$-\epsilon S_2x_1 - m\epsilon x_1 - \epsilon\frac{L}{10} = -\lambda_1 m\epsilon x_1 + \epsilon\frac{L}{10},$$

or

$$S_2x_1 + \frac{L}{5} = (\lambda_1 - 1)mx_1.$$

hence,

$$2\frac{L}{5} \geq \left\| S_2x_1 + \frac{L}{5} \right\| = (\lambda_1 - 1)m\|x_1\| = (\lambda_1 - 1)mL,$$

and

$$\frac{2}{5m} + 1 \geq \lambda_1,$$

which is a contradiction.

Therefore, all conditions of Theorem 2.7 hold and the IVP (1.1) has at least two solutions $u_1$ and $u_2$ so that

$$\|u_1\| = L < \|u_2\| < R_1,$$

or

$$r < \|u_1\| < L < \|u_2\| < R_1.$$

## 5. An example

Below, we will illustrate our main results. Let $B = \mu = \nu = 1$ and

$$R_1 = 10, \quad L = 5, \quad r = 4, \quad m = 10^{50}, \quad A = \frac{1}{5B_1}, \quad \epsilon = \frac{1}{5B_1(1+A)}.$$

let also,

$$u_0(x,y) = \frac{1}{1+x^2+y^2}, \quad (x,y) \in \mathbb{R}^2.$$

then

$$B_1 = 2 + 2 + 2 = 6,$$

and

$$AB_1 = \frac{1}{5} < B, \quad \epsilon B_1(1+A) < 1,$$

i.e. ($H4$) holds. Next,

$$r < L < R_1, \quad \epsilon > 0, \quad R_1 > \left(\frac{2}{5m} + 1\right)L, \quad AB_1 < \frac{L}{5}.$$

i.e. ($H5$) holds. Take

$$h(s) = \log\frac{1 + s^{l+1}\sqrt{2} + s^{2l+2}}{1 - s^{l+1}\sqrt{2} + s^{2l+2}}, \quad l(s) = \arctan\frac{s^{l+1}\sqrt{2}}{1 - s^{2l+2}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1.$$

then

$$\begin{aligned} h'(s) &= \frac{2\sqrt{2}(l+1)s^l\left(1 - s^{2l+2}\right)}{\left(1 - s^{l+1}\sqrt{2} + s^{2l+2}\right)\left(1 - s^{l+1}\sqrt{2} + s^{2l+2}\right)}, \\ l'(s) &= \frac{(l+1)\sqrt{2}s^l\left(1 + s^{2l+2}\right)}{1 + s^{4l+4}}, \quad s \in \mathbb{R}, \quad s \neq \pm 1. \end{aligned}$$

therefore

$$\begin{aligned} \lim_{s\to\pm\infty}\sum_{r=0}^{l+1} s^r h(s) &= \lim_{s\to\pm\infty}\frac{h(s)}{\frac{1}{\sum_{r=0}^{l+1} s^r}} = \lim_{s\to\pm\infty}\frac{h'(s)}{-\frac{\sum_{r=0}^{l}(r+1)s^r}{\left(\sum_{r=0}^{l+1} s^r\right)^2}} \\ &= -\lim_{s\to\pm\infty}\frac{2\sqrt{2}(l+1)s^l\left(1 - s^{2l+2}\right)\left(\sum_{r=0}^{l+1} s^r\right)^2}{\left(\sum_{r=0}^{l}(r+1)s^r\right)\left(1 - s^{l+1}\sqrt{2} + s^{2l+2}\right)\left(1 - s^{l+1}\sqrt{2} + s^{2l+2}\right)} \neq \pm\infty \end{aligned}$$

and

$$\begin{aligned} \lim_{s\to\pm\infty}\sum_{r=0}^{l+1} s^r l(s) &= \lim_{s\to\pm\infty}\frac{l(s)}{\frac{1}{\sum_{r=0}^{l+1} s^r}} = \lim_{s\to\pm\infty}\frac{l'(s)}{-\frac{\sum_{r=0}^{l}(r+1)s^r}{\left(\sum_{r=0}^{l+1} s^r\right)^2}} \\ &= -\lim_{s\to\pm\infty}\frac{(l+1)\sqrt{2}s^l\left(1 + s^{2l+2}\right)\left(\sum_{r=0}^{l+1} s^r\right)^2}{\left(1 + s^{4l+4}\right)\left(\sum_{r=0}^{l}(r+1)s^r\right)} \neq \pm\infty. \end{aligned}$$

consequently

$$\begin{aligned} -\infty &< \lim_{s\to\pm\infty}\left(\sum_{r=0}^{l+1} s^r\right)h(s) < \infty, \\ -\infty &< \lim_{s\to\pm\infty}\left(\sum_{r=0}^{l+1} s^r\right)l(s) < \infty. \end{aligned}$$

hence, there exists a positive constant $C_1$ so that

$$\sum_{r=0}^{l+1}|s|^r\left(\frac{1}{(4l+4)\sqrt{2}}\log\frac{1+s^{l+1}\sqrt{2}+s^{2l+2}}{1-s^{l+1}\sqrt{2}+s^{2l+2}}+\frac{1}{(2l+2)\sqrt{2}}\arctan\frac{s^{l+1}\sqrt{2}}{1-s^{2l+2}}\right)$$
$$\leq \quad C_1,$$

$s\in\mathbb{R}$. Note that $\lim_{s\to\pm1} l(s)=\frac{\pi}{2}$ and by Ref. [15] (p. 707, Integral 79), we have

$$\int\frac{dz}{1+z^4}=\frac{1}{4\sqrt{2}}\log\frac{1+z\sqrt{2}+z^2}{1-z\sqrt{2}+z^2}+\frac{1}{2\sqrt{2}}\arctan\frac{z\sqrt{2}}{1-z^2}.$$

let

$$Q(s)=\frac{s^l}{(1+s^{4l+4})},\quad s\in\mathbb{R},$$

and

$$g_1(t,x,y)=Q(t)Q(x)Q(y),\quad t\in[0,\infty),\quad (x,y)\in\mathbb{R}^2.$$

then there exists a constant $C>0$ such that

$$2^{2l+4}(l+1)!(1+t)^2\left(\sum_{r=0}^{l+1}|x|^r\right)\left(\sum_{r=0}^{l+1}|y|^r\right)$$
$$\times\int_0^t\left|\int_0^x\int_0^y g_1(t_1,x_1,y_1)dy_1dx_1\right|dt_1\leq C,$$
$$(t,x,y)\in[0,\infty)\times\mathbb{R}^2.$$

let

$$g(t,x,y)=\frac{A}{C}g_1(t,x,y),\quad (t,x,y)\in[0,\infty)\times\mathbb{R}^3.$$

then

$$2^{2l+4}(l+1)!(1+t)^2\left(\sum_{r=0}^{l+1}|x|^r\right)\left(\sum_{r=0}^{l+1}|y|^r\right)$$
$$\times\int_0^t\left|\int_0^x\int_0^y g(t_1,x_1,y_1)dy_1dx_1\right|dt_1\leq A,\quad (t,x,y)\in[0,\infty)\times\mathbb{R}^2.$$

i.e. ($H3$), holds. Therefore, for the IVP

$$\begin{aligned} \partial_x\left(\partial_t u + u\partial_x u + \mu^2\partial_x^l u\right) + \nu\partial_{yy}u &= 0, \quad t > 0, \quad (x,y)\in\mathbb{R}^2, \\ u(0,x,y) &= \frac{1}{1+x^2+y^2}, \quad (x,y)\in\mathbb{R}^2, \end{aligned}$$

are fulfilled all conditions of Theorem 1.1 and Theorem 1.2.

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