Some results on quadratic credibility premium using the balanced loss function

On quadratic
credibility
premium

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Abstract

Purpose – This paper generalizes the quadratic framework introduced by Le Courtois (2016) and Sumpf (2018), to obtain new credibility premiums in the balanced case, i.e. under the balanced squared error loss function. More precisely, the authors construct a quadratic credibility framework under the net quadratic loss function where premiums are estimated based on the values of past observations and of past squared observations under the parametric and the non-parametric approaches, this framework is useful for the practitioner who wants to explicitly take into account higher order (cross) moments of past data.

Design/methodology/approach – In the actuarial field, credibility theory is an empirical model used to calculate the premium. One of the crucial tasks of the actuary in the insurance company is to design a tariff structure that will fairly distribute the burden of claims among insureds. In this work, the authors use the weighted balanced loss function (WBLF, henceforth) to obtain new credibility premiums, and WBLF is a generalized loss function introduced by Zellner (1994) (see Gupta and Berger (1994), pp. 371-390) which appears also in Dey *et al.* (1999) and Farsipour and Asgharzadhe (2004).

Findings – The authors declare that there is no conflict of interest and the funding information is not applicable.

Research limitations/implications – This work is motivated by the following: quadratic credibility premium under the balanced loss function is useful for the practitioner who wants to explicitly take into account higher order (cross) moments and new effects such as the clustering effect to finding a premium more credible and more precise, which arranges both parts: the insurer and the insured. Also, it is easy to apply for parametric and non-parametric approaches. In addition, the formulas of the parametric (Poisson–gamma case) and the non-parametric approach are simple in form and may be used to find a more flexible premium in many special cases. On the other hand, this work neglects the semi-parametric approach because it is rarely used by practitioners.

Practical implications – There are several examples of actuarial science (credibility).

Originality/value – In this paper, the authors used the WBLF and a quadratic adjustment to obtain new credibility premiums. More precisely, the authors construct a quadratic credibility framework under the net quadratic loss function where premiums are estimated based on the values of past observations and of past squared observations under the parametric and the non-parametric approaches, this framework is useful for the practitioner who wants to explicitly take into account higher order (cross) moments of past data.

Keywords Quadratic credibility, Loss function, Credibility premium, Parametric approach,

Non-parametric approach

Paper type Research paper

JEL Classification — 62C12, 62E10

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1. Introduction and motivation

In the actuarial field, credibility theory is an empirical model used to calculate the premium. One of the crucial tasks of the actuary in the insurance company is to design a tariff structure that will fairly distribute the burden of claims among insureds.

In this sense, actuaries use credibility theory to determine the expected claims experience of an individual risk when those risks are not homogenous, given that the individual risk belongs to a heterogenous collective. The main objective of this theory is to calculate the weight which should be assigned to the individual risk data to determine a fair premium to be charged. For recent detailed introductions to credibility theory, see Refs. [1–3].

In this work, we use the weighted balanced loss function (WBLF, henceforth) to obtain new credibility premiums, WBLF is a generalized loss function introduced by Ref. [4] (see Ref. [5, pp. 371–390] and which appears also in Refs. [6, 7]. It is given by

$$L_2(P, x) = \omega h(x)(\delta_0(x) - P)^2 + (1 - \omega)h(x)(x - P)^2 \quad (1)$$

where $0 \leq \omega \leq 1$ is the relative weight given to the goodness-of-fit portion of the loss and $(1 - \omega)$ is the relative weight given to the precision of estimation portion, $h(x)$ is a positive weight function, and $\delta_0(x)$ is a function of the observed data obtained for instance from the criterion of maximum likelihood estimator, least squares, or unbiased among others (see Ref. [8]).

In this work, we assume that $h(x) = 1$ which is the case of the balanced squared error loss function:

$$L_2(P, x) = \omega(\delta_0(x) - P)^2 + (1 - \omega)(x - P)^2 \quad (2)$$

When ω is chosen to equal 0, this loss includes as a particular case the squared error loss function, i.e.

$$L_2(P, x) = 0(\delta_0(x) - P)^2 + (1 - 0)(x - P)^2 = (x - P)^2 = L_1(P, x)$$

Moreover, in the classical credibility theory, [9] overcame the prior limitation and proved that in a class of linear estimators of the form $\delta_{Lin} = c_0 + \sum_{j=1}^n c_j X_j$, an estimator $P = z\bar{X} + (1 - z)\mu$, is also a distribution free credibility formula, which minimizes $E[\mu(\theta) - \delta_{Lin}]^2$, whenever $\mu(\theta)$ is the mean of an individual risk (or $\mu(\theta) = E(X|\theta)$), characterized by risk parameter θ and $\bar{X} = \frac{X_1 + X_2 + \dots + X_n}{n}$.

Furthermore, [10] has constructed a quadratic credible framework under the net quadratic loss function where premiums are estimated based on the values of past observations and of past squared observations. The originality of our paper lies in the fact that we make a generalization of the quadratic framework introduced by Ref. [10], to obtain new credibility premiums in the balanced case, i.e. under the balanced squared error loss function. Recently, [11] generalized the credibility framework to define the p -credibility premium by adding higher exponents of the past observations in the structure of the premium. For $p = 1$, our framework reproduces the known credibility framework and for $p = 2$ the quadratic framework from Ref. [10].

This work is motivated by the following: quadratic credibility premium under the balanced loss function is useful for the practitioner who wants to explicitly take into account higher order (cross) moments and new effects such as the clustering effect to finding a premium more credible and more precise, which arranges both parts: the insurer and the insured. Also, it is easy to apply for parametric and non-parametric approaches. In addition, the formulas of the parametric (Poisson–gamma case) and the non-parametric approach are simple in form and may be used to find a more flexible premium in many special cases. On the other hand, this work neglects the semi-parametric approach because it is rarely used by practitioners.

The rest of this paper is arranged as follows. Section 2 collects some useful elements for other sections. Section 3 provides the main contribution of this article by obtaining the quadratic credibility premiums under the balanced squared error loss function. An application of the results in a parametric approach for the pair Poisson–gamma is given in Section 4 and in a non-parametric approach is presented in Section 5 with concluding remarks.

2. Preliminaries

We assume that the individual risk, X , has a density $f(x|\theta)$ indexed by a parameter $\theta \in \Theta$ which has a prior distribution with density $\pi(\theta)$. Let, now, $\pi(\theta|x)$ be the posterior density when x is observed. The following from Ref. [12] is a generalization of Lemma 2 in Ref. [8] from which the Bayes estimator of θ under prior π is obtained.

Lemma 1. [12] *Under WBLF and prior π , the risk (individual) and collective premiums are given by*

$$P_R^{L_2} = \omega \frac{E_{f(x|\theta)}[\delta_0(x)h(x)|\theta]}{E_{f(x|\theta)}[h(x)|\theta]} + (1 - \omega) \frac{E_{f(x|\theta)}[Xh(x)|\theta]}{E_{f(x|\theta)}[h(x)|\theta]} \quad (3)$$

$$P_C^{L_2} = \omega \delta_0^\times + (1 - \omega) \frac{E_\pi \left[P_R^{L_2} h \left(P_R^{L_2} \right) \right]}{E_\pi \left[h \left(P_R^{L_2} \right) \right]} \quad (4)$$

where $\delta_0^\times$ is a target estimator for the risk premium $P_R^{L_2}$.

Proof. The proof is omitted because it is very similar to the proof given in Ref. [6] which has shown that for $h(\cdot) = 1$, the Bayes estimator under loss L_2 may be expressed simply as a convex linear combination of the Bayes estimator under loss L_1 (weight $1 - \omega$) and δ_0 (weight ω).

Hence, by generalizing this result, we minimize $E_{f(x|\theta)}[L_2(\theta, P_R^{L_2})]$ under WBLF with respect to $P_R^{L_2}$ to obtain the individual premium.

Similarly, we minimize $E_{\pi(\theta)}[L_2(P_R^{L_1}, P_C^{L_2})]$ under WBLF with respect to $P_C^{L_2}$ to obtain the collective premium.

Now the Bayes premium, $P_B^{L_2}$, is obtained replacing $\pi(\theta)$ by the posterior distribution $\pi(\theta|x)$ in (4).

$$P_B^{L_2} = \omega \delta_0^\times + (1 - \omega) \frac{E_{\pi(\theta|x)} \left[P_R^{L_2} h \left(P_R^{L_2} \right) \right]}{E_{\pi(\theta|x)} \left[h \left(P_R^{L_2} \right) \right]} \quad (5)$$

Under the squared error loss function $L_1(P, x)$, [10] added a quadratic correction in credibility theory to introduce higher order terms in the frame work, he has constructed a new credibility premium $P_q^{L_1}$ which is given by:

$$P_q^{L_1} = a_0 + Z_q \bar{X} + Y_q \bar{X}^2, \quad (6)$$

where the letter q refers to the quadratic framework.

In this work, we extend his idea under the balanced squared error loss function. For that reason, we will use throughout the following notation:

$$\mu(\theta) = E_{f(x|\theta)}[X],$$

$$\mu = E_{\pi}[\mu(\theta)] = E_{\pi}[X],$$

$$v = E_{\pi}[\text{var}[X | \theta]],$$

$$\text{cov}(X_i, X_k) = a \quad \forall i \neq k,$$

and

$$\text{cov}(X_i, X_i) = \text{var}(X_i) = a + v,$$

We also consider the following conventions:

$$\text{cov}(X_i^2, X_k) = b \quad \forall i \neq k,$$

$$\text{cov}(X_i^2, X_i) = b + g,$$

$$\text{cov}(X_i^2, X_k^2) = c \quad \forall i \neq k,$$

$$\text{cov}(X_i^2, X_i^2) = \text{var}(X_i^2) = c + h,$$

$$\text{cov}(\delta_0, X_k) = \text{cov}(\bar{X}, X_k) = d_1 = a + \frac{v}{n},$$

and

$$\text{cov}(\delta_0, X_k^2) = \text{cov}(\bar{X}, X_k^2) = d_2 = b + \frac{g}{n}.$$

Remark 2. In this work, we take $\delta_0(x) = \bar{X}$.

3. Main results

Our idea consists of replacing P in $L_2(P, x)$ by an expression of the form $a_0 + \sum_{i=1}^n A_i X_i + \sum_{i=1}^n B_i X_i^2$ depending on the past claims and past squared claims. The main result of this paper is showed in the next proposition.

Proposition 3. *The quadratic credibility premium under the balanced squared error loss function giving the best predictor of X for the next period is:*

$$\begin{aligned} P_q^{L_2} &= \omega \delta_0 (2 - \omega) + (1 - \omega)^2 \left(a_0 + Z_q \bar{X} + Y_q \bar{X}^2 \right) \\ &= \omega \delta_0 (2 - \omega) + (1 - \omega)^2 P_q^{L_1}, \end{aligned} \quad (7)$$

where

$$a_0 = (1 - \omega - Z_q) \mu - Y_q (\mu^2 + a + v) + \omega E[\delta_0], \quad (8)$$

$$Z_q = n \frac{(\omega d_2 + (1 - \omega)b)(nb + g)}{(na + v)(nc + h)}, \quad (9)$$

and

$$Y_q = n \frac{\omega d_1 + (1 - \omega)a}{(nb + g)} - \frac{\omega d_2 + (1 - \omega)b}{(nc + h)}. \quad (10)$$

Proof. The objective is to solve an optimization problem under the balanced squared error loss function:

$$\min_{a_0, \{A_i\}, \{B_i\}} E \left[\begin{array}{l} \omega \left(\delta_0 - a_0 - \sum_{i=1}^n A_i X_i - \sum_{i=1}^n B_i X_i^2 \right)^2 + \\ (1 - \omega) \left(\mu(\theta) - a_0 - \sum_{i=1}^n A_i X_i - \sum_{i=1}^n B_i X_i^2 \right)^2 \end{array} \right] \quad (11)$$

Setting the derivative with respect to a_0 equal to zero, we obtain the system of equations:

$$E \left[-2\omega \left(\delta_0 - a_0 - \sum_{i=1}^n A_i X_i - \sum_{i=1}^n B_i X_i^2 \right) - 2(1 - \omega) \left(\mu(\theta) - a_0 - \sum_{i=1}^n A_i X_i - \sum_{i=1}^n B_i X_i^2 \right) \right] = 0$$

$\Rightarrow$

$$a_0 + E \left[\sum_{i=1}^n A_i X_i \right] + E \left[\sum_{i=1}^n B_i X_i^2 \right] - \omega E[\delta_0] - (1 - \omega) E[\mu(\theta)] = 0$$

$\Rightarrow$

$$a_0 + \sum_{i=1}^n A_i E[X_i] + \sum_{i=1}^n B_i E[X_i^2] = \omega E[\delta_0] + (1 - \omega) E[\mu(\theta)]. \quad (12)$$

Taking a derivative in (11) with respect to each A_k , we obtain:

$$E \left[\begin{array}{l} -2\omega X_k \left(\delta_0 - a_0 - \sum_{i=1}^n A_i X_i - \sum_{i=1}^n B_i X_i^2 \right) - \\ 2(1 - \omega) X_k \left(\mu(\theta) - a_0 - \sum_{i=1}^n A_i X_i - \sum_{i=1}^n B_i X_i^2 \right) \end{array} \right] = 0$$

$\Rightarrow$

$$\left(\begin{array}{l} -\omega \left(E[\delta_0 X_k] + a_0 E[X_k] + \sum_{i=1}^n A_i E(X_i X_k) + \sum_{i=1}^n B_i E(X_i^2 X_k) \right) - \\ (1 - \omega) \left(E[\mu(\theta) X_k] + a_0 E[X_k] + \sum_{i=1}^n A_i E(X_i X_k) + \sum_{i=1}^n B_i E(X_i^2 X_k) \right) \end{array} \right) = 0$$

$\Rightarrow$

$$a_0 E[X_k] + \sum_{i=1}^n A_i E(X_i X_k) + \sum_{i=1}^n B_i E(X_i^2 X_k) = \omega E[\delta_0 X_k] + (1 - \omega) E[\mu(\theta) X_k]. \quad (13)$$

Subtracting $E[X_k]$ times (12) from (13), we have:

$$\sum_{i=1}^n A_i cov(X_i, X_k) + \sum_{i=1}^n B_i cov(X_i^2, X_k) = \omega cov[\delta_0, X_k] + (1 - \omega) cov[\mu(\theta), X_k]. \quad (14)$$

Now, we set the derivative with respect to each B_k equal to 0:

$$E \left[-2\omega X_k^2 \left(\delta_0 - a_0 - \sum_{i=1}^n A_i X_i - \sum_{i=1}^n B_i X_i^2 \right) - 2(1-\omega) X_k^2 \left(\mu(\theta) - a_0 - \sum_{i=1}^n A_i X_i - \sum_{i=1}^n B_i X_i^2 \right) \right] = 0$$

$\Rightarrow$

$$\left(\begin{array}{l} -\omega \left(E[\delta_0 X_k^2] + a_0 E[X_k^2] + \sum_{i=1}^n A_i E(X_i X_k^2) + \sum_{i=1}^n B_i E(X_i^2 X_k^2) \right) - \\ (1-\omega) \left(E[\mu(\theta) X_k^2] + a_0 E[X_k^2] + \sum_{i=1}^n A_i E(X_i X_k^2) + B_i \sum_{i=1}^n E(X_i^2 X_k^2) \right) \end{array} \right) = 0$$

$\Rightarrow$

$$a_0 E[X_k^2] + \sum_{i=1}^n A_i E(X_i X_k^2) + \sum_{i=1}^n B_i E(X_i^2 X_k^2) = \omega E[\delta_0 X_k^2] + (1-\omega) E[\mu(\theta) X_k^2]. \quad (15)$$

Subtracting $E[X_k^2]$ times (12) from (15), we obtain:

$$\sum_{i=1}^n A_i \text{cov}(X_i, X_k^2) + \sum_{i=1}^n B_i \text{cov}(X_i^2, X_k^2) = \omega \text{cov}[\delta_0, X_k^2] + (1-\omega) \text{cov}[\mu(\theta), X_k^2]. \quad (16)$$

Or, since $X_1, X_2, \dots, X_n$ are independently and identically distributed given θ , we consider: $\forall i = 1: n, A_i = A$ and $\forall i = 1: n, B_i = B$. Then Eqns (14) and (16) are reduced to:

$$A(na + v) + B(nb + g) = \omega d_1 + (1-\omega)a, \quad (17)$$

and

$$A(nb + g) + B(nc + h) = \omega d_2 + (1-\omega)b. \quad (18)$$

Solving the two above equations, we obtain:

$$A = \frac{(\omega d_2 + (1-\omega)b)(nb + g)}{(na + v)(nc + h)}, \quad (19)$$

and

$$B = \frac{\omega d_1 + (1-\omega)a}{(nb + g)} - \frac{\omega d_2 + (1-\omega)b}{(nc + h)}. \quad (20)$$

Hence, we can write (12) as:

$$a_0 + nA\mu + nB(\mu^2 + a + v) = \omega E[\delta_0] + (1-\omega)E[\mu(\theta)],$$

because

$$E(X_i^2) = E(X_i)^2 + \text{var}(X_i) = \mu^2 + a + v.$$

Finally, denoting Z_q by nA and Y_q by nB , we have that:

$$a_0 = (1-\omega - Z_q)\mu - Y_q(\mu^2 + a + v) + \omega E[\delta_0],$$

Thus,

$$P_q^{L_2} = \omega(2 - \omega)\delta_0 + (1 - \omega)^2 P_q^{L_1}, \quad (21)$$

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4. The parametric approach: numerical application for the Poisson–Gamma case

We are now interested firstly in illustrating the methodology described in the previous section under the parametric approach. For that reason, we present the following propositions.

Proposition 4. *Suppose that the claim follows a Poisson distribution with parameter $\theta > 0$ and the prior is a gamma distribution $\pi(\theta) \propto \theta^{\alpha-1} e^{-\beta\theta}$, $\alpha > 0$, $\beta > 0$. The risk (individual), collective and Bayes weighted balanced premiums obtained under $L_2(P, x)$ with $h(x) = 1$ for the pair Poisson–gamma are given by:*

$$P_R^{L_2} = \omega\delta_0 + (1 - \omega)\theta, \quad (22)$$

$$P_C^{L_2} = \omega\delta_0^* (2 - \omega) + (1 - \omega)^2 P_C^{L_1}, \quad (23)$$

$$P_B^{L_2} = \omega\delta_0^* (2 - \omega) + (1 - \omega)^2 P_B^{L_1}. \quad (24)$$

Proof. According to Ref. [12], we have:

$$\begin{aligned} P_R^{L_2} &= \omega\delta_0 + (1 - \omega)E_{f(x|\theta)}[X] = \omega\delta_0 + (1 - \omega)\theta, \\ P_C^{L_2} &= \omega E_\pi[\delta_0(x)] + (1 - \omega)E_\pi[P_R^{L_2}] = \omega\delta_0^* + (1 - \omega)E_\pi[\omega\delta_0 + (1 - \omega)\theta] \\ &= \omega\delta_0^* (2 - \omega) + (1 - \omega)^2 E_\pi[\theta] = \omega\delta_0^* (2 - \omega) + (1 - \omega)^2 \frac{\alpha}{\beta} \\ &= \omega\delta_0^* (2 - \omega) + (1 - \omega)^2 P_C^{L_1}. \\ P_B^{L_2} &= \omega\delta_0^* + (1 - \omega)E_{\pi^*}[\omega\delta_0 + (1 - \omega)\theta] = \omega\delta_0^* (2 - \omega) + (1 - \omega)^2 E_{\pi^*}[\theta] \\ &= \omega\delta_0^* (2 - \omega) + (1 - \omega)^2 \frac{n\bar{X} + \alpha}{n + \beta} \\ &= \omega\delta_0^* (2 - \omega) + (1 - \omega)^2 P_B^{L_1}. \end{aligned}$$

For computing the quadratic credibility parameters, we present a proposition which is similar to Proposition 1.6 given in Ref. [10].

Proposition 5. *The quantities μ , v , a , g , b , c , h , d_1 , d_2 are given by:*

$$\begin{aligned} \mu &= v = E_\pi[\theta] = \frac{\alpha}{\beta}, \\ a &= \text{var}[E[X|\theta]] = \text{var}[\theta] = \frac{\alpha}{\beta^2}, \\ g &= \frac{\alpha}{\beta} + \frac{2\alpha(\alpha + 1)}{\beta^2}, \end{aligned}$$

$$\begin{aligned}
 b &= \frac{g}{\beta}, \\
 c &= \frac{2g}{\beta} - \frac{\alpha}{\beta^2} + \text{Var}(\theta^2) = \frac{2g}{\beta} - \frac{\alpha}{\beta^2} + \frac{\alpha(\alpha+1)(4\alpha+6)}{\beta^4}, \\
 h &= \frac{\alpha}{\beta} + \frac{6\alpha(\alpha+1)}{\beta^2} + \frac{4\alpha(\alpha+1)(\alpha+2)}{\beta^3}, \\
 d_1 &= \frac{\alpha}{\beta^2} + \frac{\alpha}{n\beta}, \\
 d_2 &= \frac{g}{\beta} + \frac{g}{n}.
 \end{aligned}$$

Proof. The proof is straightforward, one can refer to Proposition 1.6 in Ref. [10] to see how to calculate v , a , b , g , c , h in the conditional Poisson case. To calculate d_1 and d_2 , we can use formulas (25) and (26):

$$d_1 = \text{cov}(\delta_0, X_k) = \text{cov}(\bar{X}, X_k) \quad (25)$$

$$d_2 = \text{cov}(\delta_0, X_k^2) = \text{cov}(\bar{X}, X_k^2), \quad (26)$$

Using above formulas, we find

$$\begin{aligned}
 d_1 &= \text{cov}\left(\frac{\sum_{i=1}^n X_i}{n}, X_k\right) = \frac{1}{n} \text{cov}\left(\sum_{i=1}^n X_i, X_k\right) = a + \frac{v}{n} = \frac{\alpha}{\beta^2} + \frac{\alpha}{n\beta}, \\
 d_2 &= \text{cov}\left(\frac{\sum_{i=1}^n X_i}{n}, X_k^2\right) = \frac{1}{n} \text{cov}\left(\sum_{i=1}^n X_i, X_k^2\right) = b + \frac{g}{n} = \frac{g}{\beta} + \frac{g}{n}.
 \end{aligned}$$

■

Remark 6. We have chosen the pair Poisson–gamma for simplicity of calculations. Obviously, we can extend the above procedure to another pairs of the exponential dispersion family like exponential-gamma or geometric-beta. . .etc, under the condition that they give us flexible calculations.

Examples.

In order to compare the classical credibility premium $P_B^{L_2}$ with the quadratic credibility premium $P_q^{L_2}$, we assume that we have 8 claims which are observed in 3 years. In addition, we take $\delta_0(x) = \bar{X} = \frac{8}{3} = 2.67$, $\omega = 0.2$ and the hyperparameters of the prior distribution $\Gamma(\alpha, \beta)$ are respectively 3.1 and 1.2. Under the classical credibility theory, $P_B^{L_2}$ is given by:

$$\begin{aligned}
 P_B^{L_2} &= \omega \bar{X} (2 - \omega) + (1 - \omega)^2 \frac{n\bar{X} + \alpha}{n + \beta}, \\
 &= 0.2(2.67)(2 - 0.2) + (1 - 0.2)^2 \frac{3(2.67) + 3.1}{3 + 1.2} = 2.65415.
 \end{aligned}$$

Now, to compute $P_q^{L_2}$, we first calculate these quantities:

$$v = 2.583\,333, \quad a = 2.152\,778, \quad g = 20.236\,11,$$

$$b = 16.863\,43, \quad c = 144.355\,7, \quad h = 205.590\,3, \quad d_1 = 3.013\,889, \quad d_2 = 23.608\,8.$$

Thus,

$$a_0 = 0.723\,886\,5,$$

$$Z_q = 0.670\,146\,1,$$

and

$$Y_q = 0.012\,929\,68.$$

The table below contains the numerical values of the observed mean of past squared observations $\bar{X}^2$ and $P_q^{L_2}$ according to the three scenarios (see [Table 1](#)).

We assume now that we have 4 claims which are observed in 5 years. In addition, we take $\delta_0(x) = \bar{X} = \frac{4}{5} = 0.8$, $\omega = 0.7$ and the hyperparameters of the prior distribution $\Gamma(\alpha, \beta)$ are respectively 2.4 and 3.2.

$P_B^{L_2}$ is given by:

$$P_B^{L_2} = \omega \bar{X} (2 - \omega) + (1 - \omega)^2 \frac{n \bar{X} + \alpha}{n + \beta},$$

$$P_B^{L_2} = 0.7(0.8)(2 - 0.7) + (1 - 0.7)^2 \frac{5(0.8) + 2.4}{5 + 3.2} = 0.798\,2.$$

Now, to compute $P_q^{L_2}$, we first calculate these quantities:

$$v = 0.75, \quad a = 0.234\,375, \quad g = 2.343\,75,$$

$$b = 0.732\,421\,9, \quad c = 2.444\,458, \quad h = 9.914\,062, \quad d_1 = 0.384\,375, \quad d_2 = 1.201\,172.$$

Thus,

$$a_0 = 0.157\,060\,9, \quad Z_q = 0.748\,589\,7,$$

and

$$Y_q = 0.042\,987\,89.$$

The results of the second example are summarized in the next table: (see [Table 2](#)).

Scenarios	(1, 3, 4)	(2, 5, 1)	(6, 0, 2)
$\bar{X}^2$	8.67	10	13.33
$P_q^{L_2}$	2.641 377	2.652 383	2.679 939

Table 1.
Estimators of $P_q^{L_2}$ for
the fourth year

Scenarios	(1, 1, 0, 1, 1)	(1, 2, 1, 0, 0)	(0, 0, 0, 3, 1)
$\bar{X}^2$	0.8	1.2	2
$P_q^{L_2}$	0.799 129 1	0.800 676 6	0.803 771 8

Table 2.
Estimators of $P_q^{L_2}$ for
the sixth year

The simulation shows that the values of $P_q^{L_2}$ are distributed around the value of $P_B^{L_2}$. When the scenarios become more irregulars, $P_q^{L_2}$ is larger than $P_B^{L_2}$. We can explain this by the fact that which is due to a more consideration of the individual experience for $P_q^{L_2}$.

5. The non-parametric approach

In this section, we aim to calculate the Bühlmann, the classic and the quadratic credibility premiums. So, we must first estimate the parameters which are unknown in practice and functionals of the unobservable random variable θ . Hence, they must be estimated from the entire portfolio data.

The estimator of expected hypothetical means is

$$\hat{\mu} = \frac{1}{rn} \sum_{i=1}^r \sum_{j=1}^n X_{ij}, \quad (27)$$

and that of expected process variance is

$$\hat{v} = \frac{1}{r(n-1)} \sum_{i=1}^r \sum_{j=1}^n (X_{ij} - \bar{X}_i)^2, \quad (28)$$

where $\bar{X}_i = \frac{1}{n} \sum_{j=1}^n X_{ij}$ is the empirical mean of past observations for insured i .

Then, the estimator of the variance of hypothetical means is

$$\hat{a} = \frac{1}{r-1} \sum_{i=1}^r (\bar{X}_i - \bar{X})^2 - \frac{\hat{v}}{n}, \quad (29)$$

where $\bar{X}$ is the empirical mean of past observations for all insureds, which is equal to $\hat{\mu}$.

Now, to estimate the q-credibility premium, we need to calculate the non-parametric unbiased estimators for the quantities h, c, g and b which are already shown in Proposition 2.1 in Ref. [10]; and d_1 and d_2 which are presented how to be estimated in the next proposition.

Proposition 7. *The non-parametric estimators for the quantities d_1 and d_2 are given as follows.*

$$\hat{d}_1 = \frac{1}{r-1} \sum_{i=1}^r (\bar{X}_i - \bar{X})^2 \quad (30)$$

$$\hat{d}_2 = \frac{1}{r-1} \sum_{i=1}^r (\bar{X}_i^2 - \bar{X}^2) (\bar{X}_i - \bar{X}) \quad (31)$$

Proof. We have:

$$\begin{aligned} \hat{d}_1 &= cov(\delta_0, X_k) = cov(\bar{X}, X_k) = \frac{1}{n} cov\left(\sum_{i=1}^n X_i, X_k\right) \\ &= \hat{a} + \frac{\hat{v}}{n} = \frac{1}{r-1} \sum_{i=1}^r (\bar{X}_i - \bar{X})^2. \end{aligned}$$

$$\begin{aligned}\hat{d}_2 &= cov(\delta_0, X_k^2) = cov(\bar{X}, X_k^2) = \frac{1}{n} cov\left(\sum_{i=1}^n X_i, X_k^2\right) \\ &= \hat{b} + \frac{\hat{g}}{n} = \frac{1}{r-1} \sum_{i=1}^r (\bar{X}_i^2 - \bar{X}^2) (\bar{X}_i - \bar{X}).\end{aligned}$$

On quadratic
credibility
premium

Example.

Let us suppose a portfolio as depicted in Table 3, where each line represents a contract. The portfolio is composed of $r = 3$ contracts with an experience of $n = 6$ years.

We want to calculate $\hat{P}^{Bühlmann}$, $\hat{P}_B^{L_2}$ and $\hat{P}_q^{L_2}$ for the seventh year.

We have:

$$(\bar{X}_1, \bar{X}_2, \bar{X}_3) = (1, 3, 2) \text{ and } \bar{X} = \frac{1+3+2}{3} = 2.$$

Then, the structural parameters are given by

$$\hat{\mu} = \bar{X} = 2, \hat{v} = \frac{16}{15}, \hat{a} = \frac{37}{45}.$$

Therefore, we obtain

$$\hat{k} = \frac{\hat{v}}{\hat{a}} = \frac{48}{37} \approx 1.30.$$

and

$$\hat{z} = \frac{6}{6 + 1.30} = 0.82.$$

The Bühlmann credibility premium for the three contracts is calculated using these formulas:

$$\hat{P}_1^{Bühlmann} = \hat{z} \bar{X}_1 + (1 - \hat{z}) \hat{\mu}$$

$$\hat{P}_2^{Bühlmann} = \hat{z} \bar{X}_2 + (1 - \hat{z}) \hat{\mu}$$

$$\hat{P}_3^{Bühlmann} = \hat{z} \bar{X}_3 + (1 - \hat{z}) \hat{\mu}$$

Thus, we calculate $\hat{P}_B^{L_2}$ as follows:

$$\hat{P}_{1,B}^{L_2} = \omega \delta_0^\times (2 - \omega) + (1 - \omega)^2 P_B^{L_1} = \omega \delta_0^\times (2 - \omega) + (1 - \omega)^2 (\hat{z} \bar{X}_1 + (1 - \hat{z}) \hat{\mu}).$$

$$\hat{P}_{2,B}^{L_2} = \omega \delta_0^\times (2 - \omega) + (1 - \omega)^2 P_B^{L_1} = \omega \delta_0^\times (2 - \omega) + (1 - \omega)^2 (\hat{z} \bar{X}_2 + (1 - \hat{z}) \hat{\mu}).$$

$$\hat{P}_{3,B}^{L_2} = \omega \delta_0^\times (2 - \omega) + (1 - \omega)^2 P_B^{L_1} = \omega \delta_0^\times (2 - \omega) + (1 - \omega)^2 (\hat{z} \bar{X}_3 + (1 - \hat{z}) \hat{\mu}).$$

Years Contract	1	2	3	4	5	6
1	0	1	2	2	1	0
2	3	4	2	4	1	4
3	3	3	2	2	1	1

Table 3.
The portfolio's data

Now, in order to finding $\widehat{P}_q^{L_2}$, we consider X the matrix containing the data of the portfolio, and we calculate $X^2 = X \circ X$ which is called the element-wise product of X with itself.

$$X^2 = \begin{pmatrix} 0 & 1 & 4 & 4 & 1 & 0 \\ 9 & 16 & 4 & 16 & 1 & 16 \\ 9 & 4 & 4 & 4 & 1 & 1 \end{pmatrix}$$

We can find straightforwardly that $\bar{X}_1 = 1, \bar{X}_2 = 3, \bar{X}_3 = 2$ and $\bar{X} = 2$.

Then, the non-parametric estimators for the quantities h, c, g, b, d_1 and d_2 are:

$$\widehat{h} = \frac{308}{15}, \widehat{c} = 201.1333, \widehat{g} = \frac{7}{45}, \widehat{b} = 4.3074, \widehat{d}_1 = 1 \text{ and } \widehat{d}_2 = 4.3333.$$

Finally, taking two values of ω , 0.1 and 0.7, the corresponding premiums are presented in the following tables: (see Table 4).

For $\omega = 0.1$: we have

$$\widehat{a}_0 = 0.7999, \widehat{Z}_q = 0.0913, \widehat{Y}_q = 0.1728.$$

For $\omega = 0.7$: we obtain similarly (see Table 5).

$$\widehat{a}_0 = 0.6548, \widehat{Z}_q = 0.0916, \widehat{Y}_q = 0.1973.$$

According to the results above, it can be seen that $\widehat{P}^{B\ddot{u}hlmann}$ is based essentially on the individual experience because $\widehat{z}$ is close to 1. For this reason, it takes a value inferior than $\widehat{P}_B^{L_2}$ and $\widehat{P}_q^{L_2}$ when the individual experience is not important. However, it is superior than $\widehat{P}_B^{L_2}$ and $\widehat{P}_q^{L_2}$ for contract 2 in the opposite case, i.e. when there is an important claims history.

Now, for $\widehat{P}_B^{L_2}$ and $\widehat{P}_q^{L_2}$, we can remark that there is a better closeness between $\widehat{P}_B^{L_2}$ and $\widehat{P}_q^{L_2}$ for the contract 1. Nevertheless, for the two other contracts, the closer the value of ω is to 0 (i.e. the relative weight assigned to the precision of estimation portion of the loss is more important), the more the value of $\widehat{P}_q^{L_2}$ diverges from $\widehat{P}_B^{L_2}$.

Contract		1	2	3
Table 4. Results of $\widehat{P}^{B\ddot{u}hlmann}$, $\widehat{P}_B^{L_2}$ and $\widehat{P}_q^{L_2}$ for the seventh year ($\omega = 0.1$)	$\widehat{P}^{B\ddot{u}hlmann}$	1.18	2.82	2
	$\widehat{P}_B^{L_2}$	1.334	2.666	2
	$\widehat{P}_q^{L_2}$	1.3351	2.6960	1.8290

Contract		1	2	3
Table 5. Results of $\widehat{P}^{B\ddot{u}hlmann}$, $\widehat{P}_B^{L_2}$ and $\widehat{P}_q^{L_2}$ for the seventh year ($\omega = 0.7$)	$\widehat{P}^{B\ddot{u}hlmann}$	1.18	2.82	2
	$\widehat{P}_B^{L_2}$	1.926	2.074	2
	$\widehat{P}_q^{L_2}$	1.9168	2.0872	1.9783

Conclusion

In this paper, we used the WBLF and a quadratic adjustment to obtain new credibility premiums. Also, we have made a comparison study between $P_q^{L_2}$ and $P_B^{L_2}$ under the two most important approaches: the parametric and the non-parametric approaches. According to the results obtained in this work, we can recommend the suitable quadratic framework for a practitioner who wants to find a more flexible premium. For future studies, we can treat the semi-parametric case and make more applications in life insurance, for example, one important problem is how to recognize a change in underlying mortality rates operating in a population under study. When is a fluctuation from past experience, as evidenced by recent data, purely a random effect and when is it a change in the basic risk process?

In this work, we take $\delta_0(x) = \bar{X}$, but we can consider other choices, like a more robust one with a median estimator, or even using credibility to estimate the variance around $\bar{X}$.

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