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# Further study on the Brück conjecture and some non-linear complex differential equations

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## Abstract

**Purpose** – The purpose of this current paper is to deal with the study of non-constant entire solutions of some non-linear complex differential equations in connection to Brück conjecture, by using the theory of complex differential equation. The results generalize the results due to Pramanik *et al.*

**Design/methodology/approach** – 39B32, 30D35.

**Findings** – In the current paper, we mainly study the Brück conjecture and the various works that confirm this conjecture. In our study we find that the conjecture can be generalized for differential monomials under some additional conditions and it generalizes some works related to the conjecture. Also we can take the complex number $a$ in the conjecture to be a small function. More precisely, we obtain a result which can be restate in the following way: Let $f$ be a non-constant entire function such that $\sigma_2(f) < \infty$, $\sigma_2(f)$ is not a positive integer and $\delta(0, f) > 0$. Let $M[f]$ be a differential monomial of $f$ of degree $\gamma_M$ and $\alpha(z), \beta(z) \in S(f)$ be such that $\max\{\sigma(\alpha), \sigma(\beta)\} < \sigma(f)$. If $M[f] + \beta$ and $f^{\gamma_M} - \alpha$ share the value 0 CM, then

$$\frac{M[f] + \beta}{f^{\gamma_M} - \alpha} = c,$$

where $c \neq 0$ is a constant.

**Originality/value** – This is an original work of the authors.

**Keywords** Entire function, Brück conjecture, Small function, Differential monomial

**Paper type** Research paper

## 1. Introduction and main results

In this paper, by meromorphic function we shall always mean a meromorphic function in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [1–4]. It will be convenient to let $E$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence.

For any non-constant meromorphic function $f(z)$, we denote by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$, $r \notin E$, where $T(r, f)$ is the Nevanlinna characteristic function of $f$. A meromorphic function $\alpha$ is said to be small with respect to $f(z)$ if $T(r, \alpha) = S(r, f)$. We denote by $S(f)$ the collection of all small functions with respect to $f$. Clearly $\mathbb{C} \cup \{\infty\} \subset S(f)$ and $S(f)$ is a field over the set of complex numbers.

**JEL Classification** — 39B32, 30D35.



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For any two non-constant meromorphic functions $f$ and $g$, and $\alpha \in S(f) \cap S(g)$, we say that $f$ and $g$ share α IM(CM) provided that $f-\alpha$ and $g-\alpha$ have the same zeros ignoring(counting) multiplicities.

For any complex number $a$, the quantity defined by

$$\delta(a,f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f-a})}{T(r,f)} = 1 - \limsup_{r\to\infty} \frac{N(r, \frac{1}{f-a})}{T(r,f)}$$

is called the deficiency of $a$ with respect to the function $f(z)$.

We also need the following definitions:

**Definition 1.1.** Let $f(z)$ be a non-constant entire function, then the order $\sigma(f)$ of $f(z)$ is defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}$$

and the lower order $\mu(f)$ of $f(z)$ is defined by

$$\mu(f) = \liminf_{r\to+\infty} \frac{\log T(r,f)}{\log r} = \liminf_{r\to+\infty} \frac{\log\log M(r,f)}{\log r}.$$

The type $\tau(f)$ of an entire function $f(z)$ with $0 < \sigma(f) = \sigma < +\infty$ is defined by

$$\tau(f) = \limsup_{r\to+\infty} \frac{\log M(r,f)}{r^{\sigma}},$$

where and in the sequel

$$M(r,f) = \max_{|z|=r} |f(z)|.$$

**Definition 1.2.** Let $f$ be a non-constant meromorphic function. Then the hyper-order $\sigma_2(f)$ of $f(z)$ is defined as follows:

$$\sigma_2(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r}.$$

**Definition 1.3.** Let $f$ be a non-constant meromorphic function. A differential monomial of $f$ is an expression of the form

$$M[f] = a_0(z) f^{n_0} \left(f^{(1)}\right)^{n_1} \left(f^{(2)}\right)^{n_2} \cdots \left(f^{(k)}\right)^{n_k}, \quad (1)$$

where $n_0, n_1, n_2, \ldots, n_k$ are non-negative integers and $a_0(z) \in S(f)$. The degree of the differential monomial is given by $\gamma_M = n_0 + n_1 + n_2 + \ldots + n_k$.

Rubel and Yang [5] proved that if a non-constant entire function $f$ and its derivative $f'$ share two distinct finite complex numbers CM, then $f \equiv f'$. What will be the relation between $f$ and $f'$, if an entire function $f$ and its derivative $f'$ share one finite complex number CM? Brück [6] made a conjecture that if $f$ is a non-constant entire function satisfying $\sigma_2(f) < \infty$, where $\sigma_2(f)$ is not a positive integer and if $f$ and $f'$ share one finite complex number $a$ CM, then $f' - a = c(f-a)$ for some finite complex number $c \neq 0$. Brück [6] himself proved the conjecture for $a = 0$. Brück also proved that the conjecture is true for $a \neq 0$ provided that $f$ satisfies the additional assumption $N(r, \frac{1}{f'}) = S(r,f)$ and in this case the order restriction on $f$ can be omitted. After that many researchers [7–10] have proved the conjecture under different conditions.

In 2017, Pramanik *et al.* [11] investigated on the non-constant entire solution of some non-linear complex differential equations related to Brück conjecture and proved the following theorems:

**Theorem 1.1.** Let $f(z)$ and $\alpha(z)$ be two non-constant entire functions and satisfy $0 < \sigma(\alpha) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\alpha)$. Also, let $P(z)$ be a polynomial. If $f$ is a non-constant entire solution of the following differential equation

$$M[f] - \alpha = (f^{\gamma_M} - \alpha)e^{P(z)},$$

then $P(z)$ is a constant.

**Theorem 1.2.** Let $f(z)$ and $\alpha(z)$ be two non-constant entire functions and satisfy $0 < \sigma(\alpha) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\alpha)$. Also, let $P(z)$ be a polynomial. If $f$ is a non-constant entire solution of the following differential equation

$$M[f] + \beta(z) - \alpha(z) = (f^{\gamma_M} - \alpha(z))e^{P(z)},$$

where $\beta(z)$ is an entire function satisfying $0 < \sigma(\beta) = \sigma(f) < +\infty$ and $\tau(f) > \tau(\beta)$, then $P(z)$ is a constant.

**Theorem 1.3.** Let $f(z)$ and $\alpha(z)$ be two non-constant entire functions satisfying $\sigma(\alpha) < \mu(f)$ and $P(z)$ be a polynomial. If $f$ is a non-constant entire solution of the following differential equation

$$M[f] + \beta(z) - \alpha(z) = (f^{\gamma_M} - \alpha(z))e^{P(z)},$$

where $\beta(z)$ is an entire function satisfying $\sigma(\beta) < \mu(f)$. Then $\sigma_2(f) = \deg P$.

Regarding Theorems 1.1–1.3, one can ask the the following

(1) What will happen if $P(z)$ is an entire function?

In this paper we answer the question by proving the following theorems:

**Theorem 1.4.** Let $f(z)$ be a non-constant entire function such that $\sigma_2(f) < \infty$, $\sigma_2(f)$ is not a positive integer and $\delta(0, f) > 0$. Let $M[f]$ be a differential monomial of $f$ of degree $\gamma_M$ as defined in (1), $\phi(z)$ be an entire function and $\alpha(z) \in S(f)$ be such that $\sigma(\alpha) < \sigma(f)$. If $f$ is a solution of the following differential equation

$$M[f] - \alpha(z) = (f^{\gamma_M} - \alpha(z))e^{\phi(z)}, \tag{2}$$

then $\frac{M[f] - \alpha(z)}{f^{\gamma_M} - \alpha(z)} = c$, where $c \neq 0$ is a constant.

**Theorem 1.5.** Let $f$ be a non-constant entire function such that $\sigma_2(f) < \infty$, $\sigma_2(f)$ is not a positive integer and $\delta(0, f) > 0$. Let $M[f]$ be a differential monomial of $f$ of degree $\gamma_M$ as defined in (1), $\phi(z)$ be an entire function and $\alpha(z), \beta(z) \in S(f)$ be such that $\sigma(\alpha) < \sigma(f)$ and $\sigma(\beta) < \sigma(f)$. If $f$ is a solution of the following differential equation

$$M[f] + \beta(z) = (f^{\gamma_M} - \alpha(z))e^{\phi(z)}, \tag{3}$$

then $\frac{M[f] + \beta(z)}{f^{\gamma_M} - \alpha(z)} = c$, where $c \neq 0$ is a constant.

## 2. Preparatory lemmas

In this section we state some lemmas needed to prove the theorems.

**Lemma 2.1.** [2] Let $f(z)$ be a transcendental entire function, $\nu(r, f)$ be the central index of $f(z)$. Then there exists a set $E \subset (1, +\infty)$ with finite logarithmic measure such that

$r \notin [0, 1] \cup E$, consider $z$ with $|z| = r$ and $|f(z)| = M(r, f)$, we get

$$\frac{f^{(j)}(z)}{f(z)} = \left\{\frac{\nu(r, f)}{z}\right\}^j (1 + o(1)), \text{ for } j \in N.$$

**Lemma 2.2.** [12] Let $f(z)$ be an entire function of finite order $\sigma(f) = \sigma < +\infty$, and let $\nu(r, f)$ be the central index of $f$. Then

$$\limsup_{r \to +\infty} \frac{\log \nu(r, f)}{\log r} = \sigma(f).$$

And if $f$ is a transcendental entire function of hyper order $\sigma_2(f)$, then

$$\limsup_{r \to +\infty} \frac{\log \log \nu(r, f)}{\log r} = \sigma_2(f).$$

**Lemma 2.3.** [13] Let $f(z)$ be a transcendental entire function and let $E \subset [1, +\infty)$ be a set having finite logarithmic measure. Then there exists $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n, f), \theta_n \in [0, 2\pi), \lim_{n \to +\infty} \theta_n = \theta_0 \in [0, 2\pi], r_n \notin E$ and if $0 < \sigma(f) < +\infty$, then for any given $\varepsilon > 0$ and sufficiently large $r_n$,

$$r_n^{\sigma(f) - \varepsilon} < \nu(r_n, f) < r_n^{\sigma(f) + \varepsilon}.$$

If $\sigma(f) = +\infty$, then for any given large $K > 0$ and sufficiently large $r_n$,

$$\nu(r_n, f) > r_n^K.$$

**Lemma 2.4.** [2] Let $P(z) = b_n z^n + b_{n-1} z^{n-1} + \ldots + b_0$ with $b_n \neq 0$ be a polynomial. Then for every $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $r = |z| > r_0$ the inequalities

$$(1 - \varepsilon)|b_n| r^n \leq |P(z)| \leq (1 + \varepsilon)|b_n| r^n$$

hold.

**Lemma 2.5.** [14] Let $f(z)$ and $A(z)$ be two entire functions with $0 < \sigma(f) = \sigma(A) = \sigma < +\infty$, $0 < \tau(A) = \tau(f) < +\infty$, then there exists a set $E \subset [1, +\infty)$ that has infinite logarithmic measure such that for all $r \in E$ and a positive number $\kappa > 0$, we have

$$\frac{M(r, A)}{M(r, f)} < \exp\{-\kappa r^\sigma\}.$$

**Lemma 2.6.** [14] *Let* $g : (0, \infty) \to \mathbb{R}$, $h : (0, \infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside an exceptional set $E$ with finite linear measure, or $g(r) \leq h(r)$, $r \notin H \cup (0, 1]$, where $H \subset (1, \infty)$ is a set of finite logarithmic measure. Then for any $\alpha > 1$, there exists $r_0$ such that $g(r) \leq h(\alpha r)$ for all $r \geq r_0$.

## 3. Proof of main theorems

In this section we present the proofs of the main results of the paper.

### *3.1 Proof of Theorem 1.4*

We will consider the following two cases:

*Case I:* Let $\alpha(z) \equiv 0$. Then

$$\frac{M[f]}{f^{\gamma_M}} = e^{\phi(z)}. \tag{4}$$

Now,

$$\begin{aligned} \frac{M[f]}{f^{\gamma_M}} &= \frac{a_0(z) f^{n_0} \left(f^{(1)}\right)^{n_1} \ldots \left(f^{(k)}\right)^{n_k}}{f^{n_0+n_1+\ldots+n_k}} \\ &= a_0(z) \left(\frac{f^{(1)}}{f}\right)^{n_1} \left(\frac{f^{(2)}}{f}\right)^{n_2} \ldots \left(\frac{f^{(k)}}{f}\right)^{n_k}. \end{aligned} \tag{5}$$

From (4) and (5), it follows that

$$\begin{aligned} T(r, e^{\phi}) = m(r, e^{\phi}) &= m\left(r, \frac{M[f]}{f^{\gamma_M}}\right) \\ &\leq \sum_{i=1}^{k} n_i m\left(r, \frac{f^{(i)}}{f}\right) + m(r, a_0) \\ &= O(\log(rT(r, f))), \end{aligned}$$

outside an exceptional set $E_0$ of finite linear measure.

Thus there exists a constant $K$ such that

$$T(r, e^{\phi}) \leq K \log(rT(r, f)) \text{ for } r \notin E_0.$$

By Lemma 2.6 there exists $r_0 > 0$ such that for $r \geq r_0$, we have

$$T(r, e^{\phi}) \leq K \log(\eta r T(\eta r, f)) \text{ for } \eta > 1. \tag{6}$$

From (6), we can deduce that $\sigma(e^{\phi}) \leq \sigma_2(f) < \infty$ and hence $\phi(z)$ is a polynomial.

Proceeding similarly as in [11], Theorem 3, we obtain that $\sigma_2(f) = \deg\phi$, which is a contradiction to our assumption that $\sigma_2(f)$ is not a positive integer. Hence $\phi(z)$ is only a constant.

*Case II:* Let $\alpha(z) \not\equiv 0$ and $d = \gamma_M$. Taking the logarithmic derivative of (2), we get

$$\phi'(z) = \frac{M'[f] - \alpha'(z)}{M[f] - \alpha(z)} - \frac{df^{d-1}f' - \alpha'(z)}{f^d - \alpha(z)}. \tag{7}$$

*Subcase I:* Let $\phi'(z) \equiv 0$. Then $\phi(z) = c_1$, $c_1$ is a constant.

*Subcase II:* Let $\phi'(z) \not\equiv 0$. Then it follows from (7) that

$$m(r, \phi') = S(r, f). \tag{8}$$

We can rewrite (7) in the following form:

$$\begin{aligned} \phi' &= f^d \left[\frac{M[f]}{f^d} . \frac{1}{M[f]} . \frac{M'[f] - \alpha'(z)}{M[f] - \alpha(z)} - \frac{1}{f^d} \frac{df^{d-1}f' - \alpha'(z)}{f^d - \alpha(z)}\right] \\ &= \frac{f^d}{\alpha(z)} \left[\frac{M[f]}{f^d} . \frac{M'[f] - \alpha'(z)}{M[f] - \alpha(z)} - \frac{M'[f]}{f^d} - \frac{df^{d-1}f' - \alpha'(z)}{f^d - \alpha(z)} + \frac{df'}{f}\right]. \end{aligned} \tag{9}$$

We set

$$\psi = \frac{M[f]}{f^d}.\frac{M'[f]-\alpha'(z)}{M[f]-\alpha(z)} - \frac{M'[f]}{f^d} - \frac{df^{d-1}f'-\alpha'(z)}{f^d-\alpha(z)} + \frac{df'}{f}. \tag{10}$$

Then we have

$$m(r,\psi) = S(r,f).$$

Therefore it follows from (9) and (10) that

$$\frac{\alpha(z)}{f^d} = \frac{\psi(z)}{\phi'(z)}. \tag{11}$$

Since $\phi$ is an entire function, then we have

$$\begin{aligned} m\left(r,\frac{1}{f^d}\right) &\le m\left(r,\frac{\alpha(z)}{f^d}\right) + m\left(r,\frac{1}{\alpha(z)}\right) \\ &\le m\left(r,\frac{\psi(z)}{\phi'(z)}\right) + S(r,f) \\ &\le m(r,\psi(z)) + m\left(r,\frac{1}{\phi'(z)}\right) + S(r,f) \\ &= T(r,\phi'(z)) + S(r,f) \\ &= m(r,\phi') + S(r,f) \\ &= S(r,f) \\ \Rightarrow m\left(r,\frac{1}{f}\right) &= S(r,f). \end{aligned} \tag{12}$$

It follows from (12) that

$$\delta(0,f) = \liminf_{r\to\infty} \frac{m(r,\frac{1}{f})}{T(r,f)} = 0,$$

which contradicts our hypothesis.

Thus the proof is completed.

*3.2 Proof of Theorem 1.5*

We will consider the following two cases:

*Case I:* Let $\alpha(z)\equiv 0$. Then from (3) it follows that

$$\begin{aligned} M[f] + \beta(z) &= f^{\gamma_M} e^{\phi(z)} \\ \Rightarrow e^{\phi(z)} &= \frac{M[f]+\beta(z)}{f^{\gamma_M}}. \end{aligned}$$

Proceeding similarly as in Case I of Theorem 1.4, we can prove that $\phi(z)$ is a constant.

*Case II:* Let $\alpha(z)\not\equiv 0$ and $d=\gamma_M$. Eliminating $e^{\phi}$ from (3) and its derivative, we get

$$\phi' = \frac{M'[f]+\beta'(z)}{M[f]+\beta(z)} - \frac{df^{d-1}f'-\alpha'(z)}{f^d-\alpha(z)}. \tag{13}$$

*Subcase I:* Let $\phi'(z)\equiv 0$. Then $\phi(z)=c_2$, $c_2$ is a constant.

*Subcase II:* Let $\phi'(z) \not\equiv 0$. Then it follows from (13) that

$$m(r, \phi') = S(r, f). \tag{14}$$

Now,

$$\begin{aligned}\frac{M'[f]+\beta'(z)}{M[f]+\beta(z)} &= \frac{f^d}{\beta(z)}\left[\frac{M[f]}{f^d}\left[\frac{1}{M[f]}-\frac{1}{M[f]+\beta(z)}\right](M'[f]+\beta'(z))\right]\\ &= \frac{f^d}{\beta(z)}\left[\frac{M'[f]+\beta'(z)}{f^d}-\frac{M[f]}{f^d}.\frac{M'[f]+\beta'(z)}{M[f]+\beta(z)}\right],\end{aligned} \tag{15}$$

and

$$\begin{aligned}\frac{df^{d-1}f'-\alpha'(z)}{f^d-\alpha(z)} &= \frac{f^d}{\alpha(z)}\left[\frac{1}{f^d-\alpha(z)}-\frac{1}{f^d}\right]\left(df^{d-1}f'-\alpha'(z)\right)\\ &= \frac{f^d}{\alpha(z)}\left[\frac{df^{d-1}f'-\alpha'(z)}{f^d-\alpha(z)}-\frac{df'}{f}+\frac{\alpha'(z)}{f^d}\right].\end{aligned} \tag{16}$$

Therefore from (13), (15) and (16) we have

$$\begin{aligned}\phi' &= \frac{f^d}{\beta(z)}\left[\frac{M'[f]}{f^d}-\frac{M[f]}{f^d}.\frac{M'[f]+\beta'(z)}{M[f]+\beta(z)}+\frac{\beta'(z)}{f^d}\right]\\ &-\frac{f^d}{\alpha(z)}\left[\frac{df^{d-1}f'-\alpha'(z)}{f^d-\alpha(z)}-\frac{df'}{f}+\frac{\alpha'(z)}{f^d}\right]\\ &=\frac{f^d}{\beta(z)}\left[\frac{M'[f]}{f^d}-\frac{M[f]}{f^d}.\frac{M'[f]+\beta'(z)}{M[f]+\beta(z)}\right]-\frac{f^d}{\alpha(z)}\left[\frac{df^{d-1}f'-\alpha'(z)}{f^d-\alpha(z)}-\frac{df'}{f}\right]\\ &+\frac{\beta'(z)}{\beta(z)}-\frac{\alpha'(z)}{\alpha(z)}.\end{aligned}$$

$$\begin{aligned}\Rightarrow \phi'-\frac{\beta'(z)}{\beta(z)}+\frac{\alpha'(z)}{\alpha(z)} &= \frac{f^d}{\beta(z)}\left[\frac{M'[f]}{f^d}-\frac{M[f]}{f^d}.\frac{M'[f]+\beta'(z)}{M[f]+\beta(z)}\right]\\ &-\frac{f^d}{\alpha(z)}\left[\frac{df^{d-1}f'-\alpha'(z)}{f^d-\alpha(z)}-\frac{df'}{f}\right].\end{aligned} \tag{17}$$

Let

$$\psi_1 = \frac{M'[f]}{f^d}-\frac{M[f]}{f^d}.\frac{M'[f]+\beta'(z)}{M[f]+\beta(z)}$$

and

$$\psi_2 = \frac{df^{d-1}f'-\alpha'(z)}{f^d-\alpha(z)}-\frac{df'}{f}.$$

Then we have $m(r, \psi_1) = S(r, f)$ and $m(r, \psi_2) = S(r, f)$.

Thus it follows from (17) that

$$\phi' - \frac{\beta'(z)}{\beta(z)} + \frac{\alpha'(z)}{\alpha(z)} = f^d\left[\frac{\psi_1}{\beta(z)} - \frac{\psi_2}{\alpha(z)}\right]$$

$$\Rightarrow \frac{1}{f^d} = \frac{\left[\frac{\psi_1}{\beta(z)} - \frac{\psi_2}{\alpha(z)}\right]}{\phi' - \frac{\beta'(z)}{\beta(z)} + \frac{\alpha'(z)}{\alpha(z)}}. \tag{18}$$

Since $\phi$ is an entire function, from (18) we have

$$\begin{aligned} m\left(r, \frac{1}{f^d}\right) &\le m\left(r, \frac{\psi_1}{\beta(z)} - \frac{\psi_2}{\alpha(z)}\right) + m\left(r, \frac{1}{\phi' - \frac{\beta'(z)}{\beta(z)} + \frac{\alpha'(z)}{\alpha(z)}}\right) \\ &\le m(r, \psi_1) + m(r, \psi_2) + T(r, \phi') + S(r, f) \\ &= S(r, f) \\ \Rightarrow m\left(r, \frac{1}{f}\right) &= S(r, f). \end{aligned} \tag{19}$$

It follows from (19) that

$$\delta(0, f) = \liminf_{r\to\infty} \frac{m(r, \frac{1}{f})}{T(r, f)} = 0,$$

which is a contradiction.

Hence the proof is completed.

**Corollary 3.1.** Let $f(z)$ be a non-constant entire function such that $\sigma_2(f) < \infty$, $\sigma_2(f)$ is not a positive integer and $\delta(0, f) > 0$. Let $M[f]$ be a differential monomial of $f$ of degree $\gamma_M$ as defined in (1), $\phi(z)$ be an entire function and $\alpha(z) \in S(f)$ be such that $\sigma(\alpha) < \mu(f)$. If $f$ is a solution of the following differential equation

$$M[f] - \alpha(z) = (f^{\gamma_M} - \alpha(z))e^{\phi(z)},$$

then $\frac{M[f] - \alpha(z)}{f^{\gamma_M} - \alpha(z)} = c$, where $c \neq 0$ is a constant.

**Corollary 3.2.** Let $f$ be a non-constant entire function such that $\sigma_2(f) < \infty$, $\sigma_2(f)$ is not a positive integer and $\delta(0, f) > 0$. Let $M[f]$ be a differential monomial of $f$ of degree $\gamma_M$ as defined in (1), $\phi(z)$ be an entire function and $\alpha(z), \beta(z) \in S(f)$ be such that $\sigma(\alpha) < \mu(f)$ and $\sigma(\beta) < \mu(f)$. If $f$ is a solution of the following differential equation

$$M[f] + \beta(z) = (f^{\gamma_M} - \alpha(z))e^{\phi(z)},$$

then $\frac{M[f]+\beta(z)}{f^{\gamma_M} - \alpha(z)} = c$, where $c \neq 0$ is a constant.

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