Introduction of new Picard–S hybrid iteration with application and some results for nonexpansive mappings

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Abstract

Purpose – In this paper, Picard–S hybrid iterative process is defined, which is a hybrid of Picard and S-iterative process. This new iteration converges faster than all of Picard, Krasnoselskii, Mann, Ishikawa, S-iteration, Picard–Mann hybrid, Picard–Krasnoselskii hybrid and Picard–Ishikawa hybrid iterative processes for contraction mappings and to find the solution of delay differential equation, using this hybrid iteration also proved some results for Picard–S hybrid iterative process for nonexpansive mappings.

Design/methodology/approach – This new iteration converges faster than all of Picard, Krasnoselskii, Mann, Ishikawa, S-iteration, Picard–Mann hybrid, Picard–Krasnoselskii hybrid, Picard–Ishikawa hybrid iterative processes for contraction mappings.

Findings – Showed the fastest convergence of this new iteration and then other iteration defined in this paper. The author finds the solution of delay differential equation using this hybrid iteration. For new iteration, the author also proved a theorem for nonexpansive mapping.

Originality/value – This new iteration converges faster than all of Picard, Krasnoselskii, Mann, Ishikawa, S-iteration, Picard–Mann hybrid, Picard–Krasnoselskii hybrid, Picard–Ishikawa hybrid iterative processes for contraction mappings and to find the solution of delay differential equation, using this hybrid iteration also proved some results for Picard–S hybrid iterative process for nonexpansive mappings.

Keywords Fixed point, Iteration, Contraction, Nonexpansive mappings

Paper type Research paper

1. Introduction

Let E be a normed linear space and C be a non-empty convex subset of E. A mapping $T: C \to C$ is called contraction if

$$\|Tx - Ty\| \leq \delta\|x - y\| \tag{1.1}$$

for all $x, y \in C$ and $\delta \in (0, 1)$.

Let C be a non-empty subset of a normed linear space E and $T: C \to E$ a mapping. Then T is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \text{ for all } x, y \in C$$

A sequence $x_n \subset C$ is an approximating fixed point sequence of T if $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. We say that $x \in C$ is a fixed point of T if $T(x) = x$ and denote $F(T)$ the set of all fixed points of T.

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In this paper, N denotes the set of all positive integers.

The Picard iterative process [1] is defined by the sequence $\{u_n\}$ as follows:

$$\begin{aligned} u_1 &= u \in C \\ u_{n+1} &= Tu_n, \quad n \in N \end{aligned} \tag{1.2}$$

The Krasnoselskii iterative process [2] is defined by the sequence $\{v_n\}$:

$$\begin{aligned} v_1 &= v \in C \\ v_{n+1} &= (1-\lambda)v_n + \lambda T v_n, \quad n \in N \end{aligned} \tag{1.3}$$

where $\lambda \in (0,\ 1)$.

The Mann iteration [3] is defined by the sequence $\{w_n\}$:

$$\begin{aligned} w_1 &= w \in C \\ w_{n+1} &= (1-\alpha_n)w_n + \alpha_n T w_n, \quad n \in N \end{aligned} \tag{1.4}$$

where $\{\alpha_n\} \subset (0,\ 1)$ satisfies certain appropriate conditions.

The Ishikawa iterative process [4] is defined by the sequence $\{z_n\}$:

$$\begin{aligned} z_1 &= z \in C \\ z_{n+1} &= (1-\alpha_n)z_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)z_n + \beta_n T z_n, \quad n \in N \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,\ 1)$ satisfies certain appropriate conditions.

S-iterative process [5] is defined by the sequence $\{q_n\}$:

$$\begin{aligned} q_1 &= q \in C \\ q_{n+1} &= (1-\alpha_n)Tq_n + \alpha_n T y_n \\ y_n &= (1-\beta_n)q_n + \beta_n T q_n, \quad n \in N \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}, \{\beta_n\} \subset (0,\ 1)$ satisfies certain appropriate conditions.

Many important non-linear problems of applied mathematics are usually constructed in the form of fixed point equation. These problems are related with physical problem of applied sciences and engineering.

The Picard iteration is the simple iteration for approximate solution of fixed point equation for non-linear contraction mapping. Some results based on Picard iteration are introduced by Chidume and Olaleru [6].

Khan [7] introduced the Picard–Mann hybrid iterative process defined by the sequence $\{s_n\}$:

$$\begin{aligned} s_1 &= s \in C \\ s_{n+1} &= T y_n \\ y_n &= (1-\alpha_n)s_n + \alpha_n T s_n, \quad n \in N \end{aligned} \tag{1.7}$$

where $\{\alpha_n\}$ is a real sequence in $(0, 1)$.

Okeke and Abbas [8] introduced the Picard–Krasnoselskii hybrid iterative process defined by the sequence $\{m_n\}$:

$$\begin{aligned} m_1 &= m \in C \\ m_{n+1} &= T y_n \\ y_n &= (1-\lambda)m_n + \lambda T m_n, \quad n \in N \end{aligned} \tag{1.8}$$

where $\lambda \in (0,\ 1)$.

Okeke [9] introduced the Picard–Ishikawa hybrid iterative process defined by the sequence $\{t_n\}$:

$$\begin{aligned} t_1 &= t \in C \\ t_{n+1} &= Tv_n v \\ v_n &= (1-\alpha_n)t_n + \alpha_n Tu_n \\ u_n &= (1-\beta_n)t_n + \beta_n Tt_n, \quad n \in N \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in (0,1). Using hybridization with Picard, now I introduce Picard–S hybrid iterative process defined by the sequence $\{x_n\}$:

$$\begin{aligned} x_1 &= x \in C \\ x_{n+1} &= Tz_n \\ z_n &= (1-\alpha_n)Tx_n + \alpha_n Ty_n \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n, \quad n \in N \end{aligned} \tag{1.10}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0, 1)$ satisfying condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n) = \infty \tag{1.11}$$

Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration processes that converge to a certain fixed point p of a given operator T. The sequence $\{u_n\}$ is better than $\{v_n\}$ if

$$||u_n - p|| \le ||v_n - p||$$

for all $n \in N$ (given by Rhodes [10]).

2. Preliminaries

Definition 2.1. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b, respectively. If

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0$$

then $\{a_n\}$ converges faster than $\{b_n\}$.

Definition 2.2. Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iterative processes, both converge to fixed point p of a given operator T. Suppose that the error estimates.

$$\begin{aligned} ||u_n - p|| &\le a_n, \quad \text{for all } n \in N \\ ||v_n - p|| &\le b_n, \quad \text{for all } n \in N \end{aligned}$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to 0. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to p.

Definition 2.3. Let X be a Banach space. Then a function $\delta_X : [0, 2] \to [0, 1]$ is said to be the modulus of convexity of X if

$$\delta_X(\epsilon) = inf\left\{1 - \left\|\frac{x+y}{2}\right\| : ||x|| \le 1,\ ||y|| \le 1,\ ||x-y|| \ge \epsilon\right\}$$

It is easy to see that $\delta_X(0) = 0$ and $\delta_X(t) \ge 0$ for all $t \ge 0$. References [10–19] dealing with rate of convergence of iterative process. Some authors analyse its stability. We need following lemma to prove result.

Lemma 2.1. Let $\{s_n\}$ be a sequence of positive real numbers which satisfies

$$s_{n+1} \leq (1-\mu_n)s_n$$

$$If \{\mu_n\} \subset (0,1) \text{ and } \sum_{n=1}^{\infty}\mu_n = \infty, \text{ then } \lim_{n\to\infty} s_n = 0.$$

The aim of this paper is to introduce the Picard–S hybrid iterative process and to show that this new iterative process is faster than all of Picard, Krasnoselskii, Mann, Ishikawa in sense of Berinde [20], S-iteration in sense of Agarwal [5], Picard–Mann hybrid in sense of Khan [7], Picard–Krasnoselskii hybrid in sense of Okeke [8] and Picard–Ishikawa hybrid in the sense of Okeke [9].

Okeke already proved that Picard–Krasnoselskii hybrid iterative process converges faster than Picard, Krasnoselskii, Mann and Ishikawa. Khan [7] proved that Picard–Mann hybrid iterative process converges faster than Picard, Mann, Ishikawa iterative processes. Therefore, I show that my new Picard–S hybrid iterative process converges faster than S-iteration, Picard–Mann hybrid iteration, Picard–Krasnoselskii hybrid iteration and Picard–Ishikawa hybrid iterative process in the topic Rate of Convergence. In 2020, Zhao [21] proved existence and uniqueness of pseudo almost periodic solution for a class of iterative functional differential equations with delays depending on state. In next section, I find the solution of delay differential equation using Picard–S hybrid iterative process. Aynur Sahin [22] proved some strong convergence results of Picard–Krasnoselskii hybrid iterative process for a general class of contractive-like operator in hyperbolic space. In next section, I prove some results of Picard–S hybrid iterative process for nonexpansive mappings in uniformly convex Banach space.

3. Rate of convergence

Proposition 3.1. Let C be a non-empty closed convex subset of a normed space E and let T be a contraction of C into itself. Suppose that each of the iterative process 1.6, 1.7, 1.8, 1.9 and 1.10 converges to the same fixed point p of T where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0.1) such that $0 < \lambda \leq \alpha_n, \beta_n < 1$ for all $n \in N$ and for some λ and $\delta \in (0, 1)$ is a Lipschitz constant for contraction mapping T. Then Picard–S hybrid iterative process defined by (1.10) converges faster than all the other four iterations.

Proof: Suppose that p is the fixed point of the operator T. Using (1.1) and S-iterative process (1.6), we have

$$\begin{aligned}
\|q_{n+1}-p\| &\leq (1-\alpha_n)\delta\|q_n-p\| + \alpha_n\delta\|y_n-p\| \\
&\leq \delta[(1-\alpha_n)\|q_n-p\| + \alpha_n\{(1-\beta_n)\|q_n-p\| + \delta\beta_n\|q_n-p\|\}] \\
&= \delta[1-(1-\delta)\alpha_n\beta_n]\|q_n-p\| \\
&= \left[\delta - \delta\alpha_n\beta_n + \delta^2\alpha_n\beta_n\right]\|q_n-p\| \\
&\leq [\delta - \delta\alpha_n\beta_n + \delta\alpha_n\beta_n]\|q_n-p\| \\
&= \delta\|q_n-p\| \\
&\leq \delta^2\|q_{n-1}-p\| \\
&\ \ \vdots \\
&\leq \delta^n\|q_1-p\|
\end{aligned} \tag{3.1}$$

Let $a_n = \delta^n\|q_1-p\|$

Now using (1.1) and Picard–Mann hybrid iterative process (1.7), we have

$$\begin{aligned}
\|s_{n+1} - P\| &= \|Ty_n - p\| \\
&\leq \delta\|y_n - p\| \\
&= \delta[(1-\alpha_n)\|s_n - p\| + \alpha_n\|Ts_n - p\|] \\
&\leq \delta[(1-\alpha_n)\|s_n - p\| + \alpha_n\delta\|s_n - p\|] \\
&= \delta[(1-\alpha_n + \alpha_n\delta)\|s_n - p\|] \\
&= \delta(1-(1-\delta)\alpha_n)\|s_n - p\| \\
&\leq \delta\big(1-(1-\delta)\lambda^2\big)\|s_n - p\| \\
&\vdots \\
&\leq \big[\delta\big(1-(1-\delta)\lambda^2\big)\big]^n\|s_1 - p\| \\
\text{Let} \quad b_n &= \big[\delta\big(1-(1-\delta)\lambda^2\big)\big]^n\|s_1 - p\|
\end{aligned} \tag{3.2}$$

Using (1.1) and Picard–Krasnoselskii hybrid iteration (1.8)

$$\begin{aligned}
\|m_{n+1} - p\| &= \|Ty_n - p\| \\
&\leq \delta\|y_n - p\| \\
&\leq \delta\|(1-\lambda)(m_n - p) + \lambda(Tm_n - p)\| \\
&\leq \delta[(1-\lambda)\|m_n - p\| + \lambda\delta\|m_n - p\|] \\
&= \delta\big[\big(1-(1-\delta)\lambda^2\big)\big]\|m_n - p\| \\
&\vdots \\
&\leq \big[\delta\big(1-(1-\delta)\lambda^2\big)\big]^n\|m_1 - p\| \\
\text{Let} \quad c_n &= \big[\delta\big(1-(1-\delta)\lambda^2\big)\big]^n\|m_1 - p\|
\end{aligned} \tag{3.3}$$

Using (1.1) and Picard–Ishikawa hybrid iterative process (1.9), we have

$$\begin{aligned}
\|t_{n+1} - p\| &= \|Tv_n - p\| \\
&\leq \delta\|v_n - p\| \\
\|u_n - p\| &= \|(1-\beta_n)t_n + \beta_n Tt_n - p\| \\
&\leq (1-\beta_n)\|t_n - p\| + \beta_n\|Tt_n - p\| \\
&\leq (1-\beta_n)\|t_n - p\| + \beta_n\delta\|t_n - p\| \\
&= (1-\beta_n + \beta_n\delta)\|t_n - p\| \\
&= [1-\beta_n(1-\delta)]\|t_n - p\| \\
\|v_n - p\| &= \|(1-\alpha_n)t_n + \alpha_n Tu_n - p\| \\
&\leq (1-\alpha_n)\|t_n - p\| + \alpha_n\|Tu_n - p\| \\
&\leq (1-\alpha_n)\|t_n - p\| + \alpha_n\delta\|u_n - p\| \\
&\leq (1-\alpha_n)\|t_n - p\| + \alpha_n\delta[1-\beta_n(1-\delta)]\|t_n - p\| \\
&= [1-\alpha_n + \alpha_n\delta\{1-\beta_n(1-\delta)\}]\|t_n - p\| \\
&= [1-\alpha_n + \alpha_n\delta - \alpha_n\beta_n(1-\delta)]\|t_n - p\| \\
&= [1-\alpha_n(1-\delta) - \alpha_n\beta_n(1-\delta)]\|t_n - p\| \\
&\leq [1-\alpha_n(1-\delta)]\|t_n - p\| \\
\text{Now,} \quad \|t_{n+1} - p\| &\leq \delta[1-\alpha_n(1-\delta)]\|t_n - p\| \\
&\leq \delta\big[1-\lambda^2(1-\delta)\big]\|t_n - p\| \\
&\vdots \\
&\leq \delta^n\big[1-\lambda^2(1-\delta)\big]^n\|t_1 - p\| \\
\text{Let} \quad d_n &= \delta^n\big[1-\lambda^2(1-\delta)\big]^n\|t_1 - p\|
\end{aligned} \tag{3.4}$$

Using (1.1) and Picard–S hybrid iterative process (1.10), we have

$$
\begin{aligned}
\|x_{n+1} - p\| &= \|Tz_n - p\| \\
&\leq \delta\|z_n - p\| \\
&\leq \delta\|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\| \\
&\leq \delta\|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\| \\
&\leq \delta^2(1-\alpha_n)\|x_n - p\| + \delta^2\alpha_n\|y_n - p\| \\
&= \delta^2[(1-\alpha_n)\|x_n - p\| + \alpha_n\{(1-\beta_n)\|x_n - p\| + \beta_n\delta\|x_n - p\|\}] \\
&= \delta^2[1 - \alpha_n + \alpha_n(1-\beta_n) + \alpha_n\beta_n\delta]\|x_n - p\| \\
&= \delta^2[1 - (1-\delta)\alpha_n\beta_n]\|x_n - p\| \\
&\leq \delta^2\left[1 - (1-\delta)\lambda^2\right]\|x_n - p\| \\
&\vdots \\
&\leq \left[\delta^2\left(1 - (1-\delta)\lambda^2\right)\right]^n\|x_1 - p\|
\end{aligned} \tag{3.5}
$$

Let $e_n = \left[\delta^2\left(1 - (1-\delta)\lambda^2\right)\right]^n\|x_1 - p\|$

Now compute the rate of convergence of Picard–S iterative process (1.10) as follows:

$$
\begin{aligned}
(i) \quad \frac{e_n}{a_n} &= \frac{[\delta^2(1-(1-\delta)\lambda^2)]^n}{\delta^n\|q_1 - p\|}\|x_1 - p\| \\
&= \delta^n\left(1-(1-\delta)\lambda^2\right)^n\frac{\|x_1 - p\|}{\|q_1 - p\|} \\
&\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\end{aligned}
$$

Thus, $\{x_n\}$ converges faster than $\{q_n\}$ to p, i.e. the Picard–S hybrid iterative process (1.10) converges faster than the S-iterative process:

$$
\begin{aligned}
(ii) \quad \frac{e_n}{b_n} &= \frac{[\delta^2(1-(1-\delta)\lambda^2)]^n}{[\delta(1-(1-\delta)\lambda^2)]^n}\frac{\|x_1 - p\|}{\|s_1 - p\|} \\
&= \delta^n\frac{\|x_1 - p\|}{\|s_1 - p\|} \\
&\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\end{aligned}
$$

Thus, $\{x_n\}$ converges faster than $\{s_n\}$ to p, i.e. the Picard–S hybrid iterative process (1.10) converges faster than the Picard–Mann hybrid iterative process.

$$
\begin{aligned}
(iii) \quad \frac{e_n}{c_n} &= \frac{[\delta^2(1-(1-\delta)\lambda^2)]^n}{[\delta(1-(1-\delta)\lambda^2)]^n}\frac{\|x_1 - p\|}{\|m_1 - p\|} \\
&= \delta^n\frac{\|x_1 - p\|}{\|m_1 - p\|} \\
&\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\end{aligned} \tag{3.6}
$$

Thus, $\{x_n\}$ converges faster than $\{m_n\}$ to p, i.e. the Picard–S hybrid iterative process (1.10) converges faster than the Picard–Krasnoselskii hybrid iterative process.

$$
\begin{aligned}
(iv) \quad \frac{e_n}{d_n} &= \frac{[\delta^2(1-(1-\delta)\lambda^2)]^n}{\delta^n[1-\lambda^2(1-\delta)]^n}\frac{\|x_1 - p\|}{\|t_1 - p\|} \\
&= \delta^n\frac{\|x_1 - p\|}{\|t_1 - p\|} \\
&\rightarrow 0 \quad \text{as} \quad n \rightarrow \infty
\end{aligned} \tag{3.7}
$$

Thus, $\{x_n\}$ converges faster than $\{t_n\}$ to p, i.e. Picard–S hybrid iterative process converges faster than Picard–Ishikawa hybrid iterative process. This completes the proof of the proposition. □

In [8], Okeke proved that the rate of convergence of Picard–Krasnoselskii hybrid iterative process is faster than Picard, Krasnoselskii, Mann and Ishikawa iterations. Agarwal *et al.* [5]

proved that S-iteration converges faster than Picard, Krasnoselskii, Mann and Ishikawa iterative processes, and Okeke [9] proved that rate of convergence of Picard–Ishikawa hybrid iterative process is faster than Picard–Mann hibrid and Picard–Krasnoselskii iterations. Therefore, I give an example to show that rate of convergence of Picard–S hybrid iterative process is faster than Picard–Mann hybrid, Picard–Krasnoselskii hybrid and S-iteration. This will show that Picard–S hybrid defined by (1.10) converges to fixed point faster than all other iterations defined in this paper.

Example 3.1. Let $X = R$ and $C = [1, 10] \subset X$ and $T : C \to C$ be an operator defined by $Tx = \sqrt[3]{2x+4}$ for all $x \in C$. Choose $\alpha_n = \beta_n = \lambda = \frac{1}{2}$ for each $n \in N$ with initial value $x_1 = 5$. For $\delta = \frac{1}{\sqrt[3]{4}}$, T is a contraction mapping. All the processes converge to the same fixed point 2. It is clear from Table 1 and graphs that our Picard–S hybrid iterative process converges faster than Picard–Ishikawa hybrid, Picard–Mann hybrid, Picard–Krasnoselskii hybrid and S-iteration.

4. Application to delay differential equation

Here, I use this new Picard–S hybrid iterative process to find the solution of delay differential equations.

Let $C[a, b]$ be a space of all continuous real valued function on a closed interval $[a, b]$ be endowed with the Chebyshev norm:

$$\|x - y\|_\infty = \max_{t \in [a,b]} |x(t) - y(t)|.$$

Space $(C[a, b], \|\cdot\|_\infty)$ is known as Banach Space. In this section, the following delay differential equation has been taken:

$$x'(t) = f(t, x(t), x(t-\tau)) \quad t \in [a, b] \tag{4.1}$$

with initial condition

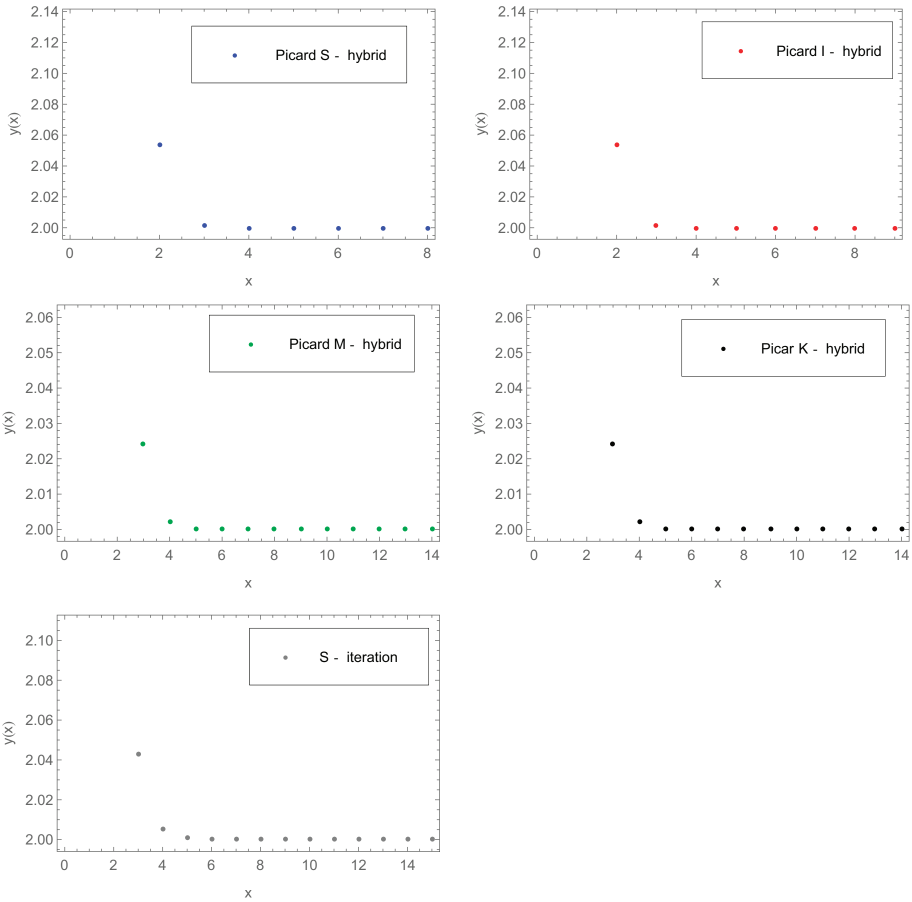
$$x(t) = \phi(t) \qquad t \in [t_o - \tau, t_o] \tag{4.2}$$

By the solution of above problem, we mean a function $x \in C([t_o - \tau, b], R) \cap (C^1[t_o, b], R)$ satisfying (4.1) and (4.2). Assume that the following conditions are satisfied.

(C_1) $t_o, b \in R$, $\tau > 0$;

Step	Picard–S hybrid	Picard–Ishikawa hybrid	Picard–Mann hybrid	Picard–Krasnoselskii hybrid	S-iteration
0	5.0000000000000	5.0000000000000	5.000000000000	5.000000000000	5.000000000000
1	2.053665985829	2.053665985829	2.251284354073	2.251284354073	2.330713309124
2	2.001174310362	2.001174310362	2.024068969098	2.024068969098	2.042698929425
3	2.000024999687	2.000024999687	2.002336639386	2.002336639386	2.005602850705
4	2.000000433332	2.000000549760	2.000227141158	2.000227141158	2.000738954904
5	2.000000009450	2.000000012089	2.000022082864	2.000022082864	2.000097495596
6	2.000000000207	2.000000000265	2.000002146942	2.000002146942	2.000012863908
7	2.000000000000	2.000000000005	2.000000208730	2.000000208730	2.000001697319
8		2.000000000000	2.000000020293	2.000000020293	2.000000223951
9			2.000000001973	2.000000001973	2.000000029549
10			2.000000000191	2.000000000191	2.000000003898
11			2.000000000018	2.000000000018	2.000000000514
12			2.000000000002	2.000000000002	2.000000000067
13			2.000000000000	2.000000000000	2.000000000008
14					2.000000000000

Table 1. A comparison of Picard–S hybrid with other iterative processes



(C_2) $f \in C([t_o, b] \times R^2, R)$;
(C_3) $\phi \in C([t_o - \tau, b], R)$;
(C_4) there exists $L_f > 0$ such that

$$|f(t, u_1, u_2) - f(t, v_1, v_2)| \leq L_f \sum_{i=1}^{2} |u_i - v_i| \quad \forall u_i, v_i \in R, i = 1, 2; t \in [t_o, b]$$

(C_5) $2L_f(b - t_o) < 1$;
Now we can reformulate problems (4.1) and (4.2) by the following integral equation:

$$x(t) = \begin{cases} \phi(t) & t \in [t_o - \tau, t_o] \\ \phi(t_o) + \int_{t_o}^{t} f(s, x(s), x(s-\tau))ds & t \in [t_o, b] \end{cases}$$

Coman [23] *et al.* established the following result.

Theorem 4.1. Assume that the conditions $(C_1 - C_5)$ are satisfied. Then problem (4.1) with initial condition (4.2) has unique solution p (say) in $C([t_o - \tau, b], R) \cap C^1([t_o, b], R)$ and

$$p = \lim_{n\to\infty} T^n(x) \quad \text{for any } x \in C([t_o - \tau, b], R).$$

Using Picard–S hybrid iterative process, I prove the following result.

Theorem 4.2. Assume that $(C_1) - (C_5)$ are satisfied. Then problem (4.1) with initial condition (4.2) has unique solution p (say) in $C([t_o - \tau, b], R) \cap C^1([t_o, b], R)$ and the Picard–S hybrid iterative process (1.10) converges to p.

Proof: Let $\{x_n\}$ be an iterative sequence generated by the Picard–S hybrid iterative process (1.10) for an operator defined by

$$Tx(t) = \begin{cases} \phi(t) & t \in [t_o - \tau, t_o] \\ \phi(t_o) + \int_{t_o}^{t} f(s, x(s), x(s-\tau))ds & t \in [t_o, b] \end{cases}$$

Let p be a fixed point of T, now I prove that $x_n \to p$ as $n \to \infty$. It is easy to see that $x_n \to p$ for each $t \in [t_o - \tau, t_o]$. Now for each $t \in [t_o, b]$, we have

$$\begin{aligned}
\|x_{n+1} - p\|_\infty &= \|Tz_n - Tp\|_\infty \\
&= \max_{t\in[t_o-\tau,b]} \left| \int_{t_o}^{t} f(s, z_n(s), z_n(s-\tau)) - f(s, p(s), p(s-\tau))ds \right| \\
&\le \max_{t\in[t_o-\tau,b]} \int_{t_o}^{t} |f(s,z_n(s),z_n(s-\tau)) - f(s,p(s),p(s-\tau))|ds \\
&\le \max_{t\in[t_o-\tau,b]} \int_{t_o}^{t} L_f(|z_n(s) - p(s)| + |z_n(s-\tau) - p(s-\tau)|)ds \\
&\le \int_{t_o}^{t} L_f\left(\max_{t\in[t_o-\tau,b]} |z_n(s) - p(s)| + \max_{t\in[t_o-\tau,b]} |z_n(s-\tau) - p(s-\tau)| \right) ds \qquad (4.3) \\
&= \int_{t_o}^{t} L_f(\|z_n - p\|_\infty + \|z_n - p\|_\infty)ds \\
&= 2\|z_n - p\|_\infty \int_{t_o}^{t} L_f ds \\
&= 2\|z_n - p\|_\infty L_f(t - t_o) \\
&\le 2L_f(b - t_o)\|z_n - p\|_\infty
\end{aligned}$$

Now,

$$\begin{aligned}
\|z_n - p\|_\infty &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\|_\infty \\
&= \|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\|_\infty \qquad (4.4) \\
&\le (1-\alpha_n)\|Tx_n - p\|_\infty + \alpha_n\|Ty_n - p\|_\infty
\end{aligned}$$

$$
\begin{aligned}
\|Tx_n - p\|_\infty &= \max_{t\in[t_o-\tau,b]}|Tx_n(t) - Tp(t)| \\
&= \max_{t\in[t_o-\tau,b]}\left|\int_{t_o}^{t} f(s,x_n(s),x_n(s-\tau))ds - \int_{t_o}^{t} f(s,p(s),p(s-\tau))ds\right| \\
&\le \max_{t\in[t_o-\tau,b]}\int_{t_o}^{t}|f(s,x(s),x_n(s-\tau)) - f(s,p(s),p(s-\tau))|ds \\
&= \max_{t\in[t_o-\tau,b]}\int_{t_o}^{t} L_f(|x_n(s)-p(s)| + |x_n(s-\tau)-p(s-\tau)|)ds \\
&\le \int_{t_o}^{t} L_f\left(\max_{t\in[t_o-\tau,b]}|x_n(s)-p(s)| + \max_{t\in[t_o-\tau,b]}|x_n(s-\tau)-p(s-\tau)|\right)ds \\
&\le \int_{t_o}^{t} L_f(\|x_n-p\|_\infty + \|x_n-p\|_\infty)ds \\
&= 2L_f(t-t_o)\|x_n-p\|_\infty \\
&\le 2L_f(b-t_o)\|x_n-p\|_\infty
\end{aligned} \tag{4.5}
$$

Now,

$$
\begin{aligned}
\|y_n - p\|_\infty &= \|(1-\beta_n)x_n + \beta_n Tx_n - p\|_\infty \\
&= \|(1-\beta_n)(x_n-p) + \beta_n(Tx_n-p)\|_\infty \\
&\le (1-\beta_n)\|x_n-p\|_\infty + \beta_n\|Tx_n-p\|_\infty
\end{aligned} \tag{4.6}
$$

Now,

$$
\begin{aligned}
\|Ty_n - p\|_\infty &= \|Ty_n - Tp\|_\infty \\
&= \max_{t\in[t_o-\tau,b]}\left|\int_{t_o}^{t} f(s, y_n(s), y_n(s-\tau))ds - \int_{t_o}^{t} f(s, p(s), p(s-\tau))ds\right| \\
&\le \max_{t\in[t_o-\tau,b]}\int_{t_o}^{t}|f(s, y_n(s), y_n(s-\tau)) - f(s, p(s), p(s-\tau))|ds \\
&\le \max_{t\in[t_o-\tau,b]}\int_{t_o}^{t} L_f(|y_n(s)-p(s)| + |y_n(s-\tau)-p(s-\tau)|)ds \\
&\le \int_{t_o}^{t} L_f\left(\max_{t\in[t_o-\tau,b]}|y_n(s)-p(s)| + \max_{t\in[t_o-\tau,b]}|y_n(s-\tau)-p(s-\tau)|\right)ds \\
&= \int_{t_o}^{t} L_f(\|y_n-p\|_\infty + \|y_n-p\|_\infty)ds \\
&= 2\|y_n-p\|_\infty\int_{t_o}^{t} L_f ds \\
&= 2\|y_n-p\|_\infty L_f(t-t_o) \\
&\le 2L_f(b-t_o)\|y_n-p\|_\infty
\end{aligned} \tag{4.7}
$$

Using (4.6) in (4.7), we get

$$\begin{aligned}\|Ty_n - p\|_\infty &= \|Ty_n - Tp\|_\infty\\ &\le 2L_f(b-t_0)\{(1-\beta_n)\|x_n - p\|_\infty + \beta_n\|Tx_n - p\|_\infty\}\\ &= 2(1-\beta_n)L_f(b-t_o)\|x_n - p\|_\infty + 2\beta_n L_f(b-t_o)\|Tx_n - p\|_\infty\end{aligned} \tag{4.8}$$

From (4.5) we get

$$\begin{aligned}\|Ty_n - p\|_\infty &\le 2(1-\beta_n)L_f(b-t_o)\|x_n - p\|_\infty + 2\beta_n L_f(b-t_o)2L_f(b-t_o)\|x_n - p\|_\infty\\ &= 2L_f(b-t_o)\{(1-\beta_n) + 2\beta_n L_f(b-t_o)\}\|x_n - p\|_\infty\end{aligned} \tag{4.9}$$

Using (4.5) and (4.9) in (4.4), we get

$$\begin{aligned}\|z_n - p\|_\infty &\le (1-\alpha_n)2L_f(b-t_o)\|x_n - p\|_\infty + \alpha_n[2L_f(b-t_0)\{(1-\beta_n) + 2\beta_n L_f(b-t_o)\}\|x_n - p\|_\infty]\\ &= 2L_f(b-t_o)[(1-\alpha_n) + \alpha_n(1-\beta_n) + 2\alpha_n\beta_n L_f(b-t_o)]\|x_n - p\|_\infty\\ &= 2L_f(b-t_o)[1-\alpha_n\beta_n + 2\alpha_n\beta_n L_f(t-t_o)]\|x_n - p\|_\infty\end{aligned} \tag{4.10}$$

Note that $2L_f(b-t_o)[1-\alpha_n\beta_n + 2\alpha_n\beta_n L_f(t-t_o)] = \mu_n < 1$ and $\|x_n - P\|_\infty = S_n$. Thus, all conditions of lemma 2.1 are satisfied. Hence, $\lim_{n\to\infty}\|x_n - P\|_\infty = 0$. This completes the proof of above theorem. □

5. Picard–S hybrid iterative process for nonexpansive mappings

Lemma 5.1. Let E be a normed space, C a non-empty convex subset of E and $T : C \to C$ a nonexpansive mapping. If $\{x_n\}$ is the iterative process defined by (1.10), then $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists.

Proof: Set $a_n = x_n - Tx_n$ for all $n \in N$. Then we have

$$\begin{aligned}\|a_{n+1}\| &= \|x_{n+1} - Tx_{n+1}\|\\ &= \|Tz_n - Tx_{n+1}\|\\ &\le \|z_n - x_{n+1}\|\\ &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - Tz_n\|\\ &= \|\{(1-\alpha_n)(Tx_n - Tz_n)\} + \{\alpha_n(Ty_n - Tz_n)\}\|\\ &\le (1-\alpha_n)\|Tx_n - Tz_n\| + \alpha_n\|Ty_n - Tz_n\|\\ &\le (1-\alpha_n)\|x_n - z_n\| + \alpha_n\|y_n - z_n\|\end{aligned} \tag{5.1}$$

$$\begin{aligned}\|x_n - z_n\| &= \|x_n - \{(1-\alpha_n)Tx_n + \alpha_n Ty_n\}\|\\ &= \|(x_n - Tx_n) + \alpha_n(Tx_n - Ty_n)\|\\ &\le \|x_n - Tx_n\| + \alpha_n\|Tx_n - Ty_n\|\\ &\le \|x_n - Tx_n\| + \alpha_n\|x_n - y_n\|\\ &= \|a_n\| + \alpha_n\|x_n - y_n\|\end{aligned} \tag{5.2}$$

Now,

$$\begin{aligned}\|x_n - y_n\| &= \|x_n - \{(1-\beta_n)x_n + \beta_n Tx_n\}\|\\ &= \|x_n - x_n + \beta_n x_n - \beta_n Tx_n\|\\ &= \beta_n\|x_n - Tx_n\|\\ &= \beta_n\|a_n\|\end{aligned} \tag{5.3}$$

From inequality (5.2) and (5.3), we have

$$\begin{aligned} \|x_n - z_n\| &\leq \|a_n\| + \alpha_n\beta_n\|\alpha_n\| \\ &= (1+\alpha_n\beta_n)\|\alpha_n\| \end{aligned} \tag{5.4}$$

Now,

$$\begin{aligned} \|y_n - z_n\| &= \|y_n - \{(1-\alpha_n)Tx_n + \alpha_n Ty_n\}\| \\ &= \|(1-\alpha_n)y_n + \alpha_n y_n - (1-\alpha_n)Tx_n - \alpha_n Ty_n\| \\ &= \|(1-\alpha_n)(y_n - Tx_n) + \alpha_n(y_n - Ty_n)\| \\ &\leq (1-\alpha_n)\|y_n - Tx_n\| + \alpha_n\|y_n - Ty_n\| \end{aligned} \tag{5.5}$$

Now,

$$\begin{aligned} \|y_n - Tx_n\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - Tx_n\| \\ &= \|(1-\beta_n)x_n - (1-\beta_n)Tx_n\| \\ &\leq (1-\beta_n)\|x_n - Tx_n\| \\ &= (1-\beta_n)\|a_n\| \end{aligned} \tag{5.6}$$

$$\begin{aligned} \|y_n - Ty_n\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - Ty_n\| \\ &= \|(1-\beta_n)x_n + \beta_n Tx_n - (1-\beta_n+\beta_n)Ty_n\| \\ &= \|(1-\beta_n)(x_n - Ty_n) + \beta_n(Tx_n - Ty_n)\| \\ &\leq (1-\beta_n)\|x_n - Ty_n\| + \beta_n\|Tx_n - Ty_n\| \\ &= (1-\beta_n)\|x_n - Ty_n\| + \beta_n\|x_n - y_n\| \end{aligned} \tag{5.7}$$

Using (5.3) in (5.7), we get

$$\begin{aligned} \|y_n - Ty_n\| &\leq (1-\beta_n)\|x_n - Ty_n\| + \beta_n\beta_n\|a_n\| \\ &= (1-\beta_n)\|x_n - Ty_n\| + \beta_n^2\|a_n\| \end{aligned} \tag{5.8}$$

Using (5.6) and (5.8) in (5.5), we get

$$\begin{aligned} \|y_n - z_n\| &\leq (1-\alpha_n)(1-\beta_n)\|a_n\| + \alpha_n\{(1-\beta_n)\|x_n - Ty_n\| + \beta_n^2\|a_n\|\} \\ &= (1-\alpha_n)(1-\beta_n)\|a_n\| + \alpha_n\{(1-\beta_n)\|x_n - Tx_n + Tx_n - Ty_n\| + \beta_n^2\|a_n\|\} \\ &\leq (1-\alpha_n)(1-\beta_n)\|a_n\| + \alpha_n\{(1-\beta_n)(\|x_n - Tx_n\| + \|Tx_n - Ty_n\|) + \beta_n^2\|a_n\|\} \\ &\leq (1-\alpha_n)(1-\beta_n)\|a_n\| + \alpha_n\{((1-\beta_n)\|x_n - Tx_n\| + (1-\beta_n)\|x_n - y_n\|) + \beta_n^2\|a_n\|\} \\ &= (1-\alpha_n)(1-\beta_n)\|a_n\| + \alpha_n\{((1-\beta_n)\|a_n\| + (1-\beta_n)\|x_n - y_n\|) + \beta_n^2\|a_n\|\} \end{aligned} \tag{5.9}$$

Using (5.3) in (5.9), we get

$$\begin{aligned} \|y_n - z_n\| &\leq (1-\alpha_n)(1-\beta_n)\|a_n\| + \alpha_n\{((1-\beta_n)\|a_n\| + (1-\beta_n)\beta_n\|a_n\|) + \beta_n^2\|a_n\|\} \\ &= \left[(1-\alpha_n)(1-\beta_n) + \alpha_n(1-\beta_n)(1+\beta_n) + \alpha_n\beta_n^2\right]\|a_n\| \\ &= (1-\beta_n+\alpha_n\beta_n)\|a_n\| \end{aligned} \tag{5.10}$$

From (5.1), (5.4) and (5.10), we get

$$\begin{aligned} \|a_{n+1}\| &\leq [(1-\alpha_n)(1+\alpha_n\beta_n)]\|a_n\| + [\alpha_n(1-\beta_n+\alpha_n\beta_n)]\|a_n\| \\ &= [(1-\alpha_n)(1+\alpha_n\beta_n) + \alpha_n(1-\beta_n+\alpha_n\beta_n)]\|a_n\| \\ &= [1+\alpha_n\beta_n - \alpha_n - \alpha_n^2\beta_n + \alpha_n - \alpha_n\beta_n + \alpha_n^2\beta_n]\|a_n\| \\ &= \|a_n\| \end{aligned}$$

So that $\{\|a_n\|\}$ is nonincreasing and hence, $\lim_{n\to\infty}\|a_n\|$ exists. □

Theorem 5.2. [5] Let X be a Banach space with modulus of convexity δ_X. Then

$$\|(1-t)x+ty\| \leq 1-2t(1-t)\delta_X(\|x-y\|)$$

for all $x, y \in X$ with $\|x\| \leq 1,\ \|y\| \leq 1$ and all $t \in [0, 1]$.

Theorem 5.3. Let C be a non-empty closed convex (not necessary bounded) subset of a uniformly convex Banach space X and $T: C \to C$ a nonexpansive mapping. Let $\{x_n\}$ be the sequence defined by (1.10) with the restriction:

$$\lim_{n\to\infty} \alpha_n\beta_n(1-\alpha_n) \text{ exists and } \lim_{n\to\infty} \alpha_n\beta_n(1-\beta_n) \neq 0$$

Then, for arbitrary initial value $x_1 \in C$, $\{\|x_n - Tx_n\|\}$ converges to some constant $\gamma_C(T) = \inf\{\|x - Tx\| : x \in C\}$, which is independent of the choice of the initial value $x_1 \in C$.

Proof: Lemma (5.1) implies that $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists and denote $\gamma(x_1) = \lim_{n\to\infty}\|x_n - Tx_n\|$. Let $\{x_n^*\}$ be another iterative sequence generated by (1.10) with the same restriction on parameters $\{\alpha_n\}$ and $\{\beta_n\}$ of iteration as the sequence $\{x_n\}$ but with the initial value $x_1^* \in C$. It follows from lemma (5.1) that

$$\lim_{n\to\infty}\|x_n^* - Tx_n^*\| = \gamma(x_1^*) \tag{5.11}$$

Observe that

$$\begin{aligned} \|Ty_n - Ty_n^*\| &\leq \|y_n - y_n^*\| \\ &\leq (1-\beta_n)\|x_n - x_n^*\| + \beta_n\|Tx_n - Tx_n^*\| \\ &\leq (1-\beta_n)\|x_n - x_n^*\| + \beta_n\|x_n - x_n^*\| \\ &\leq \|x_n - x_n^*\| \end{aligned} \tag{5.12}$$

Now

$$\begin{aligned} \|x_{n+1} - x_{n+1}^*\| &= \|Tz_n - Tz_n^*\| \\ &\leq \|z_n - z_n^*\| \\ &= \|(1-\alpha_n)(Tx_n - Tx_n^*) + \alpha_n(Ty_n - Ty_n^*)\| \\ &\leq (1-\alpha_n)\|Tx_n - Tx_n^*\| + \alpha_n\|Ty_n - Ty_n^*\| \\ &\leq (1-\alpha_n)\|x_n - x_n^*\| + \alpha_n\|Ty_n - Ty_n^*\| \end{aligned} \tag{5.13}$$

Using (5.12) in (5.13), we have

$$\|x_{n+1} - x_{n+1}^*\| \leq (1-\alpha_n)\|x_n - x_n^*\| + \alpha_n\|x_n - x_n^*\| \tag{5.14}$$

$$= \|x_n - x_n^*\| \tag{5.15}$$

This shows that $\lim_{n\to\infty}\|x_n - x_n^*\|$ exists.

Let $\lim_{n\to\infty}\|x_n - x_n^*\| = d$ for some $d > 0$.

Let

$$\begin{aligned} \|y_n - y_n^*\| &= \|(1-\beta_n)(x_n - x_n^*) + \beta_n(Tx_n - Tx_n^*)\| \\ &= \left\|\frac{(1-\beta_n)(x_n - x_n^*)}{\|x_n - x_n^*\|} + \frac{\beta_n(Tx_n - Tx_n^*)}{\|x_n - x_n^*\|}\right\|\|x_n - x_n^*\| \end{aligned} \tag{5.16}$$

Since

$$\left\|\frac{(x_n - x_n^*)}{\|x_n - x_n^*\|}\right\| = \frac{\|x_n - x_n^*\|}{\|x_n - x_n^*\|} = 1$$

$$\left\|\frac{(Tx_n - Tx_n^*)}{\|x_n - x_n^*\|}\right\| = \frac{\|Tx_n - Tx_n^*\|}{\|x_n - x_n^*\|} \leq \frac{\|x_n - x_n^*\|}{\|x_n - x_n^*\|} = 1$$

Using theorem (5.2) and (5.16), we obtain that

$$\|y_n - y_n^*\| \leq 1 - 2\beta_n(1-\beta_n)\delta_X\left(\frac{\|x_n - x_n^* - (Tx_n - Tx_n^*)\|}{\|x_n - x_n^*\|}\right)\|x_n - x_n^*\|$$

It follows from (5.14) that

$$\begin{aligned}\|x_{n+1} - x_{n+1}^*\| &\leq (1-\alpha_n)\|x_n - x_n^*\| + \alpha_n\|x_n - x_n^*\|\left[1 - 2\beta_n(1-\beta_n)\delta_X\left(\frac{\|(x_n - x_n^*) - (Tx_n - Tx_n^*)\|}{\|x_n - x_n^*\|}\right)\right] \\ &= \|x_n - x_n^*\| - 2\alpha_n\beta_n(1-\beta_n)\|x_n - x_n^*\|\delta_X\left(\frac{\|(x_n - x_n^*) - (Tx_n - Tx_n^*)\|}{\|x_1 - x_1^*\|}\right)\end{aligned} \tag{5.17}$$

This gives us

$$2\alpha_n\beta_n(1-\beta_n)\|x_n - x_n^*\|\delta_X\left(\frac{\|(x_n - x_n^*) - (Tx_n - Tx_n^*)\|}{\|x_n - x_n^*\|}\right) \leq \|x_n - x_n^*\| - \|x_{n+1} - x_{n+1}^*\|$$

Or

$$2\alpha_n\beta_n(1-\beta_n)\|x_n - x_n^*\|\delta_X\left(\frac{\|(x_n - x_n^*) - (Tx_n - Tx_n^*)\|}{\|x_n - x_n^*\|}\right) \leq \|x_1 - x_1^*\|$$

Using restriction $\lim_{n\to\infty} \alpha_n\beta_n(1-\beta_n) \neq 0$ and $\lim_{n\to\infty}\|x_n - x_n^*\| = d > 0$.
Therefore,

$$\lim_{n\to\infty}\delta_X\left(\frac{\|(x_n - x_n^*) - (Tx_n - Tx_n^*)\|}{\|x_n - x_n^*\|}\right) = 0$$

δ_X is strictly increasing and continuous and $\lim_{n\to\infty}\|x_n - x_n^*\| = d > 0$.
We have

$$\lim_{n\to\infty}\|(x_n - x_n^*) - (Tx_n - Tx_n^*)\| = 0$$

Observe that

$$\big|\|x_n - Tx_n\| - \|x_n^* - Tx_n^*\|\big| \leq \|(x_n - Tx_n) - (x_n^* - Tx_n^*)\|$$

which implies that

$$\lim_{n\to\infty}\big|\|x_n - Tx_n\| - \|x_n^* - Tx_n^*\|\big| = 0$$

Thus, $\gamma(x_1) = \gamma(x_1^*)$. Because

$$\|x_{n+1} - Tx_{n+1}\| \leq \|x_n - Tx_n\| \leq \|x_1 - Tx_1\|$$

for all $n \in N$ and $x_1 \in C$

It follows that

$$\gamma_C(T) = \inf\{\|x - Tx\| : x \in C\}$$ □

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