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Fourier coefficients for Laguerre–Sobolev type orthogonal polynomials

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Abstract

Purpose – In this paper, the authors take the first step in the study of constructive methods by using Sobolev polynomials.
Design/methodology/approach – To do that, the authors use the connection formulas between Sobolev polynomials and classical Laguerre polynomials, as well as the well-known Fourier coefficients for these latter.
Findings – Then, the authors compute explicit formulas for the Fourier coefficients of some families of Laguerre–Sobolev type orthogonal polynomials over a finite interval. The authors also describe an oscillatory region in each case as a reasonable choice for approximation purposes.
Originality/value – In order to take the first step in the study of constructive methods by using Sobolev polynomials, this paper deals with Fourier coefficients for certain families of polynomials orthogonal with respect to the Sobolev type inner product. As far as the authors know, this particular problem has not been addressed in the existing literature.

Keywords Fourier coefficients, Sobolev type orthogonal polynomials, Laguerre polynomials
Paper type Research paper

1. Introduction

Within the framework of spectral approximation, and to recover values of smooth functions with exponential accurate, it is customary to use Fourier series for periodic problems and series of classical orthogonal polynomials for nonperiodic problems. Nevertheless, if it deals with piecewise smooth function, estimates by means of partial sums are unhealthy; oscillations do not decrease near discontinuities with partial sums of higher order; and far of them, convergence order is low. Thus, the global properties from Fourier coefficients are not enough to obtain local information. This lack of uniform convergence is known as Gibbs phenomenon. A priori, this is a serious issue considering the large number of applications modeled through piecewise smooth function. In literature, methods to face the Gibbs phenomenon in reconstruction of piecewise smooth functions from partial sums have been widely studied. For instance, in Refs. [1, 2], the problem to construct piecewise smooth function values with exponential accuracy at all points is solved by means of approximations with Fourier–Gegenbauer coefficients expansions. These are the so-called *Gegenbauer reconstruction methods* where the expansion of Gegenbauer polynomials in its Fourier series

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is crucial. In Ref. [3], the Gegenbauer reconstruction methods are revisited and analyzed in order to prove that Gegenbauer reconstruction is also effective for Fourier–Bessel series. To do that, the author obtains coefficients Fourier for Jacobi polynomials and also for classical orthogonal polynomials with unbounded support (Laguerre, Hermite).

On the other hand, consider a vector of Borel positive measures $(\mu_0, \mu_1, \ldots, \mu_m)$, on the real line, with finite moments and μ_0 with continuous support. Then, we define the *Sobolev inner product* on the space of polynomials with real coefficients.

$$\langle f, g\rangle_S = \int_{\mathbb{R}} f(x)g(x)d\mu_0 + \sum_{k=1}^{m}\int_{\mathbb{R}} f^{(k)}(x)g^{(k)}(x)d\mu_k. \tag{1.1}$$

A sequence of polynomials $\{S_n\}_{n\geq 0}$, $\deg\ S_n = n$, is orthogonal with respect to (1.1) if

$$\langle S_n, S_m\rangle_S = K_n\delta_{n,m},\ K_n > 0.$$

The sequence $\{S_n\}_{n\geq 0}$ is said to be a sequence of *Sobolev polynomials* orthogonal with respect to (1.1). If μ_k is discrete, for $k = 1, \ldots, m$, the above inner product and the sequence $\{S_n\}_{n\geq 0}$ are said to be of Sobolev type. Sobolev orthogonal polynomials have been widely studied in the last three decades. The first publication on Sobolev polynomials goes back to 1962 in Ref. [4], which deals with certain extremal problem related to smooth polynomial approximation whose solution is posed by means of Sobolev–Legendre polynomials. Such a problem is formulated previously in Ref. [5], although not in terms of Sobolev orthogonality. It has been documented as the approximations with Sobolev–Fourier series from smooth functions in the corresponding Sobolev space improve approximations made through standard families of orthogonal polynomials (see Ref. [6]). Additional applications include spectral methods in numerical analysis for ordinary differential equations and partial differential equations, and generalization of Gauss quadrature formulas, among others. The nice surveys [7, 8] are highly recommended, as well as the paper [9] and references therein. In order to take the first step in the study of constructive methods by using Sobolev polynomials, this paper deals with Fourier coefficients for certain families of polynomials orthogonal with respect to the Sobolev type inner product (1.1) when μ_0 is the classical and absolutely continuous Laguerre measure on $[0, \infty)$. In the next section, we propose the basic background with respect to Laguerre polynomials, and we present the particular Sobolev–Laguerre type families of polynomials to be discussed. In Section 3, we obtain the respective Fourier coefficients by using of similar techniques as the presented in Ref. [10]. Since the orthogonality interval for Laguerre polynomials is unbounded, we will turn special attention to oscillation regions for the Sobolev polynomials.

2. Preliminaries

Let $\mathbb{P}$ be the space of polynomials with real coefficients

2.1 Classical Laguerre polynomials and generalities

The classical Laguerre polynomials $\{L_n^{\alpha}\}_{n\geq 0}$, with $\alpha > -1$, are orthogonal with respect to the inner product:

$$\langle p, q\rangle_{\alpha} := \int_0^{\infty} p(x)q(x)e^{-x}x^{\alpha}dx, \quad p, q \in \mathbb{P}.$$

For an arbitrary polynomial p, $k(p)$ will denote the leading coefficient of p. In the sequel, to normalize Laguerre polynomials, we assume that $k(L_n^{\alpha}) := (-1)^n/n$. These polynomials satisfy the three terms recurrence relation (TTRR in short),

$$(n+1)L_{n+1}^{\alpha}(x) = (2n+\alpha+1-x)L_n^{\alpha}(x) - (n+\alpha)L_{n-1}^{\alpha}(x), \tag{2.1}$$

for $n \geq 0$ with the initial conditions $L_{-1}^{\alpha} := 0$ and $L_0^{\alpha} = 1$. For $n \geq 1$, the zeros of every L_n^{α} are all real, simple and are located in $(0, \infty)$ (see Ref. [11]). In the sequel, $\{x_{n,i}\}_{i=1}^{n}$ will denote the zeros of L_n^{α} ordered in increasing order.

Definition 1. Let p be a polynomial with real zeros. An oscillatory region I for p is any bounded interval containing their zeros, in such a way that p is monotone outside I.

With respect to an oscillatory region of classical Laguerre polynomials, we get the next.

Proposition 1. *([11]). For $\alpha > -1$ and $n > 0$, the n zeros of L_n^{α} are into $[0, \zeta_{n,\alpha}]$, where*

$$\zeta_{n,\alpha} = 2n+\alpha+1+\sqrt{(2n+\alpha+1)^2+\frac{1}{4}-\alpha^2}.$$

We consider, for a nonnegative integer m, the functions $\beta_m^{a,b}$ defined as (see Ref. [10]),

$$\beta_m^{a,b}(z) = \int_a^b x^m e^{-zx}dx = z^{-m-1}m!\left(e^{-az}e_m(az) - e^{-bz}e_m(bz)\right), \tag{2.2}$$

where e_m is the $m-th$ partial sum of the Maclaurin series for the exponential function and $[a, b]$ is a bounded interval.

As a consequence of this definition, it is possible to show that if $x = \varepsilon\xi + \delta$, $\varepsilon = \frac{b-a}{2}$ and $\delta = \frac{b+a}{2}$, we get

$$\int_{-1}^{1} (\varepsilon\xi+\delta)^m e^{-i\pi k\xi}d\xi = \frac{1}{\varepsilon}e^{ik\delta/\varepsilon}\beta_m^{a,b}\left(\frac{i\pi k}{\varepsilon}\right). \tag{2.3}$$

In this way, the next result for the Fourier series for Laguerre polynomials is presented in Ref. [10].

Theorem 1. *Let $[a, b]$ be an interval with $-\infty < a < b < \infty$ and $\xi \in [-1, 1]$, $\varepsilon = \frac{b-a}{2}$, $\delta = \frac{b+a}{2}$. The Fourier coefficients for $L_n^{\alpha}(\varepsilon\xi+\delta)$, in the local variable ξ, are given by*

$$\widehat{L}_n^{\alpha}(k) = \frac{1}{2\varepsilon}e^{ik\delta/\varepsilon}\sum_{t=0}^{n}\frac{(-1)^t}{t!}\binom{n+\alpha}{n-t}\beta_t^{a,b}\left(\frac{i\pi k}{\varepsilon}\right). \tag{2.4}$$

2.2 Quasi-orthogonality and zeros

Let $\{P_n\}_{n\geq 0}$ be a sequence of polynomials orthogonal with respect to a positive Borel measure μ supported on $[a, b]$, with $-\infty \leq a < b \leq \infty$, i.e.

$$\int_a^b P_n(x)x^k d\mu = \delta_{n,k}K_n, \quad K_n > 0, \quad k = 1, \ldots, n.$$

Definition 2. Let r be a nonnegative integer and R_n a polynomial with degree $n \geq r$ satisfying $\int_a^b R_n(x)x^k d\mu = 0$ for $k = 0, 1, 2, \ldots, n-r-1$, and $\int_a^b R_n(x)x^{n-r}d\mu \neq 0$. Then, R_n is said to be quasi-orthogonal of order r on $[a, b]$ and with respect to μ.

Of course, if $r = 0$, then the orthogonality is recovered. The next result describes a necessary and sufficient condition for quasi-orthogonality.

Proposition 2. *([12]). R_n is quasi-orthogonal of order r on $[a, b]$ with respect to μ if and only if there exist numbers $b_{n,i}$, $i = 0, 1, \ldots, r$, with $b_{n,0}b_{n,r} \neq 0$, such that*

$$R_n(x) = \sum_{k=0}^{r} b_{n,k} P_{n-k}. \tag{2.5}$$

With respect to zeros of quasi-orthogonal polynomials, the next result is well known.

Proposition 3. *([12]). If R_n is quasi-orthogonal of order r with respect to μ on $[a, b]$, then R_n has $n - r$ simple zeros on (a, b).*

Suppose that R_n is quasi-orthogonal of order r with respect to μ on $[a, b]$ and R_n and P_n are monic. It is well known that the monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ can be obtained by means of a TTRR:

$$P_{n+1}(x) = \left(\widehat{A}_{n+1}x + \widehat{B}_{n+1}\right)P_n(x) - \widehat{C}_{n+1}P_{n-1}(x), \quad n \geq 0, \tag{2.6}$$

and we define $B_{n+1} := \frac{k(P_n)\widehat{B}_{n+1}}{k(P_{n+1})}$ and $C_{n+1} := \frac{k(P_{n-1})\widehat{C}_{n+1}}{k(P_{n+1})}$. In the particular case, when $r = 2$, from (2.5), we get $R_n(x) = b_{n,0}P_n(x) + b_{n,1}P_{n-1}(x) + b_{n,2}P_{n-2}(x)$, and we also define $a_n := \frac{k(P_{n-1})b_{n,1}}{k(R_n)}$ and $b_n := \frac{k(P_{n-2})b_{n,2}}{k(R_n)}$. The next results refer to behavior and localization of zeros of quasi-orthogonal polynomials for $r = 2$.

Theorem 2. *([13]). If $b_n \leq C_n$, the n zeros of R_n are real and simples.*

Theorem 3. *([13]). Suppose that $\{x_{n,i}\}_{i=1}^{n}$ and $\{y_{n,i}\}_{i=1}^{n}$ are the zeros of P_n and R_n, respectively, and ordered in increasing order.*

(1) $b_n < C_n$ *if and only if*

$$y_{n,1} < x_{n-1,1} < y_{n,2} < x_{n-1,2} < \cdots < y_{n,n-1} < x_{n-1,n-1} < y_{n,n}. \tag{2.7}$$

(2) $0 < b_n < C_n$ *and* $a_n > -\frac{b_n\left(x_{n,1}+B_n\right)}{C_n}$ *if and only if*

$$y_{n,1} < x_{n,1} < y_{n,2} < x_{n,2} < \cdots < x_{n,n-1} < y_{n,n} < x_{n,n}. \tag{2.8}$$

(3) $0 < b_n < C_n$ *and* $a_n < -\frac{b_n\left(x_{n,n}+B_n\right)}{C_n}$ *if and only if*

$$x_{n,1} < y_{n,1} < x_{n,2} < y_{n,2} < \cdots < y_{n,n-1} < x_{n,n} < y_{n,n}. \tag{2.9}$$

Theorem 4. *([13]). Suppose that $b_n < C_n$ and $b_{n+1} < C_{n+1}$ and we define*

$$f_n(x) = \frac{k(P_{n-1})P_n(x)}{k(P_n)P_{n-1}(x)}. \tag{2.10}$$

For $i = 1, 2, \ldots, n$

$$f_{n+1}\left(y_{n,i}\right)f_n\left(y_{n,i}\right) + a_{n+1}f_n\left(y_{n,i}\right) + b_{n+1} < 0,$$

if and only if

$$y_{n+1,1} < y_{n,1} < y_{n+1,2} < y_{n,2} < \cdots < y_{n+1,n} < y_{n,n} < y_{n+1,n+1}, \tag{2.11}$$

2.3 Laguerre–Sobolev type orthogonal polynomials, nondiagonal case

If $p \in \mathbb{P}$ and $\mathbf{P}(x) := (p(x), p'(x))^t$, we define the *Laguerre–Sobolev type inner product*

$$\langle p, q\rangle_{S_1} = \langle p, q\rangle_{\alpha} + \mathbf{P}(0)^t A\mathbf{Q}(0), \tag{2.12}$$

where $A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}$, with $M_0 M_1 \geq 0$ and λ such that $\det A \geq 0$. Let $\{S_n^\alpha\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to (2.12) such that $k(S_n^\alpha) = k(L_n^\alpha)$ for $n \geq 0$.

Theorem 5. *([14]). For every $n \in \mathbb{N}$*

$$S_n^\alpha(x) = L_n^{\alpha+2}(x) + A_{n,\alpha} L_{n-1}^{\alpha+2}(x) + B_{n,\alpha} L_{n-2}^{\alpha+2}(x), \tag{2.13}$$

where

$$A_{n,\alpha} \sim \frac{(\alpha+1)(\alpha+2)}{n} - 2, \qquad B_{n,\alpha} \sim 1 - \frac{(\alpha+1)(\alpha+2)}{n}.$$

2.3.1 Laguerre–Sobolev type polynomials of higher order derivatives. Let $\left\{S_{n,m}^{\alpha,W}\right\}_{n\geq 0}$ be orthogonal with respect to Sobolev inner product

$$\langle p, q\rangle_{S,m}^{\alpha,W} = \langle p, q\rangle_\alpha + W p^{(m)}(0) q^{(m)}(0), \tag{2.14}$$

with $W > 0$, m a nonnegative integer and $p, q \in \mathbb{P}$. Moreover $k(S_{n,m}^{\alpha,W}) = k(L_n^\alpha)$.

Theorem 6. *([15]). For $n > m$*

$$S_{n,m}^{\alpha,W}(x) = \sum_{k=0}^{m+1} A_{n,k} L_{n-k}^{(\alpha+k)}(x), \tag{2.15}$$

where

$$A_{n,0} = 1 + W \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} \sum_{k=1}^{m+1} (-1)^{k+1} \binom{n+\alpha}{n-m-k} \binom{n-k}{m+1-k},$$

and for $k = 1, \ldots, m+1$

$$A_{n,k} = (-1)^k W \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} \binom{n+\alpha}{n-m} \binom{n-k}{m+1-k}.$$

With respect to zeros of every $S_{n,m}^{\alpha,W}$, we enunciate the next results.

Theorem 7. *(See Ref. [16]). For every n, the zeros of $S_{n,m}^{\alpha,W}$ are real, simple and at most one of them is outside $(0, \infty)$. If $S_{n.m}^{\alpha,W}$ has a zero in $(-\infty, 0]$ then $n \geq m+1$. In addition, if for n_0, $S_{n_0,m}^{\alpha,W}$ has a negative zero, then $S_{n,m}^{\alpha,W}$ has a negative zero for $n > n_0$.*

Theorem 8. *Assume that $n \geq m+1$. If $\{\upsilon_{n,i}\}_{i=1}^n$ are the zeros of $S_{n,m}^{\alpha,W}$, ordered in increasing order, then $\upsilon_{n,i} < x_{n,i}$ for $i = 1, \ldots, n$.*

Theorem 9. *If ρ_n is the negative zero of $S_{n,m}^{\alpha,W}$ then $-m\tilde{\upsilon}_{n,m+1} < \rho_n$, where $\tilde{\upsilon}_{n,m+1}$ denotes the $m-th$ positive zero of $S_{n,m}^{\alpha,W}$.*

2.3.2 Christoffel transformations and Laguerre–Sobolev type inner product with mass outside support. Given $\xi \leq 0$, and an integer $k \geq 1$, we consider the weight $\omega_{\alpha,k}(x) = (x-\xi)^k e^{-x} x^\alpha$, on $[0, \infty)$. This is a *Christoffel perturbation* of the classical Laguerre measure (see Ref. [11]). $\left\{L_n^{(\alpha,k)}\right\}_{n\geq 0}$ denotes the respective sequence of orthogonal polynomials, where

$k(L_n^{(\alpha,k)}) = k(L_n^\alpha)$ for every n and $L_n^{(\alpha,0)} := L_n^\alpha$. An algebraic connection between polynomials orthogonal with respect to the weight $\omega_{\alpha,k}(x)$ is as follows (see Ref. [11]),

$$(x-\xi)L_n^{(\alpha,k)}(x) = L_{n+1}^{(\alpha,k-1)}(x) - \frac{L_{n+1}^{(\alpha,k-1)}(\xi)}{L_n^{(\alpha,k-1)}(\xi)} L_n^{(\alpha,k-1)}(x). \tag{2.16}$$

Assume $\left\{x_{n,i}^{[k]}\right\}_{i=1}^{n}$ are the zeros of $L_n^{(\alpha,k)}$ in increasing order, with $x_{n,i}^{[0]} := x_{n,i}$.

Proposition 4. *([17]). For $i = 1, \ldots, n$*

$$x_{n,i}^{[k-1]} < x_{n,i}^{[k]} < x_{n+1,i+1}^{[k-1]}. \tag{2.17}$$

Now we consider the Sobolev–Laguerre type inner product:

$$\langle p, q\rangle_{\alpha,M,N} = \langle p, q\rangle_\alpha + Mp(\xi)q(\xi) + Np'(\xi)q'(\xi), \tag{2.18}$$

with $M, N \geq 0$, $\xi \leq 0$. Let $\left\{S_n^{\alpha,M,N}\right\}_{n\geq 0}$ be the respective sequence of orthogonal polynomials such that $k(S_n^{\alpha,M,N}) = k(L_n^\alpha)$ for $n \geq 0$.

Theorem 10. *([17]). There exist constants $D_{n,0}$, $D_{n,1}$ and $D_{n,2}$ such that*

$$S_n^{\alpha,M,N}(x) = D_{n,0}L_n^\alpha(x) + D_{n,1}(x-\xi)L_{n-1}^{(\alpha,2)}(x) + D_{n,2}(x-\xi)^2 L_{n-2}^{(\alpha,4)}(x), \tag{2.19}$$

where:

(1) *If $M, N > 0$, then $D_{n,0} \sim \frac{8\xi n^\alpha}{M(L^\alpha(\xi))^2}$, $D_{n,1} \sim \frac{32(-\xi)^{3/2} n^{\alpha-1/2}}{M(L^\alpha(\xi))^2}$ and $D_{n,2} \sim \frac{1}{n^2}$.*

(2) *If $M = 0$ and $N > 0$, then $D_{n,0} \sim \frac{1}{4\sqrt{-\xi n}}$, $D_{n,1} \sim -\frac{1}{n}$ and $D_{n,2} \sim \frac{1}{4n^2\sqrt{-\xi n}}$.*

(3) *If $N = 0$ and $M > 0$, then $D_{n,0} \sim \frac{\sqrt{-\xi}}{Mn^{1/2-\alpha}(L^\alpha(\xi))^2}$, $D_{n,1} \sim -\frac{1}{n}$ and $D_{n,2} = 0$.*

Let $\{v_{n,i}\}_{i=1}^n$ be the zeros of $S_n^{\alpha,M,N}$ in increasing order. To describe results on zeros of every $S_n^{\alpha,M,N}$, we present the next results.

Proposition 5. *([17]). The zeros of $S_n^{\alpha,M,N}$ are real, simple and at most one of them is outside $[\xi, \infty)$.*

Proposition 6. *([18]). If $\xi < v_{n,1}$ then*

$$v_{n,1} < x_{n,1} < \cdots < v_{n,n} < x_{n,n}. \tag{2.20}$$

Proposition 7. *([17]). Suppose that $v_{n,1} < \xi$. Then*

$$2\xi - x_{n-1,1}^{[2]} < v_{n,1} < \xi < v_{n,2} < x_{n-1,2}^{[2]} < \cdots < v_{n,n} < x_{n-1,n-1}^{[2]}. \tag{2.21}$$

3. Fourier coefficients for Laguerre–Sobolev type polynomials

In this section, we describe the Fourier coefficients associated to Laguerre–Sobolev type polynomials presented in the above section, computed on any finite interval $[a, b]$. For approximation purposes, we will find an oscillatory region for every family of Sobolev–Laguerre polynomials, in order to exhibit a reasonable choose for the interval $[a, b]$.

3.1 Nondiagonal case

Let $\{S_n^\alpha\}_{n\geq 0}$ be the sequence of Sobolev polynomials orthogonal with respect to the inner product (2.12). From (2.13), this sequence is quasi-orthogonal of order 2 with respect to the classical Laguerre polynomials with parameter $\alpha+2$. Then, we consider (2.1) for $\alpha+2$, and from (2.13) we define

$$a_n = A_{n,\alpha}, \quad b_n = B_{n,\alpha},$$

and

$$B_{n+1} = -(2n+\alpha+3), \quad C_{n+1} = n(n+\alpha+2).$$

Then, in the language of Theorem 3, we get the next result.

Corollary 1. *If*

$$0 < B_{n+1,\alpha} < n(n+\alpha+2), \quad A_{n+1,\alpha} < -\frac{B_{n+1,\alpha}\left(x_{n+1,n+1}^{\alpha+2} - (2n+\alpha+3)\right)}{n(n+\alpha+2)}, \tag{3.1}$$

$$0 < B_{n,\alpha} < (n-1)(n+\alpha+1), \quad A_{n,\alpha} > -\frac{B_{n,\alpha}\left(x_{n,1}^{\alpha+2} - (2n+\alpha+1)\right)}{(n-1)(n+\alpha+1)} \tag{3.2}$$

and

$$f_{n+1}(y_{n,i})f_n(y_{n,i}) + A_{n+1,\alpha}f_n(y_{n,i}) + B_{n+1,\alpha} < 0, \tag{3.3}$$

for $i = 1, 2, \ldots, n$, then S_n^α has n real and simple zeros in the interval $[0, \zeta_{n,\alpha+2}]$. Here, $\{y_{n,i}\}_{i=1}^n$ and $\left\{x_{n,i}^{\alpha+2}\right\}_{i=1}^n$ represent the zeros of S_n^α and $L_n^{\alpha+2}$, respectively, ordered in increasing order.

Proof. According to Theorem 3, Part 3, inequalities in (3.1) are equivalent to

$$x_{n+1,1}^{\alpha+2} < y_{n+1,1} < x_{n+1,2}^{\alpha+2} < y_{n+1,2} < \cdots < y_{n+1,n} < x_{n+1,n+1}^{\alpha+2} < y_{n+1,n+1}.$$

and from Part 2, inequalities in (3.2), are equivalent to

$$y_{n,1} < x_{n,1}^{\alpha+2} < y_{n,2} < x_{n,2}^{\alpha+2} < \cdots < x_{n,n-1}^{\alpha+2} < y_{n,n} < x_{n,n}^{\alpha+2}.$$

In the other hand, from Theorem 4, (3.1) is equivalent to

$$y_{n+1,1} < y_{n,1} < y_{n+1,2} < y_{n,2} < \cdots < y_{n+1,n} < y_{n,n} < y_{n+1,n+1}.$$

Finally, since the zeros of $L_n^{\alpha+2}$ and $L_{n+1}^{\alpha+2}$ are interlaced, we obtain

$$x_{n,n}^{\alpha+2} < x_{n+1,n+1}^{\alpha+2}.$$

The above inequalities imply that

$$x_{n+1,1}^{\alpha+2} < y_{n+1,1} < y_{n,1} < \cdots < y_{n,n} < x_{n,n}^{\alpha+2} < x_{n+1,n+1}^{\alpha+2}.$$

From the Proposition 1, we get the result. □

On the other hand, we suppose that $x \in [a, b]$, and we make the transformation $x = \varepsilon\xi + \delta$, where $\xi \in [-1, 1]$, $\varepsilon = \frac{b-a}{2}$ and $\delta = \frac{b+a}{2}$. Then

$$S_n^\alpha(\varepsilon\xi+\delta)=\sum_{k=0}^{\infty}\widehat{S}_n^\alpha(k)e^{ik\pi\xi},$$

and by using of (2.13) we get

$$\begin{aligned}\widehat{S}_n^\alpha(k) &= \frac{1}{2}\int_{-1}^{1}\left(L_n^{\alpha+2}(\varepsilon\xi+\delta)+A_{n,\alpha}L_{n-1}^{\alpha+2}(\varepsilon\xi+\delta)+B_{n,\alpha}L_{n-2}^{\alpha+2}(\varepsilon\xi+\delta)\right)e^{-ik\pi\xi}d\xi\\
&=\frac{1}{2}\int_{-1}^{1}L_n^{\alpha+2}(\varepsilon\xi+\delta)e^{-ik\pi\xi}d\xi+\frac{1}{2}A_{n,\alpha}\int_{-1}^{1}L_{n-1}^{\alpha+2}(\varepsilon\xi+\delta)e^{-ik\pi\xi}d\xi\\
&+\frac{1}{2}B_{n,\alpha}\int_{-1}^{1}L_{n-2}^{\alpha+2}(\varepsilon\xi+\delta)e^{-ik\pi\xi}d\xi,\\
&=\frac{1}{2}\widehat{L}_n^{\alpha+2}(k)+\frac{1}{2}A_{n,\alpha}\widehat{L}_{n-1}^{\alpha+2}k)+\frac{1}{2}B_{n,\alpha}\widehat{L}_{n-2}^{\alpha+2}(k),\end{aligned}$$

and by using of (2.4) we obtain

$$\begin{aligned}&\widehat{S}_n^\alpha(k)\\
&=\frac{1}{2}\widehat{L}_n^{\alpha+2}(k)+\frac{1}{2}A_{n,\alpha}\widehat{L}_{n-1}^{\alpha+2}k)+\frac{1}{2}B_{n,\alpha}\widehat{L}_{n-2}^{\alpha+2}(k),\\
&=\frac{1}{4\varepsilon}e^{ik\delta/\varepsilon}\left[\sum_{t=0}^{n}\frac{(-1)^t}{t!}\binom{n+\alpha+2}{n-t}\beta_t^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)\right.\\
&+A_{n,\alpha}\sum_{t=0}^{n-1}\frac{(-1)^t}{t!}\binom{n+\alpha+1}{n-1-t}\beta_t^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)\\
&\left.+B_{n,\alpha}\sum_{t=0}^{n-2}\frac{(-1)^t}{t!}\binom{n+\alpha}{n-2-t}\beta_t^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)\right]\\
&=\frac{1}{4\varepsilon}e^{ik\delta/\varepsilon}\left(\frac{(-1)^n}{n!}\left[\beta_n^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)-n(n+\alpha+2)\beta_{n-1}^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)-A_{n,\alpha}n\beta_{n-1}^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)\right]\right.\\
&\left.+\sum_{t=0}^{n-2}\frac{(-1)^t}{t!}\beta_t^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)\left[\binom{n+\alpha+2}{n-t}+A_{n,\alpha}\binom{n+\alpha+1}{n-1-t}+B_{n,\alpha}\binom{n+\alpha}{n-2-t}\right]\right).\end{aligned}$$

We summarize in the next.

Proposition 8. *Let* $[a,b]$ *be a bounded interval. Assume* x *in* $[a,b]$, *and* $x=\varepsilon\xi+\delta$, *where* $\xi\in[-1,1]$, $\varepsilon=\frac{b-a}{2}$ *and* $\delta=\frac{b+a}{2}$. *The coefficients of Fourier for* $S_n^\alpha(\varepsilon\xi+\delta)$, *in the local variable* ξ, *are giving by*

$$\begin{aligned}&\widehat{S}_n^\alpha(k)\\
&=\frac{1}{4\varepsilon}e^{ik\delta/\varepsilon}\left[\frac{(-1)^n}{n!}\left[\beta_n^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)-n(n+\alpha+2)\beta_{n-1}^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)-A_{n,\alpha}n\beta_{n-1}^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)\right]\right.\\
&\left.+\sum_{t=0}^{n-2}\frac{(-1)^t}{t!}\beta_t^{a,b}\left(\frac{i\pi k}{\varepsilon}\right)\left[\binom{n+\alpha+2}{n-t}+A_{n,\alpha}\binom{n+\alpha+1}{n-1-t}+B_{n,\alpha}\binom{n+\alpha}{n-2-t}\right]\right].\end{aligned}$$

3.2 Higher order derivatives

$\left\{S_{n,m}^{\alpha,W}\right\}_{n\geq 0}$ represents the sequence of polynomials orthogonal with respect to (2.14). As before, we propose a bounded interval that containing the n zeros of $S_{n,m}^{\alpha,W}$ for n large enough.

Corollary 2. *For $n \geq m+1$, the zeros of $S_{n,m}^{\alpha,W}$ are located in $[-mx_{n,m+1}, \zeta_{n,\alpha}]$.*

Proof. From Theorem 8, $\tilde{v}_{n,m+1} < x_{n,m+1}$, and from Theorem 9, the result is

$$-mx_{n,m+1} < \rho_n,$$

□

As before, we assume $x \in [a,b]$ and $x = \varepsilon\xi + \delta$, where $\xi \in [-1,1]$, $\varepsilon = \frac{b-a}{2}$ and $\delta = \frac{b+a}{2}$.
From (2.15) we have

$$\begin{aligned}
\widehat{S}_{n,m}^{\alpha,W}(k) &= \frac{1}{2}\int_{-1}^{1} S_{n,m}^{\alpha,W}(\varepsilon\xi+\delta)e^{-ik\pi\xi}d\xi \\
&= \frac{1}{2}\sum_{q=0}^{m+1} A_{n,q}\left(\int_{-1}^{1} L_{n-q}^{(\alpha+q)}(\varepsilon\xi+\delta)e^{-ik\pi\xi}d\xi\right) \\
&= \frac{1}{2}\sum_{q=0}^{m+1} A_{n,q}\widehat{L}_{n-q}^{\alpha+q}(k),
\end{aligned}$$

and from (2.4) we arrive to the next.

Proposition 9. *Let $[a,b]$ a bounded interval and $n \geq m+1$. Consider the transformation $x = \varepsilon\xi + \delta$, where $x \in [a,b]$, $\xi \in [-1,1]$, $\varepsilon = \frac{b-a}{2}$ and $\delta = \frac{b+a}{2}$. The Fourier series for $S_{n,m}^{\alpha,W}$, with the local variable ξ, is given by*

$$S_{n,m}^{\alpha,W}(\varepsilon\xi+\delta) = \sum_{k=-\infty}^{\infty} \widehat{S}_{n.m}^{\alpha,W}(k)e^{ik\pi\xi},$$

where

$$\widehat{S}_{n,m}^{\alpha,W}(k) = \frac{1}{4\varepsilon}e^{ik\delta/\epsilon}\sum_{q=0}^{m+1}\sum_{t=0}^{n-q}\frac{(-1)^t}{t!}A_{n,q}\binom{n+\alpha}{n-q-t}\beta_t^{a,b}\left(\frac{i\pi k}{\varepsilon}\right).$$

3.3 Mass outside support

Assume that

$$L_n^{(\alpha,k)}(x) = \sum_{j=0}^{n} a_{n,j}^{\alpha,[k]}x^j, \tag{3.4}$$

with $a_{n,j}^{\alpha,[0]} := a_{n,j}^{\alpha} = \frac{(-1)^j}{j!}\binom{n+\alpha}{n-j}$ (see Ref. [11]). From (2.16), for $k \geq 1$,

$$(x-\xi)L_n^{(\alpha,k)}(x) = L_{n+1}^{(\alpha,k-1)}(x) - d_{n+1,\xi}^{(\alpha,k-1)}L_n^{(\alpha,k-1)}(x), \tag{3.5}$$

with

$$d_{n+1,\xi}^{(\alpha,k-1)} = \frac{L_{n+1}^{(\alpha,k-1)}(\xi)}{L_n^{(\alpha,k-1)}(\xi)}.$$

Then, replacing (3.4) in (3.5) we obtain

$$\sum_{j=1}^{n+1} a_{n,j-1}^{\alpha,[k]} x^j - \sum_{j=0}^{n} \xi a_{n,j}^{\alpha,[k]} x^j = a_{n+1,n+1}^{\alpha,[k-1]} x^{n+1} + \sum_{j=0}^{n} \left(a_{n+1,j}^{\alpha,[k-1]} - d_{n+1,\xi}^{(\alpha,k-1)} a_{n,j}^{\alpha,[k-1]} \right) x^j,$$

or equivalently

$$\begin{aligned} & a_{n,n}^{\alpha,[k]} x^{n+1} + \sum_{j=1}^{n} \left(a_{n,j-1}^{\alpha,[k]} - \xi a_{n,j}^{\alpha,[k]} \right) x^j - \xi a_{n,0}^{\alpha,[k]} \\ & = a_{n+1,n+1}^{\alpha,[k-1]} x^{n+1} + \sum_{j=1}^{n} \left(a_{n+1,j}^{\alpha,[k-1]} - d_{n+1,\xi}^{(\alpha,k-1)} a_{n,j}^{\alpha,[k-1]} \right) x^j + \left(a_{n+1,0}^{\alpha,[k-1]} - d_{n+1,\xi}^{(\alpha,k-1)} a_{n,0}^{\alpha,[k-1]} \right), \end{aligned}$$

then we get the equations

$$\begin{aligned} a_{n,n}^{\alpha,[k]} &= a_{n+1,n+1}^{\alpha,[k-1]}, \\ \xi a_{n,0}^{\alpha,[k]} &= \left(d_{n+1,\xi}^{(\alpha,k-1)} a_{n,0}^{\alpha,[k-1]} - a_{n+1,0}^{\alpha,[k-1]} \right), \end{aligned}$$

and

$$\xi a_{n,j}^{\alpha,[k]} = a_{n,j-1}^{\alpha,[k]} + d_{n+1,\xi}^{(\alpha,k-1)} a_{n,j}^{\alpha,[k-1]} - a_{n+1,j}^{\alpha,[k-1]}, \quad j = 1, \ldots, n.$$

Lemma 1. *If*

$$L_n^{(\alpha,k)}(x) = \sum_{j=0}^{n} a_{n,j}^{\alpha,[k]} x^j, \qquad a_{n,j}^{\alpha,[0]} := a_{n,j}^{\alpha},$$

and $\xi < 0$*, then*

$$a_{n,n}^{\alpha,[k]} = a_{n+1,n+1}^{\alpha,[k-1]},$$

and the coefficients $a_{n,j}^{\alpha,[k]}$*, with* $j = 1, \ldots, n-1$*, can be obtained recurrently by means of*

$$a_{n,j}^{\alpha,[k]} = \xi^{-1} \left(a_{n,j-1}^{\alpha,[k]} + d_{n+1,\xi}^{(\alpha,k-1)} a_{n,j}^{\alpha,[k-1]} - a_{n+1,j}^{\alpha,[k-1]} \right),$$

with the initial condition

$$a_{n,0}^{\alpha,[k]} = \xi^{-1} \left(d_{n+1,\xi}^{(\alpha,k-1)} a_{n,0}^{\alpha,[k-1]} - a_{n+1,0}^{\alpha,[k-1]} \right).$$

If $\xi = 0$ *then*

$$a_{n,n}^{\alpha,[k]} = a_{n+1,n+1}^{\alpha,[k-1]},$$

and for $j = 1, \ldots, n$

$$a_{n,j-1}^{\alpha,[k]} = a_{n+1,j}^{\alpha,[k-1]} - d_{n+1,0}^{(\alpha,k-1)} a_{n,j}^{\alpha,[k-1]}.$$

According to (2.17), we can deduce that

$$x_{n,1} < x_{n,1}^{[1]} < x_{n,1}^{[2]} < x_{n,1}^{[3]} < \cdots < x_{n,1}^{[k]} < \cdots,$$

and

$$x_{n,n}^{[k]} < x_{n+1,n+1}^{[k-1]} < x_{n+2,n+2}^{[k-2]} < \cdots < x_{n+k,n+k},$$

and as consequence we get the next.

Lemma 2. *The zeros of* $L_n^{(\alpha,k)}$ *are located in* $[x_{n,1}, \beta_{n+k,\alpha}]$.

Since the Fourier series for $L_n^{(\alpha,k)}(\varepsilon\eta+\delta)$ in the local variable η is determined by the coefficients

$$\widehat{L}_n^{(\alpha,k)}(k) = \frac{1}{2}\int_{-1}^{1} L_n^{(\alpha,k)}(\varepsilon\eta+\delta)e^{-ik\pi\eta}d\eta,$$

by using of Lemma 1, (2.2) and (2.3) we have the next.

Proposition 10. *Fourier coefficients for* $L_n^{(\alpha,k)}$ *on a finite interval* $[a,b]$, *in the local variable* η, *with* $x=\varepsilon\eta+\delta$, $\eta\in[-1,1]$, $\varepsilon=\frac{b-a}{2}$ *and* $\delta=\frac{b+a}{2}$, *are defined by*

$$\widehat{L}_n^{(\alpha,k)}(k) = \frac{e^{ik\delta/\varepsilon}}{2\varepsilon}\sum_{j=0}^{n} a_{n,j}^{\alpha,[k]}\beta_j^{a,b}\left(\frac{i\pi k}{\varepsilon}\right). \tag{3.6}$$

Let $\left\{S_n^{\alpha,M,N}\right\}_{n\geq 0}$ be the sequence of polynomials orthogonal with respect to (2.18).

Corollary 3. *For* $n \geq 2$, *the zeros of* $S_n^{\alpha,M,N}$ *are into* $[2\xi - x_{n+1,3}, x_{n+1,n+1}]$.

Proof. From (2.17) we get

$$x_{n-1,1}^{[1]} < x_{n-1,1}^{[2]} < x_{n,2}^{[1]}, \quad \text{and} \quad x_{n,2} < x_{n,2}^{[1]} < x_{n+1,3},$$

thus

$$x_{n-1,1}^{[2]} < x_{n+1,3}. \tag{3.7}$$

In the same way, $x_{n-1,n-1}^{[1]} < x_{n-1,n-1}^{[2]} < x_{n,n}^{[1]}$ and $x_{n,n} < x_{n,n}^{[1]} < x_{n+1,n+1}$, thus

$$x_{n-1,n-1}^{[2]} < x_{n+1,n+1}. \tag{3.8}$$

On the other hand, from (2.20), (2.21), (3.7) and (3.8), we obtain

$$2\xi - x_{n+1,3} < 2\xi - x_{n-1,1}^{[2]} < v_{n,1} < \cdots < v_{n,n} < x_{n+1,n+1}$$

□

On the one hand, and on a finite interval $[a,b]$, we compute the Fourier coefficients for every $S_n^{\alpha,M,N}$ and in terms of the local variable η. Indeed, if

$$S_n^{\alpha,M,N}(\varepsilon\eta+\delta) = \sum_{k=0}^{\infty}\widehat{S}_n^{\alpha,M,N}(k)e^{ik\pi\eta},$$

by using of (2.19)

$$\widehat{S}_n^{\alpha,M,N}(k)$$
$$= \frac{1}{2} D_{n,0} \widehat{L}_n^{\alpha}(k) + D_{n,1} \frac{1}{2} \int_{-1}^{1} (x-\xi) L_{n-1}^{(\alpha,2)}(x) e^{-ik\pi\xi} d\xi$$
$$+ D_{n,2} \frac{1}{2} \int_{-1}^{1} (x-\xi)^2 L_{n-2}^{(\alpha,4)}(x) e^{-ik\pi\xi} d\xi$$

Then, from (3.5), for $k = 2$ and $k = 4$, we obtain

$$(x-\xi) L_{n-1}^{(\alpha,2)}(x) = L_n^{(\alpha,1)}(x) - d_{n,\xi}^{(\alpha,1)} L_{n-1}^{(\alpha,1)}(x),$$

and

$$\begin{aligned} (x-\xi)^2 L_{n-2}^{(\alpha,4)}(x) &= (x-\xi) L_{n-1}^{(\alpha,3)}(x) - d_{n-1,\xi}^{(\alpha,3)} (x-\xi) L_{n-2}^{(\alpha,3)}(x) \\ &= L_n^{(\alpha,2)}(x) - \left(d_{n,\xi}^{(\alpha,2)} + d_{n-1,\xi}^{(\alpha,3)} \right) L_{n-1}^{(\alpha,2)}(x) + d_{n-1,\xi}^{(\alpha,3)} d_{n-1,\xi}^{(\alpha,2)} L_{n-2}^{(\alpha,2)}(x), \end{aligned}$$

respectively.

As a consequence, (2.19) can be written as

$$\begin{aligned} S_n^{\alpha,M,N}(x) &= D_{n,0} L_n^{\alpha}(x) + D_{n,1} \left(L_n^{(\alpha,1)}(x) - d_{n,\xi}^{(\alpha,1)} L_{n-1}^{(\alpha,1)}(x) \right) \\ &+ D_{n,2} \left(L_n^{(\alpha,2)}(x) - \left(d_{n,\xi}^{(\alpha,2)} + d_{n-1,\xi}^{(\alpha,3)} \right) L_{n-1}^{(\alpha,2)}(x) + d_{n-1,\xi}^{(\alpha,3)} d_{n-1,\xi}^{(\alpha,2)} L_{n-2}^{(\alpha,2)}(x) \right) \\ &= D_{n,0} L_n^{\alpha}(x) + D_{n,1} L_n^{(\alpha,1)}(x) - D_{n,1} d_{n,\xi}^{(\alpha,1)} L_{n-1}^{(\alpha,1)}(x) \\ &+ D_{n,2} L_n^{(\alpha,2)}(x) + \gamma_{n,\xi,1}^{\alpha,2} L_{n-1}^{(\alpha,2)}(x) + \gamma_{n,\xi,2}^{\alpha,2} L_{n-2}^{(\alpha,2)}(x), \end{aligned}$$

where

$$\gamma_{n,\xi,1}^{\alpha,2} = -\left(d_{n,\xi}^{(\alpha,2)} + d_{n-1,\xi}^{(\alpha,3)} \right), \quad \gamma_{n,\xi,2}^{\alpha,2} = D_{n,2} d_{n-1,\xi}^{(\alpha,3)} d_{n-1,\xi}^{(\alpha,2)}. \tag{3.9}$$

Then, Fourier coefficients are given by

$$\widehat{S}_n^{\alpha,M,N}(m)$$
$$= \frac{1}{2} \int_{-1}^{1} \Big[D_{n,0} L_n^{\alpha}(\varepsilon\eta+\delta) + D_{n,1} L_n^{(\alpha,1)}(\varepsilon\eta+\delta) - D_{n,1} d_{n,\xi}^{(\alpha,1)} L_{n-1}^{(\alpha,1)}(\varepsilon\eta+\delta)$$
$$+ D_{n,2} L_n^{(\alpha,2)}(\varepsilon\eta+\delta) + \gamma_{n,\xi,1}^{\alpha,2} L_{n-1}^{(\alpha,2)}(\varepsilon\eta+\delta) + \gamma_{n,\xi,2}^{\alpha,2} L_{n-2}^{(\alpha,2)}(\varepsilon\eta+\delta) \Big] e^{-im\pi\eta} d\eta$$
$$= D_{n,0} \widehat{L}_n^{\alpha}(m) + D_{n,1} \widehat{L}_n^{(\alpha,1)}(m) - D_{n,1} d_{n,\xi}^{(\alpha,1)} \widehat{L}_{n-1}^{(\alpha,1)}(m)$$
$$+ D_{n,2} \widehat{L}_n^{(\alpha,2)}(m) + \gamma_{n,\xi,1}^{\alpha,2} \widehat{L}_{n-1}^{(\alpha,2)}(m) + \gamma_{n,\xi,2}^{\alpha,2} \widehat{L}_{n-2}^{(\alpha,2)}(m)$$

and if we use (2.4) and (3.6) we get

$$
\begin{aligned}
&\widehat{S}_n^{\alpha,M,N}(m)\\
&= D_{n,0}\frac{e^{ik\delta/\epsilon}}{2\varepsilon}\sum_{j=0}^{n} a_{n,j}^{\alpha}\beta_j^{a,b}\left(\frac{i\pi m}{\epsilon}\right) + D_{n,1}\frac{e^{ik\delta/\epsilon}}{2\varepsilon}\sum_{j=0}^{n} a_{n,j}^{\alpha,[1]}\beta_j^{a,b}\left(\frac{i\pi m}{\varepsilon}\right)\\
&-D_{n,1}d_{n,\xi}^{(\alpha,1)}\frac{e^{ik\delta/\epsilon}}{2\varepsilon}\sum_{j=0}^{n-1} a_{n-1,j}^{\alpha,[1]}\beta_j^{a,b}\left(\frac{i\pi m}{\varepsilon}\right)\\
&+D_{n,2}\frac{e^{ik\delta/\epsilon}}{2\varepsilon}\sum_{j=0}^{n} a_{n,j}^{\alpha,[2]}\beta_j^{a,b}\left(\frac{i\pi m}{\varepsilon}\right) + \gamma_{n,\xi,1}^{\alpha,2}\frac{e^{ik\delta/\epsilon}}{2\varepsilon}\sum_{j=0}^{n-1} a_{n-1,j}^{\alpha,[2]}\beta_j^{a,b}\left(\frac{i\pi m}{\varepsilon}\right)\\
&+\gamma_{n,\xi,2}^{\alpha,2}\frac{e^{ik\delta/\epsilon}}{2\varepsilon}\sum_{j=0}^{n-2} a_{n-2,j}^{\alpha,[2]}\beta_j^{a,b}\left(\frac{i\pi m}{\varepsilon}\right).
\end{aligned}
$$

We summarize in the next.

Theorem 11. *Consider $x = \varepsilon\eta + \delta$, where $\eta \in [-1, 1]$, $\varepsilon = \frac{b-a}{2}$ and $\delta = \frac{b+a}{2}$. For every $n \geq 2$, the Fourier coefficients for the polynomial $S_n^{\alpha,M,N}$ defined in (2.19), and in the local variable η, are given by*

$$
\begin{aligned}
\widehat{S}_n^{\alpha,M,N}(m) &= \frac{e^{ik\delta/\epsilon}}{2\varepsilon}\Bigg[\Phi_{n,n,\xi}^{\alpha,[1,2]} + \Phi_{n,n-1,\xi}^{\alpha,[1,2]} + \Omega_{n-1,n-1,\xi}^{\alpha,[1,2]}\\
&\quad + \sum_{j=0}^{n-2}\left(\Phi_{n,j,\xi}^{\alpha,[1,2]} + \Omega_{n-1,j,\xi}^{\alpha,[1,2]} + \gamma_{n,\xi,2}^{\alpha,2} a_{n-2,j}^{\alpha,[2]}\beta_j^{a,b}\left(\frac{i\pi m}{\varepsilon}\right)\right)\Bigg],
\end{aligned}
$$

where

$$
\Phi_{n,j,\xi}^{\alpha,[1,2]} = \left(D_{n,0}a_{n,j}^{\alpha} + D_{n,1}a_{n,j}^{\alpha,[1]} + D_{n,2}a_{n,j}^{\alpha,[2]}\right)\beta_j^{a,b}\left(\frac{i\pi m}{\varepsilon}\right),
$$

$$
\Omega_{n-1,j,\xi}^{\alpha,[1,2]} = \left(-D_{n,1}d_{n,\xi}^{(\alpha,1)}a_{n-1,j}^{\alpha,[1]} + \gamma_{n,\xi,1}^{\alpha,2}a_{n-1,j}^{\alpha,[2]}\right)\beta_j^{a,b}\left(\frac{i\pi m}{\varepsilon}\right),
$$

and $\gamma_{n,\xi,1}^{\alpha,2}$, $\gamma_{n,\xi,2}^{\alpha,2}$ are given in (3.9).

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