Arithmetic properties of (2, 3)-regular overcubic bipartitions

S. Shivaprasada Nayaka
Department of Mathematics,
JSS Banashankari Arts, Commerce and S. K. Gubbi Science College, Dharwad, India

Abstract
Purpose – Let \( b_{2,3}(n) \), which enumerates the number of (2, 3)-regular overcubic bipartition of \( n \). The purpose of the paper is to describe some congruences modulo 8 for \( b_{2,3}(n) \). For example, for each \( \alpha \geq 0 \) and \( n \geq 1 \),
\[
\begin{align*}
b_{2,3}(8n + 5) &\equiv 0 \pmod{8}, \\
b_{2,3}(2 \cdot 3^{\alpha+3}n + 4 \cdot 3^{\alpha+2}) &\equiv 0 \pmod{8}.
\end{align*}
\]
Design/methodology/approach – H.C. Chan has studied the congruence properties of cubic partition function \( a(n) \), which is defined by
\[
P \sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q; q)_{\infty}} \left( q^{23} \right).
\]
Findings – To establish several congruence modulo 8 for \( b_{2,3}(n) \), here the author keeps to the classical spirit of \( q \)-series techniques in the proofs.
Originality/value – The results established in the work are extension to those proved in \( \ell \)-regular cubic partition pairs.
Keywords Congruences, Dissections, (2, 3)-regular overcubic bipartitions

1. Introduction
A partition \( \lambda \) of a natural number \( n \) is a finite non-increasing sequence of positive integer parts \( \lambda_i \) \( (1 \leq i \leq m) \) such that
\[
n = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_m.
\]
In this case, we write \( |\lambda| = n \). The number of partitions of \( n \) is denoted by \( p(n) \) and the generating function is given by as follows:
\[
\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_{\infty}}.
\]
Ramanujan’s three famous congruences of \( p(n) \) are as follows:
\[
\begin{align*}
p(5n + 4) &\equiv 0 \pmod{5}, \\
p(7n + 5) &\equiv 0 \pmod{7}, \\
p(11n + 6) &\equiv 0 \pmod{11}.
\end{align*}
\]

JEL Classification — 05A17, 11P83.
© S. Shivaprasada Nayaka. Published in Arab Journal of Mathematical Sciences. Published by Emerald Publishing Limited. This article is published under the Creative Commons Attribution (CC BY 4.0) license. Anyone may reproduce, distribute, translate and create derivative works of this article (for both commercial and non-commercial purposes), subject to full attribution to the original publication and authors. The full terms of this license may be seen at http://creativecommons.org/licenses/by/4.0/legalcode
In [1–3], H.C. Chan has studied the congruence properties of cubic partition function $a(n)$, which is defined by as follows:

$$
\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{(q; q)_{\infty}(q^2; q^2)_{\infty}}.
$$

B. Kim [4] studied its overpartition analog, the overcubic partition function $\tilde{a}(n)$, which is defined by as follows:

$$
\sum_{n=0}^{\infty} \tilde{a}(n)q^n = \frac{(-q; q)_{\infty}(-q^2; q^2)_{\infty}}{(q; q)_{\infty}(q^2; q^2)_{\infty}}.
$$

In [5], M.D. Hirschhorn obtained the results satisfied by $\tilde{a}(n)$, which appeared in Kim’s paper [4], and Sellers [6] has proved a number of arithmetic properties of $\tilde{a}(n)$ by employing elementary generating function methods. Zhao and Zhong [7] studied cubic partition pairs, which are denoted by $b(n)$, and the generating function is as follows:

$$
\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q; q)_{\infty}(q^2; q^2)_{\infty}}.
$$

Recently, Kim [8] studied congruence properties of $\tilde{b}(n)$, which denotes overcubic partition pairs of $n$, whose generating function is given by as follows:

$$
\sum_{n=0}^{\infty} \tilde{b}(n)q^n = \frac{(-q; q)_{\infty}(-q^2; q^2)_{\infty}}{(q; q)_{\infty}(q^2; q^2)_{\infty}}.
$$

More recently, Lin [9] studied various arithmetic properties of $\tilde{b}(n)$ modulo 3 and 5. For example, for any $\alpha \geq 2, n \geq 0$,

$$
\tilde{b}(3^\alpha(3n+2)) \equiv 0 \pmod{3},
$$

for $\alpha \geq 0$,

$$
\tilde{b}(380 \cdot 5^n) \equiv 0 \pmod{3}.
$$

In [10], Naika and Nayaka have established some congruences for $\ell$-regular cubic partition pairs. Let $\tilde{b}_{2,3}(n)$ denote the number of $(2, 3)$-regular overcubic bipartitions of $n$, whose generating function is given by as follows:

$$
\tilde{b}_{2,3}(n)q^n = \frac{(q^2; q^2)_{\infty}^4 (q^4; q^4)_{\infty}^4 (q^{24}; q^{24})_{\infty}^2}{(q; q)_{\infty}^4 (q^6; q^6)_{\infty}^2 (q^8; q^8)_{\infty}^2 (q^{12}; q^{12})_{\infty}^4}.
$$

In this paper, we establish several congruences modulo 8 for $\tilde{b}_{2,3}(n)$. These results can be found in Theorems (3.1), and we keep to the classical spirit of $q$-series techniques in our proofs.

2. Preliminaries

For $|ab| < 1$, Ramanujan’s general theta function $f(a, b)$ is defined as follows:

$$
f(a, b) := \sum_{n=-\infty}^{\infty} a^{(n+1)/2} b^{n/2}.
$$
Some special cases of $f(a, b)$ are as follows:

$$\varphi(q) := f(q, q) = \sum_{n=0}^{\infty} q^{n^2} = (-q; q^2)_\infty (q^2; q^2)_\infty,$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = (q^2; q^2)_\infty (q; q^2)_\infty$$

and

$$f(-q) := f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q; q)_\infty.$$  

Where the product representation of $f(a, b)$ arises from Jacobi’s triple product identity

[11, p. 35, Entry 19] as follows:

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

The following dissection formulas to prove our main results.

**Lemma 2.1.** For each prime $p$ and $n \geq 1$,

$$(q; q)_\infty^p \equiv (q^2; q^2)_\infty^{p-1} \pmod {p^n}. \quad (2.1)$$

**Lemma 2.2.** The following 2-dissections holds:

$$\frac{(q; q)_\infty^2}{(q; q)_\infty^2} = \frac{(q^2; q^2)_\infty^2 (q^8; q^8)_\infty^5}{(q^4; q^4)_\infty^2 (q^{16}; q^{16})_\infty^2} - 2q \frac{(q^2; q^2)_\infty (q^{16}; q^{16})_\infty^2}{(q^8; q^8)_\infty}. \quad (2.2)$$

$$\frac{1}{(q; q)_\infty^2} = \frac{(q^8; q^8)_\infty^5}{(q^2; q^2)_\infty^2 (q^{16}; q^{16})_\infty^2} + 2q \frac{(q^2; q^2)_\infty (q^{16}; q^{16})_\infty^2}{(q^8; q^8)_\infty}. \quad (2.3)$$

**Lemma 2.2** is a consequence of dissection formulas of Ramanujan, which is collected in Berndt’s book [11, p. 40, Entry 25].

**Lemma 2.3.** The following 2-dissections holds:

$$\frac{(q^3; q^3)_\infty^3}{(q; q)_\infty^3} = \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty} + q \frac{(q^{12}; q^{12})_\infty^3}{(q^4; q^4)_\infty}. \quad (2.4)$$

$$\frac{(q; q)_\infty^3}{(q^2; q^2)_\infty^3} = \frac{(q^4; q^4)_\infty^3}{(q^{12}; q^{12})_\infty} - 3q \frac{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty^3}{(q^4; q^4)_\infty (q^6; q^6)_\infty^2}. \quad (2.5)$$

$$\frac{(q^3; q^3)_\infty^3}{(q^2; q^2)_\infty^3} = \frac{(q^4; q^4)_\infty^3 (q^6; q^6)_\infty^2}{(q^2; q^2)_\infty^2 (q^{12}; q^{12})_\infty^2} + 3q \frac{(q^4; q^4)_\infty^2 (q^6; q^6)_\infty (q^{12}; q^{12})_\infty^2}{(q^2; q^2)_\infty^3}. \quad (2.6)$$
Hirschhorn, Garvan and Borwein [12] proved (2.4) and (2.5). For proof of (2.6), see [13].

**Lemma 2.4.** The following 2-dissections holds:

\[
\frac{1}{(q;q)\infty (q^3;q^3)\infty} = \frac{(q^8;q^8)\infty (q^{12};q^{12})\infty}{(q^2;q^2)\infty (q^4;q^4)\infty (q^6;q^6)\infty (q^{24};q^{24})\infty}
\]

\[+ q\frac{(q^4;q^4)\infty (q^{24};q^{24})\infty}{(q^2;q^2)\infty (q^6;q^6)\infty (q^{12};q^{12})\infty} \quad (2.7)\]

\[
(q;q)\infty (q^3;q^3)\infty = \frac{(q^2;q^2)\infty (q^6;q^6)\infty (q^{12};q^{12})\infty}{(q^4;q^4)\infty (q^6;q^6)\infty (q^{24};q^{24})\infty}
\]

\[+ q\frac{(q^4;q^4)\infty (q^6;q^6)\infty (q^{24};q^{24})\infty}{(q^2;q^2)\infty (q^6;q^6)\infty (q^{12};q^{12})\infty} = \quad (2.8)\]

Eqn (2.7) was proved by Baruah and Ojah [14]. Replacing \(q\) by \(-q\) in (2.7) and using the fact that \((-q;-q)\infty = \frac{\sqrt{5}}{2}/\sqrt{3}\) we get (2.8).

**Lemma 2.5.** The following 3-dissection hold:

\[
(q;q)\infty (q^2;q^2)\infty = \frac{(q^6;q^6)\infty (q^9;q^9)\infty}{(q^3;q^3)\infty (q^{18};q^{18})\infty} - q(q^6;q^6)\infty (q^{18};q^{18})\infty
\]

\[- 2q^2\frac{(q^2;q^2)\infty (q^{18};q^{18})^4}{(q^6;q^6)\infty (q^9;q^9)\infty} \quad (2.9)\]

One can see this identity in [15].

**Lemma 2.6.** [11, p. 345, Entry 1 (iv)] We have the following 3-dissection

\[
(q;q)^3 = (q^9;q^9)^3 (\zeta^{-1} - 3q + 4q^3\zeta^2), \quad (2.10)
\]

where

\[
\zeta = \frac{(q^3;q^3)\infty (q^{18};q^{18})^3}{(q^6;q^6)\infty (q^9;q^9)^3} \quad (2.11)
\]

3. Congruences modulo 8 for \(\bar{b}_{2,3}(n)\)

**Theorem 3.1.** For each \(\alpha \geq 0\) and \(n \geq 1\), we have

\[
\bar{b}_{2,3}(6n + 5) \equiv 0 \pmod{8}, \quad (3.1)
\]
\[ b_{2,3}(8n + 5) \equiv 0 \pmod{8}, \tag{3.2} \]
\[ b_{2,3}(12n + 7) \equiv 0 \pmod{8}, \tag{3.3} \]
\[ b_{2,3}(18n + 15) \equiv 0 \pmod{8}, \tag{3.4} \]
\[ b_{2,3}(36n + 21) \equiv 0 \pmod{8}, \tag{3.5} \]
\[ b_{2,3}(72n + 38) \equiv 0 \pmod{8}, \tag{3.6} \]
\[ b_{2,3}(72n + 51) \equiv 0 \pmod{8}, \tag{3.7} \]
\[ b_{2,3}(72n + 57) \equiv 0 \pmod{8}, \tag{3.8} \]
\[ b_{2,3}(216n + 99) \equiv 0 \pmod{8}, \tag{3.9} \]
\[ b_{2,3}(216n + 171) \equiv 0 \pmod{8}, \tag{3.10} \]
\[ b_{2,3}(8 \cdot 9^{n+2}n + 57 \cdot 9^{n+1}) \equiv 0 \pmod{8}, \tag{3.11} \]
\[ b_{2,3}(8 \cdot 9^{n+2}n + 35 \cdot 9^{n+2}) \equiv 0 \pmod{8}, \tag{3.12} \]
\[ b_{2,3}(2 \cdot 3^{n+3}n + 4 \cdot 3^{n+2}) \equiv 0 \pmod{8}, \tag{3.13} \]
\[ b_{2,3}(72n + 3) \equiv b_{2,3}(24n + 1) \pmod{8}, \tag{3.14} \]
\[ b_{2,3}(36n + 3) \equiv b_{2,3}(12n + 1) \pmod{8}. \tag{3.15} \]

**Proof.** Employing (2.4) and (2.5) in (1.1), we have

\[
\sum_{n=0}^{\infty} b_{2,3}(n)q^n = \frac{(q^4; q^4)_{\infty}^4(q^6; q^6)_{\infty}^3(q^{24}; q^{24})_{\infty}^2}{(q^2; q^2)_{\infty}^6(q^8; q^8)_{\infty}^2(q^{12}; q^{12})_{\infty}^3}
+ 4q \frac{(q^4; q^4)_{\infty}^9(q^6; q^6)_{\infty}^6(q^{24}; q^{24})_{\infty}^2}{(q^2; q^2)_{\infty}^{12}(q^8; q^8)_{\infty}^2(q^{12}; q^{12})_{\infty}^3}
+ 3q^2 \frac{(q^4; q^4)_{\infty}^{15}(q^{12}; q^{12})_{\infty}(q^{24}; q^{24})_{\infty}^2}{(q^2; q^2)_{\infty}^{24}(q^6; q^6)_{\infty}(q^8; q^8)_{\infty}^3}, \tag{3.16}
\]

which implies the generating function as follows:

\[
\sum_{n=0}^{\infty} b_{2,3}(2n + 1)q^n = 4 \frac{(q^2; q^2)_{\infty}^9(q^3; q^3)_{\infty}(q^{12}; q^{12})_{\infty}^2}{(q; q)_{\infty}^7(q^4; q^4)_{\infty}^2(q^6; q^6)_{\infty}^3}, \tag{3.18}
\]
Invoking (2.1) in (3.18), we obtain the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(2n + 1)q^n \equiv 4(q; q)_{\infty}(q^{3}; q^{3})_{\infty}(q^{2}; q^{3})_{\infty}(q^{6}; q^{3})_{\infty} \pmod{8}. \]  

(3.19)

Substituting (2.8) into (3.19), we get the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(2n + 1)q^n \equiv 4 \frac{(q^{2}; q^{4})_{\infty}(q^{6}; q^{3})_{\infty}(q^{12}; q^{12})_{\infty}}{(q^{4}; q^{4})_{\infty}(q^{24}; q^{24})_{\infty}} + 4q \frac{(q^{6}; q^{6})_{\infty}(q^{4}; q^{4})_{\infty}(q^{24}; q^{24})_{\infty}}{(q^{8}; q^{8})_{\infty}(q^{12}; q^{12})_{\infty}} \]  

(3.20)

Extracting the terms in which powers of \( q \) are congruent to 1 modulo 2 from (3.20), we have the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(4n + 3)q^n \equiv 4(q^{3}; q^{3})_{\infty}(q^{2}; q^{2})_{\infty}(q^{6}; q^{6})_{\infty} \]  

(3.21)

Invoking (2.1) in (3.21), we obtain as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(4n + 3)q^n \equiv 4(q^{3}; q^{3})_{\infty}(q^{6}; q^{6})_{\infty} \]  

(3.22)

Extracting the terms involving \( q^{3n} \) from (3.22), replacing \( q^{3} \) by \( q \), we have the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(12n + 3)q^n \equiv 4(q; q)_{\infty}(q^{2}; q^{2})_{\infty} \]  

(3.23)

Employing (2.2) into (3.23), we find the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(12n + 3)q^n \equiv 4 \frac{(q^{2}; q^{2})_{\infty}(q^{8}; q^{8})_{\infty}}{(q^{4}; q^{4})_{\infty}(q^{16}; q^{16})_{\infty}} \]  

(3.24)

Extracting the terms involving \( q^{2n} \) from (3.24), replacing \( q^{2} \) by \( q \), we have the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(24n + 3)q^n \equiv 4 \frac{(q; q)_{\infty}(q^{4}; q^{4})_{\infty}}{(q^{2}; q^{2})_{\infty}(q^{8}; q^{8})_{\infty}} \]  

(3.25)

Invoking (2.1) in (3.25), we get the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(24n + 3)q^n \equiv 4 \frac{(q^{2}; q^{2})_{\infty}}{(q; q)_{\infty}} \]  

(3.26)
Ramanujan recorded the following identity in his third note book; for proof, one can see [11, p. 49].

$$
\psi(q) = \frac{(q^2; q^2)_\infty}{(q; q)_\infty} = \frac{(q^6; q^6)_\infty(q^9; q^9)_\infty}{(q^3; q^3)_\infty(q^{18}; q^{18})_\infty} + q \frac{(q^{18}; q^{18})_\infty}{(q^9; q^9)_\infty}. \quad (3.27)
$$

Substituting (3.27) into (3.26), we obtain the generating function as follows

$$
\sum_{n=0}^{\infty} b_{2,3}(24n + 3)q^n \equiv 4 \frac{(q^6; q^6)_\infty}{(q^3; q^3)_\infty}(q^9; q^9)_\infty + 4q \frac{(q^{18}; q^{18})_\infty}{(q^9; q^9)_\infty} \pmod{8}. \quad (3.28)
$$

Congruence (3.7) follows from (3.28).

Extracting the terms in which powers of $q$ are congruent to 1 modulo 3 from (3.28), we have the generating function as follows:

$$
\sum_{n=0}^{\infty} \tilde{b}_{2,3}(72n + 27)q^n \equiv 4 \frac{(q^6; q^6)_\infty}{(q^3; q^3)_\infty} \pmod{8}. \quad (3.29)
$$

The results (3.9) and (3.10) follow from (3.29).

From (3.29), we obtain the generating function as follows:

$$
\sum_{n=0}^{\infty} \tilde{b}_{2,3}(216n + 27)q^n \equiv 4 \frac{(q^2; q^2)_\infty}{(q; q)_\infty}(q^3; q^3)_\infty \pmod{8}. \quad (3.30)
$$

Using the congruences (3.30) and (3.26), we can see that

$$
\tilde{b}_{2,3}(216n + 27) \equiv \tilde{b}_{2,3}(24n + 3) \pmod{8}. \quad (3.31)
$$

By mathematical induction on $\alpha$, we find that

$$
\tilde{b}_{2,3}(216 \cdot 9^\alpha n + 27 \cdot 9^\alpha) \equiv \tilde{b}_{2,3}(24n + 3) \pmod{8}. \quad (3.32)
$$

Using (3.7) in (3.31), we get (3.12).

Extracting the terms involving $q^{3n}$ from (3.28), replacing $q^3$ by $q$, we have the generating function as follows:

$$
\sum_{n=0}^{\infty} \tilde{b}_{2,3}(72n + 3)q^n \equiv 4 \frac{(q^2; q^2)_\infty}{(q; q)_\infty}(q^3; q^3)_\infty \pmod{8}. \quad (3.33)
$$

Invoking (2.1) in (3.32), we get the generating function as follows:

$$
\sum_{n=0}^{\infty} \tilde{b}_{2,3}(72n + 3)q^n \equiv 4(q; q)_\infty \pmod{8}. \quad (3.34)
$$

From (3.20), we can see that

$$
\sum_{n=0}^{\infty} \tilde{b}_{2,3}(4n + 1)q^n \equiv 4 \frac{(q^2; q^2)^2(q^4; q^4)^2}{(q^2; q^2)^2(q^4)^2(q^{12}; q^{12})^2} \pmod{8}. \quad (3.35)
$$
Invoking (2.1) in (3.34), we have the generating function as follows:
\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(4n+1)q^n \equiv 4(q^2; q^2)_{\infty}(q^4; q^4)_{\infty} \pmod{8}. \tag{3.35}
\]

Congruence (3.2) follows from (3.34).

Extracting the terms involving \(q^{3n}\) from (3.35), replacing \(q^2\) by \(q\), we have the generating function as follows:
\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(8n+1)q^n \equiv 4(q; q)_{\infty}(q^2; q^2)_{\infty} \pmod{8}. \tag{3.36}
\]

Employing (2.9) into (3.36), we obtain the generating function as follows:
\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(8n+1)q^n \equiv 4 \frac{(q^6; q^6)_{\infty}(q^9; q^9)_{\infty}^4}{(q^3; q^3)_{\infty}(q^{18}; q^{18})_{\infty}^2} + 4q(q^9; q^9)_{\infty}(q^{18}; q^{18})_{\infty} \pmod{8}. \tag{3.37}
\]

Extracting the terms in which powers of \(q\) are congruent to 1 modulo 3 from (3.37), we have the generating function as follows:
\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(24n+9)q^n \equiv 4(q^2; q^3)_{\infty}(q^6; q^6)_{\infty} \pmod{8}. \tag{3.38}
\]

The results (3.6) and (3.8) follow from (3.38).

From (3.38), we have the generating function as follows:
\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(24n+9)q^n \equiv 4(q; q)_{\infty}(q^2; q^2)_{\infty} \pmod{8}. \tag{3.39}
\]

Using the congruences (3.39) and (3.36), we can see that
\[
\bar{b}_{2,3}(72n+9) \equiv \bar{b}_{2,3}(8n+1) \pmod{8}.
\]

By mathematical induction on \(a\), we obtain the generating function as follows:
\[
\bar{b}_{2,3}(8 \cdot 9^{a+1} + 9^{a+1}) \equiv \bar{b}_{2,3}(8n+1) \pmod{8}. \tag{3.40}
\]

Using (3.8) in (3.40), we get (3.11).

Extracting the terms involving \(q^{3n}\) from (3.37), replacing \(q^3\) by \(q\), we have the generating function as follows:
\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(24n+1)q^n \equiv 4 \frac{(q^3; q^3)_{\infty}(q^9; q^9)_{\infty}^4}{(q; q)_{\infty}(q^6; q^6)_{\infty}^2} \pmod{8}. \tag{3.41}
\]

Invoking (2.1) in (3.41), we get the generating function as follows:
\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(24n+1)q^n \equiv 4(q; q)_{\infty} \pmod{8}. \tag{3.42}
\]
Using the congruences (3.33) and (3.42), we obtain (3.14).

From (3.19), it can be rewritten as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(2n + 1)q^n \equiv 4(q; q)_\infty(q^2; q^3)_\infty(q^3; q^3)^3 \pmod{8}. \]  

(3.43)

Employing (2.9) into (3.43), we obtain the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(2n + 1)q^n \equiv 4 \frac{(q^3; q^3)_\infty^2(q^6; q^6)_\infty(q^9; q^9)_\infty^4}{(q^{18}; q^{18})_\infty^2} \]
\[ + 4q(q^3; q^3)_\infty^3(q^6; q^6)_\infty(q^9; q^9)_\infty^3 \pmod{8}. \]  

(3.44)

Congruence (3.1) follows from (3.44).

Extracting the terms in which powers of \( q \) are congruent to 1 modulo 3 from (3.44), we have the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(6n + 3)q^n \equiv 4(q; q)_\infty^3(q^3; q^3)_\infty^3(q^6; q^6)_\infty^3 \pmod{8}. \]  

(3.45)

which implies as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(6n + 3)q^n \equiv 4(q; q)_\infty^3(q^3; q^3)^3 \pmod{8}. \]  

(3.46)

Substituting (2.10) into (3.46), we obtain the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(6n + 3)q^n \equiv 4 \frac{(q^3; q^3)_\infty^2(q^6; q^6)_\infty(q^9; q^9)_\infty^6}{(q^{18}; q^{18})_\infty^3} \]
\[ + 4q(q^3; q^3)_\infty^3(q^6; q^6)_\infty(q^9; q^9)_\infty^3 \pmod{8}. \]  

(3.47)

Congruence (3.4) follows from (3.47).

Extracting the terms in which powers of \( q \) are congruent to 1 modulo 3 from (3.47), we get the generating function as follows:

\[ \sum_{n=0}^{\infty} \bar{b}_{2,3}(18n + 9)q^n \equiv 4(q; q)_\infty^3(q^3; q^3)_\infty^3 \pmod{8}. \]  

(3.48)

Using the congruences (3.48) and (3.46), we find that

\[ \bar{b}_{2,3}(18n + 9) \equiv \bar{b}_{2,3}(6n + 3) \pmod{8}. \]

By mathematical induction on \( \alpha \), we obtain the generating function as follows:

\[ \bar{b}_{2,3}(2 \cdot 3^{\alpha+2} + 3^{\alpha+2}) \equiv \bar{b}_{2,3}(6n + 3) \pmod{8}. \]  

(3.49)
Using (3.4) in (3.49), we get (3.13).

From (3.47), we have the generating function as follows:

\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(18n + 3)q^n \equiv 4 \frac{(q; q)_{\infty}^2 (q^2; q^2)_{\infty} (q^3; q^3)_{\infty}^6}{(q^6; q^6)_{\infty}^2} \quad (\text{mod } 8). \tag{3.50}
\]

Invoking (2.1) in (3.50), we find that

\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(18n + 3)q^n \equiv 4(q^2; q^2)_{\infty}^2 \quad (\text{mod } 8). \tag{3.51}
\]

Congruence (3.5) follows from (3.51).

Extracting the terms involving \(q^{3n}\) from (3.43), replacing \(q^3\) by \(q\), we have the generating function as follows:

\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(6n + 1)q^n \equiv 4 \frac{(q; q)_{\infty}^2 (q^2; q^2)_{\infty} (q^3; q^3)_{\infty}^4}{(q^6; q^6)_{\infty}^2} \quad (\text{mod } 8). \tag{3.52}
\]

Invoking (2.1) in (3.52), we obtain the generating function as follows:

\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(6n + 1)q^n \equiv 4(q^2; q^2)_{\infty}^2 \quad (\text{mod } 8). \tag{3.53}
\]

Congruence (3.3) easily follows from (3.53).

From (3.51) and (3.53), we have the generating function as follows:

\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(36n + 3)q^n \equiv 4(q; q)_{\infty}^2 \quad (\text{mod } 8) \tag{3.54}
\]

and

\[
\sum_{n=0}^{\infty} \bar{b}_{2,3}(12n + 1)q^n \equiv 4(q; q)_{\infty}^2 \quad (\text{mod } 8). \tag{3.55}
\]

Using the congruences (3.54) and (3.55), we get internal congruence (3.15).

References

Corresponding author
S. Shivaprasada Nayaka can be contacted at: shivprasadnayaks@gmail.com

For instructions on how to order reprints of this article, please visit our website: www.emeraldgrouppublishing.com/licensing/reprints.htm
Or contact us for further details: permissions@emeraldinsight.com