Irreversible $k$-threshold conversion number of some graphs

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Abstract

**Purpose** – This paper aims to study Irreversible conversion processes, which examine the spread of a one way change of state (from state 0 to state 1) through a specified society (the spread of disease through populations, the spread of opinion through social networks, etc.) where the conversion rule is determined at the beginning of the study. These processes can be modeled into graph theoretical models where the vertex set $V(G)$ represents the set of individuals on which the conversion is spreading.

**Design/methodology/approach** – The irreversible $k$-threshold conversion process on a graph $G=(V,E)$ is an iterative process which starts by choosing a set $S_0 \subseteq V$, and for each step $t$ ($t = 1, 2, \ldots$), $S_t$ is obtained from $S_{t-1}$ by adjoining all vertices that have at least $k$ neighbors in $S_{t-1}$. $S_0$ is called the seed set of the $k$-threshold conversion process and is called an irreversible $k$-threshold conversion set (IkCS) of $G$ if $S_t = V(G)$ for some $t = 0$. The minimum cardinality of all the IkCSs of $G$ is referred to as the irreversible $k$-threshold conversion number of $G$ and is denoted by $C_k(G)$.

**Findings** – In this paper the authors determine $C_k(G)$ for generalized Jahangir graph $J_{s,m}$ for $1 < k = m$ and $s, m$ are arbitrary. The authors also determine $C_{s,k}(G)$ for strong grids $P_2 \square P_n$ when $k = 4, 5$. Finally, the authors determine $C_{s,2}(G)$ for $P_n \square P_n$ when $n$ is arbitrary.

**Originality/value** – This work is 100% original and has important use in real life problems like Anti-Bioterrorism.

**Keywords** – Jahangir graph, Strong grid graph, Graph conversion process, $k$-threshold conversion set

**Paper type** – Research paper

1. Introduction

As usual $n = |V|$ and $m = |E|$ denote the numbers of vertices and edges at a graph $G(V,E)$, respectively. Let $Y \subseteq V$ and let $F$ be a subset of $E$ such that $F$ consists of all edges of $G$ which have endpoints in $Y$, then $H = (Y,F)$ is called an induced subgraph of $G$ by $Y$ and is denoted by $G_Y$. The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V : uv \in E\}$ while the closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v$ is denoted by $\deg(v)$ and $\deg(v) = |N(v)|$. An independent vertex set of a graph $G(V,E)$ is a subset of $V$ such that no two vertices in the subset represent and edge of $G$. The independence number, denoted by $\alpha(G)$, is the cardinality of the largest independent vertex set of $G$. The term irreversible $k$-threshold conversion problem on graphs refers to the process of finding the least number of vertices we need to initially convert in step $t = 0$ in order to get an irreversible $k$-threshold conversion process, which is an iterative process that starts by choosing a seed set $S_0 \subseteq V$, and for each step $t$ ($t = 1, 2, \ldots$), $S_t$ is obtained from $S_{t-1}$ by adjoining all vertices that have at least $k$ neighbors in $S_{t-1}$. We call $S_0$ the seed set of the $k$-threshold conversion process and if $S_t = V(G)$ for some $t \geq 0$, then $S_0$ is an irreversible $k$-threshold conversion set (IkCS) of $G$. The $k$-threshold conversion number of $G$ (denoted by $C_k(G)$) is the minimum cardinality of all the
IkCSs of $G$. It is obvious that $1 \leq k \leq \Delta(G)$ and $C_1(G) = 1$ for connected graphs. The first graph model of the Irreversible $k$-threshold conversion problem was presented by Dreyer and Roberts in Ref. [1] where they determined the value of $C_k(G)$ for paths and cycles. They also determined $C_2(G)$ and $C_3(G)$ for grid graphs $P_2 \boxtimes P_n$. In Ref. [2] Kyncl et al. found an upper bound for $C_k(G)$ of toroidal grids of size $m \times n$ if $m = 4$ or $n = 4$. In Ref. [3] Adams et al. presented an upper bound for $C_k(G)$ of the tensor product of two arbitrary graphs $G$ and $H$. In Ref. [4] Mynhardt and Wodlinger presented a lower bound for $C_k(G)$ of graphs of maximum degree $k + 1$. Frances et al. [5] studied the relationship between IkCSs and minimum decycling sets. An upper bound for $C_k(G)$ of regular graphs was presented by Mynhardt and Wodlinger in Ref. [6]. In Ref. [7] Shaheen et al. studied irreversible $k$-threshold conversion processes on circulant graphs. In Ref. [8] Shaheen et al. determined $C_2(G)$ and $C_3(G)$ for the strong grid graphs $P_m \boxtimes P_n$ when $m = 2, 3$. For further information on the irreversible $k$-threshold conversion problem on graphs see Centeno et al. [9]. Takaoka and Ueno [10], Kyncl et al. [11]. A generalized Jahangir graph $J_{s,m}$ for $m \geq 2$ is a graph on $sm + 1$ vertices, i.e. a graph consisting of a cycle $C_{sm}$ with one additional vertex which is adjacent to $m$ vertices of $C_{sm}$ at distance $s$ each other on $C_{sm}$, see Ref. [12] for more information on Jahangir graph. Let $v_{sm+1}$ be the label of the central vertex and $v_1, v_2, \ldots, v_{sm}$ be the labels of the vertices that incident clockwise on cycle $C_{sm}$ so that $\deg(v_1) = 3$. We will use this labeling for the rest of the article. The vertices that are adjacent to $v_{sm+1}$ have the labels $v_1, v_1+s, v_1+2s, \ldots, v_1+(m-1)s$. Let $P_m, P_n$ be two paths, we define the strong product of $P_m$ and $P_n$ (also called strong grid graph) as the graph $P_n \boxtimes P_m$ such that $V(P_n \boxtimes P_m) = \{(i,j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ and two vertices $(i_1, j_1), (i_2, j_2)$ are adjacent if and only if $\max\{|i_2-i_1|, |j_2-j_1|\} = 1$. See Ref. [13] for more information on strong grids.

**Proposition 1.1.** [3] For $n \geq 2$; $C_2(P_n) = \frac{n+1}{2}$.

**Proposition 1.2.** [3] For $n \geq 3$; $C_2(C_n) = \frac{n}{2}$.

**Proposition 1.3.** [13] For $n \geq 2$; $C_2(P_2 \boxtimes P_n) = 2$.

**Proposition 1.4.** [13] For $n \geq 2$; $C_3(P_2 \boxtimes P_n) = n + 1$.

**Proposition 1.5.** [6] For $m, n \geq 2$; $\alpha(P_m \boxtimes P_n) = \frac{m n}{2}$.

**Note 1:** As an immediate consequence of the definition, $C_k(G) \geq k$ for any graph $G$.

**Note 2:** As an immediate consequence of the definition, when studying an irreversible $k$-threshold conversion process on a graph $G(V, E)$ all vertices $\{v \in V : \deg(v) < k\}$ must be included in the seed set $S_0$, otherwise the process will fail because none of these vertices can satisfy the conversion rule.

**Note 3:** For Jahangir graph $J_{s,m}$, we denote the set of vertices of degree 3 which consists of $v_1, v_1+s, \ldots, v_1+(m-1)s$, by $R$. So, $R = \{v_1+i s : i = 0, 1, \ldots, m-1\}$

**Note 4:** In every figure of this article, we assign the black color to the converted vertices and the white color to unconverted ones.

**2. Main results**

In this section we determine $C_k(G)$ for generalized Jahangir graph $J_{s,m}$ for $1 \leq k \leq m$ and $s, m$ are arbitrary. We also determine $C_k(G)$ for strong grids $P_2 \boxtimes P_n$ when $k = 4, 5$. Then we determine $C_2(G)$ for $P_n \boxtimes P_n$ when $n$ is arbitrary.

**2.1 $C_k(J_{s,m})$**

In this sub-section we find $C_k(G)$ of generalized Jahangir graph $J_{s,m}$ for $1 \leq k \leq m$ and $s, m$ are arbitrary.
Theorem 2.1. For \( s, m \geq 2 \), \( C_2(J_{s,m}) = \frac{m(s-1)}{2} + 1 \).

Proof. Let \( J_{s,m} \) be a Jahangir graph on which an irreversible 2-threshold conversion process is being studied with a seed set \( S_0 \), let \( U \subseteq V - S_0 \) and \( v_{sm+1} \notin U \), then \( U \) is a 2-unconvertible set of \( J_{s,m} \) if it satisfies the following condition:

For all \( u \in U : |N(u) \cap U| \geq 2 \).

Which means each vertex \( u \in U \) is unconverted and is adjacent to at least 2 vertices of \( U \) at \( t = 0 \). Since \( \deg(u) \leq 3 \) then \( |N(u) \cap S_0| < 2 \) and the conversion cannot reach any vertex of \( U \) during any step of the process. Therefore, we try to avoid having any version of \( U \) on \( J_{s,m} \) when choosing \( S_0 \). We imply that the following sets

\( A_1 = \{a, b, c\} \) with \( \deg(a) = \deg(c) = 2 \) and \( \deg(b) = 3 \), \( B_1 = \{x, y\} \) (with \( \deg(x) = \deg(y) = 2 \)) are 2-unconvertible sets on \( J_{s,m} \). Both \( A_1 \) and \( B_1 \) are represented in Figure 1.

By Proposition 1.1, we have \( C_2(P_n) = \frac{n+1}{2} \). It is obvious that the \( \frac{n+1}{2} - \text{IkCS} \) of a path \( P_n \) must contain the end vertices \( \{v_1, v_n\} \) otherwise the spread will never reach them. The vertices of \( R \) divide \( C_{sm} \) into \( m \) paths (each of which consists of \( s - 1 \) vertices and they are separated by the vertices of \( R \)). We denote these paths as follows:

\[
\begin{align*}
P_{s-1}^{(1)} & = v_2 \ldots v_s, \\
P_{s-1}^{(2)} & = v_{2+s} \ldots v_{2s}, \\
& \vdots \\
P_{s-1}^{(m)} & = v_{2+(m-1)s} \ldots v_{ms}.
\end{align*}
\]

We consider the following subcases:

**Case 1.** \( s \) is even.

In this case, each path \( P_{s-1}^{(i)} : 1 \leq i \leq m \) contains an odd number of vertices. We divide \( V(P_{s-1}^{(i)}) \) into two sets:

\[
\begin{align*}
EP_i & = \{v_{2+(i-1)s}, v_{4+(i-1)s}, \ldots, v_{is}\} \text{ which consists of } \frac{s-1}{2} \text{ vertices.} \\
OP_i & = \begin{cases} \\
\emptyset & \text{if } s = 2; \\
\{v_{3+(i-1)s}, v_{5+(i-1)s}, \ldots, v_{is-1}\} & \text{which consists of } \frac{s-1}{2} \text{ vertices if } s \geq 4.
\end{cases}
\end{align*}
\]

We define a family of sets \( D = \{ D_i : 1 \leq i \leq m \} \), where \( D_i = \begin{cases} \\
EP_i \text{ if } i \text{ is odd; } \\
OP_i \text{ if } i \text{ is even.}
\end{cases} \)

The process goes as follows:

\[
t = 0: \text{ We convert the vertices of } S_0 = D \cup \{v_{sm+1}\}. \\
t = 1: \text{ The conversion spreads to:}
\]

---

**Figure 1.**

\( A_1 = \{a, b, c\} \), \( B_1 = \{x, y\} \) are 2-unconvertible on \( J_{s,m} \).
The vertices of \( \{OP_i : i \text{ is odd}\} \).

The vertices of \( \{EP_i = \{v_{2 + (i-1)s}, v_{is}\} : i \text{ is even}\} \).

The vertices of degree 3 (vertices of \( R \)).

\[ t = 2 : \text{The conversion spreads to } \{v_{2 + (i-1)s}, v_{is} : i \text{ is even}\}. \]

By the end of step \( t = 2 \), the conversion is spread to \( V(J_{s,m}) \) entirely and the process succeeds. It is obvious that

\[ |S_0| = \begin{cases} \frac{m}{2} \left( \frac{s-1}{2} + \frac{s-1}{2} \right) + 1 \text{ if } m \text{ is even;} \\ \frac{m}{2} \frac{s-1}{2} + \frac{m}{2} \frac{s-1}{2} + 1 \text{ if } m \text{ is odd.} \end{cases} \]

Which means:

\[ C_2(J_{s,m}) \leq \frac{m(s-1)}{2} + 1 \] (1)

Figure 2 shows that \( C_2(J_{6,4}) \leq 11 \).

We imply that the sets \( A_1 = \{a, b, c\}, B_1 = \{x, y\} \) represented in Figure 1 are 2-unconvertible on \( J_{s,m} \). We notice that \( D \) is the only \( \frac{m(s-1)}{2} \)-seed set that does not leave any versions of \( A_1 \) or \( B_1 \) on \( C_{sm} \) and every \( k \)-seed set with \( k < \frac{m(s-1)}{2} \) will leave some versions of \( A_1 \) or \( B_1 \) on \( C_{sm} \). Let us assume that \( D_0 \) is a \( 1kCS \) of cardinality \( \frac{m(s-1)}{2} \), we consider the following subcases:

**Case 1.a.** \( v_{sm+1} \not\in D_0 \), which means \( D_0 \subseteq V(C_{sm}) \), and since \( |D_0| = \frac{m(s-1)}{2} \), then \( D_0 = D \) as we found earlier. However, since \( C_2(C_{sm}) = \frac{sm}{2} \) by Proposition 1.2, it is impossible to convert all the vertices of \( C_{sm} \) depending only on \( D \). Therefore, we need to convert \( v_{sm+1} \) at some point and benefit from it being adjacent to \( m \) vertices of \( C_{sm} \). To achieve that we need at step \( t = 0 \) to choose one of three strategies:

- Convert 2 vertices of \( R \) (e.g. \( v_1, v_{1+3} \)). However, that leaves \( \frac{m(s-1)}{2} - 2 \) vertices in \( D_0 \) which means we end up with at least two versions of \( B_1 \) and the process fails as shown in Figure 3(a). Without loss of generality, any choice of the two vertices of \( R \) leads to the same result.

- Convert 1 vertex of \( R \) (e.g. \( v_1 \)), and 2 vertices that are adjacent to a vertex of \( R \) (for example \( v_{2s}, v_{2s+2s} \)), by converting the remaining \( \frac{m(s-1)}{2} - 3 \) vertices in \( D_0 \) in a similar way to \( D \), we end up with two versions of \( B_1 \) and the process fails as shown in Figure 2.
Without loss of generality, any choice of the one vertex of $R$ and the two vertices that are adjacent to a vertex of $R$ leads to the same result.

- Convert 2 pairs of vertices each of which is adjacent to one vertex of $R$ (e.g., $v_2, v_{sm}, v_s, v_{2+s}$), by converting the remaining $\frac{m(s-1)}{2} - 4$ vertices in $D_0$ in a similar way to $D$, we end up with two versions of $B_t$ and the process also fails as shown in Figure 3(c). Without loss of generality, the same result is obtained for whatever 4 vertices that each of which is adjacent to a vertex of $R$ we choose to initially convert.

All strategies end up with two versions of $B_t$ on $C_{sm}$, and without loss of generality, we get the same results by choosing different vertices that satisfy the conditions mentioned in the three strategies above, therefore $C_2(J_{s,m}) > \frac{m(s-1)}{2}$ when $s$ is even and $v_{sm+1} \notin D_0$.

Case 1.b. $v_{sm+1} \in D_0$, by converting $\frac{m(s-1)}{2}$ vertices of $C_{sm}$ we end up with two versions of $B_t$ (as shown in Figure 3(d)), and the process fails. Therefore, $C_2(J_{s,m}) > \frac{m(s-1)}{2}$ in this case as well.

From Case 1.a and Case 1.b we conclude that:

For $s$ is even, $C_2(J_{s,m}) > \frac{m(s-1)}{2}$. \hfill (2)

From (1) and (2) we conclude that for $s$ is even; $C_2(J_{s,m}) = \frac{m(s-1)}{2} + 1$. 

**Figure 3**

Irreversible $k$-threshold conversion

$C_2(J_{6,4}) > 10$
**Case 2.** $s$ is odd.

In this case, each path $P_i: 1 \leq i \leq m$ contains an even number of vertices. We define a family of sets $D = \{D_i : 1 \leq i \leq m\}$ where $D_i = \{v_{2+(i-1)s}, v_{4+(i-1)s}, \ldots, v_{is-1}\}$ which contains $\frac{m(s-1)}{2}$ vertices. The process goes as follows:

1. $t = 0$: We convert the vertices of $S_0 = D \cup \{v_{sm+1}\}$.
2. $t = 1$: The conversion spreads to the vertices of $\{D_i \setminus \{v_{is-1}\} : 1 \leq i \leq m\}$.
3. $t = 2$: The conversion spreads to $\{v_{is-1} : 1 \leq i \leq m\}$.

By the end of step $t = 2$, the conversion is spread to $V(J_{s,m})$ entirely and the process succeeds.

It is obvious that $|S_0| = \frac{m(s-1)}{2} + 1$ which means $C_2(J_{s,m}) \leq \frac{m(s-1)}{2} + 1$. In a similar way to Case 1, $S_0$ is the only IкССS of cardinality $\frac{m(s-1)}{2} + 1$ because $D$ is the only set of cardinality $\frac{m(s-1)}{2}$ that does not leave any versions of $A_1$ or $B_1$ on $C_{sm}$. By following the same discussion in Case 1 we conclude that $C_2(J_{s,m}) > \frac{m(s-1)}{2}$ if $s$ is odd, which means $C_2(J_{s,m}) = \frac{m(s-1)}{2} + 1$ if $s$ is odd.

**Figure 4** illustrates a 2-conversion process on $J_{7,4}$ starting with $|S_0| = 13$.

From Case 1 and Case 2 we conclude that $C_2(J_{s,m}) = \frac{m(s-1)}{2} + 1$ for $s \geq 2$.

**Theorem 2.2.** For $m \geq 2$, $C_3(J_{s,m}) = m(s-1) + 1$.

**Proof.** By definition all vertices with degree lower than 3 need to be added to the seed set $S_0$. However, in order to convert a vertex of degree 3, we need it to be adjacent to three converted vertices which means the conversion will not spread unless $v_{sm+1}$ is initially converted. The process goes as follows:

1. $t = 0$: We convert the vertices of $S_0 = \{V(C_{sm}) \setminus R\} \cup \{v_{sm+1}\}$, we implied that this set is unique.
2. $t = 1$: The conversion spreads to the vertices of $R$.

**Figure 4.** A 2-conversion process on $J_{7,4}$ starting with $|S_0| = 13$.
The process succeeds and $J_{s,m}$ is entirely converted by the end of step 2 which means that $C_3(J_{s,m}) \leq m(s-1) + 1$, since $S_0$ is unique and none of its vertices can be removed, then $C_3(J_{s,m}) = m(s-1) + 1$.

**Theorem 2.3.** For $4 \leq k \leq m$, $C_k(J_{s,m}) = sm$.

**Proof.** By definition all vertices with degree lower than 4 need to be included in $S_0$ which means $S_0 = V(C_{sm})$ and this set is unique. The process goes as follows:

- $t = 0$: We convert the vertices of $S_0 = V(C_{sm})$.
- $t = 1$: The conversion spreads to $v_{sm+1}$.

The process succeeds and $J_{s,m}$ is entirely converted by the end of step 2 which means that $C_k(J_{s,m}) \leq sm$, since $S_0$ is unique and none of its vertices can be removed, we conclude that $C_k(J_{s,m}) = sm$.

2.2 $C_k(P_m \boxtimes P_n)$

In this sub-section we determine $C_k(G)$ for strong grids $P_2 \boxtimes P_n$ when $k = 4, 5$. Then we determine $C_2(G)$ for $P_n \boxtimes P_n$ when $n$ is arbitrary.

**Theorem 2.4.** For $n \geq 3$: $C_4(P_2 \boxtimes P_n) = \begin{cases} n + 1 & \text{if } n \text{ is odd;} \\ n + 2 & \text{if } n \text{ is even.} \end{cases}$

**Proof.** Let $P_2 \boxtimes P_n$ be a strong grid graph on which an irreversible 4-threshold conversion process is being studied with a seed set $S_0$, let $U \subseteq V - S_0$ and $\{1, 1\}, (1, n), (2, 1), (2, n) \} \cap U = \emptyset$, then $U$ is a 2-unconvertible set of $P_2 \boxtimes P_n$ if it satisfies the following condition:

For all $u \in U : |N(u) \cap U| \geq 2$.

Which means each vertex $u \in U$ is unconverted and is adjacent to at least 2 vertices of $U$ at $t = 0$. Since $\deg(u) = 5$ then $|N(u) \cap S_0| \leq 3$ and the conversion cannot reach any vertex of $U$ during any step of the process. Therefore, we try to avoid having any version of $U$ on $P_2 \boxtimes P_n$ when choosing $S_0$. For $2 \leq j \leq n - 1$, we imply that the following sets are 4-unconvertible:

- $X_1 = \{(1, j-1), (1, j), (2, j)\}$,
- $X_2 = \{(2, j-1), (1, j), (2, j)\}$,
- $X_3 = \{(1, j), (2, j), (1, j + 1)\}$,
- $X_4 = \{(1, j), (2, j), (2, j + 1)\}$.

Figure 5 shows that for $1 \leq i \leq 4$: if $X_i \cap S_0 = \emptyset$ on $P_2 \boxtimes P_6$ then none of the vertices of $X_i$ can be converted and the process fails even if $S_0 = V - X_i$. In order to avoid having any version of $X_1$, $X_2$, $X_3$ or $X_4$ on $P_2 \boxtimes P_n$, every two adjacent columns must include at least two vertices of $S_0$ at $t = 0$.

We consider the following cases:

**Case 1.** $n$ is odd.

Let $S_0$ be a seed set of an irreversible 4-threshold conversion process on $P_2 \boxtimes P_n$, since each vertex of $W = \{(1, 1), (1, n), (2, 1), (2, n)\}$ is of degree 3, then $W \subseteq S_0$, otherwise the process fails. Since we are trying to avoid having two adjacent columns that include less than two

![Figure 5](image-url)

$X_1$, $X_2$, $X_3$ and $X_4$ are 4-unconvertible on $P_2 \boxtimes P_6$.
vertices of $S_0$ at $t = 0$, we choose $S_0 = \{(1, 2l + 1), (2, 2l + 1): 0 \leq l \leq \frac{n-1}{2}\}$, which means $S_0$ contains all the vertices of the odd indexed columns of $P_2 \boxtimes P_n$ and $|S_0| = n + 1$. The process goes as follows:

$$t = 0: S_0 = \{(1, 2l + 1), (2, 2l + 1): 0 \leq l \leq \frac{n-1}{2}\}.$$  

$$t = 1: S_1 = S_0 \cup \{(1, 2l), (2, 2l): 1 \leq l \leq \frac{n-1}{2}\} = V(P_2 \boxtimes P_n).$$

This means $S_0$ is an I4CS on $P_2 \boxtimes P_n$. Therefore, if $n$ is odd then:

$$C_4(P_2 \boxtimes P_n) \leq n + 1$$  

Figure 6 illustrates that $C_4(P_2 \boxtimes P_3) \leq 10$.

Now let us assume that $D_0$ is a 4-conversion seed set of cardinality $n$ on $P_2 \boxtimes P_n$. Since $W$ must be contained in $D_0$, this means we need to distribute the remaining $n - 4$ vertices of $D_0$ on the remaining $n - 2$ columns $(2, 3, \ldots, n - 1)$ without having two adjacent columns that include less than two vertices of $D_0$ which is impossible. Therefore, we end up with at least one version of $X_1, X_2, X_3$ or $X_4$ on $P_2 \boxtimes P_n$ and the process fails. This means if $n$ is odd:

$$C_4(P_2 \boxtimes P_n) > n$$

From (3) and (4) we conclude that $C_4(P_2 \boxtimes P_n) = n + 1$ if $n$ is odd.

**Case 2.** $n$ is even.

In a similar way to Case 1, the vertices of $W$ must be contained in the seed set $S_0$. We choose $S_0 = \{(1, 2l + 1), (2, 2l + 1): 0 \leq l \leq \frac{n}{2} - 1\} \cup \{(1, n), (2, n)\}$ which is of cardinality $n + 2$. The process goes as follows:

$$t = 0: S_0 = \{(1, 2l + 1), (2, 2l + 1): 0 \leq l \leq \frac{n}{2} - 1\} \cup \{(1, n), (2, n)\}.$$  

$$t = 1: S_1 = S_0 \cup \{(1, 2l), (2, 2l): 1 \leq l \leq \frac{n}{2} - 1\} = V(P_2 \boxtimes P_n).$$

This means $S_0$ is an I4CS on $P_2 \boxtimes P_n$. Therefore, if $n$ is even then:

$$C_4(P_2 \boxtimes P_n) \leq n + 2$$

Figure 7 shows that $C_4(P_2 \boxtimes P_{10}) \leq 12$. According to the same 4-threshold conversion process, let $D_0$ be an I4CS of cardinality $n + 1$. Since $W$ must be contained in $D_0$, it is impossible to distribute the remaining $n - 3$ vertices of $D_0$ on the $n - 2$ unconverted columns at $t = 0$ without having at least two adjacent columns that include less than two vertices of $D_0$, which means a version of $X_1, X_2, X_3$ or $X_4$ will definitely be created on $P_2 \boxtimes P_n$ and the process fails. We conclude that if $n$ is even then:

**Figure 6.**

$C_4(P_2 \boxtimes P_3) \leq 10$
\[ C_5(P_2 \Box P_n) \geq n + 1 \]  

From (5) and (6) we conclude that \( C_5(P_2 \Box P_n) = n + 2 \) if \( n \geq 4 \) and \( n \) is even. From Case 1 and Case 2 we conclude the requested.

**Theorem 2.5.** For \( n \geq 3 \), \( C_5(P_2 \Box P_n) = \begin{cases} 
\frac{3n + 1}{2} & \text{if } n \text{ is odd;} \\
\frac{3n}{2} + 1 & \text{if } n \text{ is even.}
\end{cases} \)

**Proof.** In a similar way to Theorem 2.4, the vertices of \( W = \{(1, 1), (1, n), (2, 1), (2, n)\} \) must be included in the seed set \( S_0 \). Now we try to determine which vertices of \( M = V - W \) we need to include in \( S_0 \). Since \( M = \{(1, j), (2, j): 2 \leq j \leq n - 1\} \), every vertex of \( M \) is of degree 5 which means there cannot be two adjacent vertices \( v_1, v_2 \in M - S_0 \). Otherwise, the process will fail because neither \( v_1 \) nor \( v_2 \) will be converted at any step of the conversion process. We conclude that \( M - S_0 \) must be an independent set. In order to make \( S_0 \) as small as possible, we try to make \( M - S_0 \) as large as possible. We notice that \( M \) represents the vertices of a strong grid \( P_2 \Box P_{n-2} \) with the difference that the end vertices of \( M \): \( (1, 2), (2, 2), (1, n-1), (2, n-1) \) are of degree 5 while the end vertices of a usual \( P_2 \Box P_{n-2} \) strong grid: \( (1, 1), (2, 1), (1, n), (2, n) \) are of degree 3, but this difference does not change that \( \alpha(G_M) = \alpha(P_2 \Box P_{n-2}) \) which means from Proposition 1.5, \( \alpha(G_M) = \frac{n-2}{2} \). We conclude that the minimum cardinality of \( S_0 \) that does not allow having two adjacent unconverted vertices is:

\[ |S_0| = |M| - \alpha(G_M) + |W| = 2(n - 2) - \frac{n-2}{2} + 4 = 2n - \frac{n-2}{2}. \]

We consider the following cases for \( n \):

**Case 1.** \( n \) is odd.

Since \( n \) is odd then \( \alpha(G_M) = \frac{n-1}{2} \). Therefore, \( C_5(P_2 \Box P_n) = 2n - \frac{n-1}{2} = \frac{3n+1}{2} \).

**Case 2.** \( n \) is even.

Since \( n \) is even then \( \alpha(G_M) = \frac{n-2}{2} \). Therefore, \( C_5(P_2 \Box P_n) = 2n - \frac{n-2}{2} = \frac{3n}{2} + 1 \).

From Case 1 and case 2 we conclude the requested.

**Theorem 2.6.** For \( n \geq 3 \), \( C_2(P_n \Box P_n) = 2 \).

**Proof.** It is known by definition that \( C_2(G) \geq k \) for any \( G \). Therefore, \( C_2(P_n \Box P_n) \geq 2 \). Now we prove that \( C_2(P_n \Box P_n) \leq 2 \) by finding an I2CS of cardinality 2 on \( P_n \Box P_n \). In order to make the conversion steps as few as possible, we start from the middle by choosing the seed set to be \( S_0 = \{ (\frac{n-1}{2}, \frac{n+1}{2}), (\frac{n+3}{2}, \frac{n+1}{2}) \} \). The process goes as follows:

\[ t = 0: S_0 = \left\{ \left( \frac{n-1}{2}, \frac{n+1}{2} \right), \left( \frac{n+3}{2}, \frac{n+1}{2} \right) \right\}. \]

\[ t = 1: S_1 = S_0 \cup \left\{ \left( \frac{n+1}{2}, \frac{n-1}{2} \right), \left( \frac{n+1}{2}, \frac{n+1}{2} \right), \left( \frac{n+1}{2}, \frac{n+3}{2} \right) \right\}. \]

\[ C_2(P_2 \Box P_{10}) \leq 12 \]
$t = 2$: $S_2 = S_1 \cup \left\{ \left( \frac{n-1}{2}, \frac{n-1}{2} \right), \left( \frac{n-1}{2}, \frac{n+3}{2} \right), \left( \frac{n+3}{2}, \frac{n-1}{2} \right), \left( \frac{n+3}{2}, \frac{n+3}{2} \right) \right\}$.

$t = 3$: $S_3 = S_2 \cup \left\{ \left( \frac{n-3}{2}, \frac{n-1}{2} \right), \left( \frac{n-3}{2}, \frac{n+1}{2} \right), \left( \frac{n-3}{2}, \frac{n+3}{2} \right), \left( \frac{n-1}{2}, \frac{n-3}{2} \right), \left( \frac{n+1}{2}, \frac{n-3}{2} \right), \left( \frac{n+3}{2}, \frac{n-3}{2} \right) \right\}$.

$t = 4$: $S_4 = S_3 \cup \left\{ \left( \frac{n-5}{2}, \frac{n-1}{2} \right), \left( \frac{n-5}{2}, \frac{n+1}{2} \right), \left( \frac{n-5}{2}, \frac{n+3}{2} \right), \left( \frac{n-1}{2}, \frac{n-5}{2} \right), \left( \frac{n+1}{2}, \frac{n-5}{2} \right), \left( \frac{n+3}{2}, \frac{n-5}{2} \right) \right\}$.

$5 \leq t \leq \frac{n+1}{2}$: $S_t = S_{t-1} \cup \left\{ \left( \frac{n-2t+3}{2}, \frac{n-1}{2} \right), \left( \frac{n-2t+3}{2}, \frac{n+1}{2} \right), \left( \frac{n-2t+3}{2}, \frac{n+3}{2} \right), \left( \frac{n-1}{2}, \frac{n-2t+3}{2} \right), \left( \frac{n+1}{2}, \frac{n-2t+3}{2} \right), \left( \frac{n+3}{2}, \frac{n-2t+3}{2} \right) \right\}$.

$4 \leq l \leq t$. We notice that at $t = \frac{n+1}{2}$, the spread reaches its limits horizontally and vertically (the three middle vertices of each of the first row, the last row, the first column and the last column are
converted). Therefore, in the remaining steps, the conversion spreads only diagonally as follows:

\[ t = \frac{n+1}{2} + 1: S_{n+1} = S_{n} \cup \left\{ \left( \frac{n-2l+5}{2}, \frac{n-2t+2l-3}{2} \right), \left( \frac{n-2l+3}{2}, \frac{n+2l-3}{2} \right) \right\}. \]

\[ \left( \frac{n+2l-3}{2}, \frac{n+2t-2l+5}{2} \right): 4 \leq 1 \leq t \}\right\}. \]

\[ t = \frac{n+1}{2} + 2: S_{n+2} = S_{n+1} \cup \left\{ \left( \frac{n-2l+5}{2}, \frac{n-2t+2l-3}{2} \right), \left( \frac{n-2t+2l-3}{2}, \frac{n+2l-3}{2} \right) \right\}. \]

\[ \left( \frac{n+2l-3}{2}, \frac{n+2t-2l+5}{2} \right): 5 \leq 1 \leq t - 1 \}\right\}. \]

For \(2 \leq m \leq \frac{n-3}{2}\) which means for \(\frac{n+1}{2} + 2 \leq t \leq n - 1\):

\[ S_{n+1} = S_{n+1} \cup \left\{ \left( \frac{n-2l+5}{2}, \frac{n-2t+2l-3}{2} \right), \left( \frac{n-2t+2l-3}{2}, \frac{n+2l-3}{2} \right) \right\}. \]

\[ \left( \frac{n+2l-3}{2}, \frac{n+2t-2l+5}{2} \right): m + 3 \leq 1 \leq t - m + 1 \}\right\}. \]

When we reach step \(t = n - 1\), we have \(m = \frac{n-3}{2}\) which means:

\[ S_{n-1} = S_{n-2} \cup \left\{ \left( \frac{n-2l+5}{2}, \frac{n-2(n-1)+2l-3}{2} \right), \left( \frac{n-2(n-1)+2l-3}{2}, \frac{n+2l-3}{2} \right) \right\}. \]

\[ \left( \frac{n+2l-3}{2}, \frac{n+2(n-1)-2l+5}{2} \right): \frac{n+3}{2} \leq 1 \leq \frac{n+3}{2} \}\right\}. \]

\[ = S_{n-2} \cup \{(1,1), (1,n), (n,1), (n,n)\} = V(P_n \otimes P_n) \]

we conclude that \(S_0\) is an I2CS of cardinality 2 on \(P_n \otimes P_n\). Therefore, \(C_2(P_n \otimes P_n) \leq 2\) which means \(C_2(P_n \otimes P_n) = 2\) if \(n\) is odd. Figure 8 illustrates that \(C_2(P_5 \otimes P_5) = 2\).

**Case 2.** \(n\) is even.

In a similar way to Case 1, we need to prove that \(C_2(P_n \otimes P_n) \leq 2\) by finding an I2CS of cardinality 2 on \(P_n \otimes P_n\). We start from the middle to make the conversion steps as few as possible. We choose the seed set to be \(S_0 = \left\{ \left( \frac{n}{2}, \frac{n}{2} \right), \left( \frac{n}{2} + 1, \frac{n}{2} + 1 \right) \right\}\). The process goes as follows:

\[ t = 0: S_0 = \left\{ \left( \frac{n}{2}, \frac{n}{2} \right), \left( \frac{n}{2} + 1, \frac{n}{2} + 1 \right) \right\}. \]

\[ t = 1: S_1 = S_0 \cup \left\{ \left( \frac{n}{2}, \frac{n}{2} + 1 \right), \left( \frac{n}{2} + 1, \frac{n}{2} \right) \right\}. \]
Figure 8. 
$C_2(P_9 \otimes P_9) = 2$

$t = 2$: $S_2 = S_1 \cup \{ (\frac{n}{2} - 1, \frac{n}{2}), (\frac{n}{2} - 1, \frac{n}{2} + 1), (\frac{n}{2} + 1, \frac{n}{2} - 1), (\frac{n}{2} + 2, \frac{n}{2}), (\frac{n}{2} + 2, \frac{n}{2} + 1), (\frac{n}{2} + 1, \frac{n}{2} + 2), (\frac{n}{2} + 1, \frac{n}{2} + 1) \}$. 

$t = 3$: $S_3 = S_2 \cup \{ (\frac{3}{2} - 2, \frac{3}{2}), (\frac{n}{2} - 2, \frac{n}{2} + 1), (\frac{n}{2} + 1, \frac{n}{2} - 2), (\frac{n}{2} + 1, \frac{n}{2} - 2), (\frac{n}{2} + 3, \frac{n}{2}), (\frac{n}{2} + 3, \frac{n}{2}), (\frac{n}{2} + 3, \frac{n}{2} + 3), (\frac{n}{2} - 1, \frac{n}{2} - 1), (\frac{n}{2} + 2, \frac{n}{2} - 1), (\frac{n}{2} + 2, \frac{n}{2} + 2), (\frac{n}{2} - 1, \frac{n}{2} + 2) \}$. 

$t = 4$: $S_4 = S_3 \cup \{ (\frac{4}{2} - 3, \frac{4}{2}), (\frac{n}{2} - 3, \frac{n}{2} + 1), (\frac{n}{2} + 1, \frac{n}{2} - 3), (\frac{n}{2} + 2, \frac{n}{2} - 3), (\frac{n}{2} + 3, \frac{n}{2}), (\frac{n}{2} + 4, \frac{n}{2}), (\frac{n}{2} + 1, \frac{n}{2} + 1), (\frac{n}{2} + 1, \frac{n}{2} + 4), (\frac{n}{2} - 2, \frac{n}{2} - 1), (\frac{n}{2} - 1, \frac{n}{2} - 2), (\frac{n}{2} + 3, \frac{n}{2} - 1), (\frac{n}{2} + 2, \frac{n}{2} - 2), (\frac{n}{2} - 2, \frac{n}{2} + 2), (\frac{n}{2} - 1, \frac{n}{2} + 3) \}$. 

Figure 8. 
$C_2(P_9 \otimes P_9) = 2$
When we reach step $t = \frac{n}{2}$, Figure 9 illustrates that $\text{Sn} = \text{Sn-1} \cup \{(\frac{n}{2} - t + 1, \frac{n}{2}), (\frac{n}{2} - t + 1, \frac{n}{2} + 1), (\frac{n}{2}, \frac{n}{2} - t + 1), (\frac{n}{2} + 1, \frac{n}{2} - t + 1), (\frac{n}{2} - l + 2, \frac{n}{2} - t + l - 1), (\frac{n}{2} + l - 1, \frac{n}{2} - t + l - 1), (\frac{n}{2} - t + l - 1, \frac{n}{2} + l - 1), (\frac{n}{2} + l - 1, \frac{n}{2} + t - l + 2); 3 \leq l \leq t\}$.

We notice that at $t = \frac{n}{2}$, the spread reaches its limits horizontally and vertically (the three middle vertices of each of the first row, the last row, the first column and the last column are converted). Therefore, in the remaining steps, the conversion spreads only diagonally as follows:

For $1 \leq m \leq \frac{n}{2} - 1$ which means for $\frac{n+1}{2} + 1 \leq t \leq n - 1$:

$$\text{Sn} = \text{Sn-1} \cup \{(\frac{n}{2} - l + 2, \frac{n}{2} - t + l - 1), (\frac{n}{2} + l - 1, \frac{n}{2} - t + l - 1), (\frac{n}{2} - t + l - 1, \frac{n}{2} + t - l + 2); m + 2 \leq t \leq t - m + 1\}.$$

When we reach step $t = n - 1$, we have $m = \frac{n}{2} - 1$ which means $l = \frac{n}{2} + 1$ therefore:

$$\text{Sn} = \text{Sn-1} \cup \{(\frac{n}{2} - (\frac{n}{2} + 1) + 2, \frac{n}{2} - (n - 1) + (\frac{n}{2} + 1) - 1), (\frac{n}{2} + (\frac{n}{2} + 1) - 1, \frac{n}{2} - (n - 1) + (\frac{n}{2} + 1) - 1, \frac{n}{2} + (\frac{n}{2} + 1) - 1)\}$$

$$\text{Sn} = \{1, 1, (n, 1), (1, n), (n, n)\} = V(P_n \otimes P_n).$$

We conclude that $\text{Sn}$ is an I2CS and $C_2(P_n \otimes P_n) \leq 2$ which means $C_2(P_n \otimes P_n) = 2$ if $n$ is even. Figure 9 illustrates that $C_2(P_8 \otimes P_8) = 2$. From Case 1 and Case 2 we conclude the requested.
References


Further reading


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