Generalized cyclic contractions and coincidence points involving a control function on partial metric spaces

Generalized
cyclic
contractions

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Abstract

Purpose – In this paper, we use the notion of cyclic representation of a nonempty set with respect to a pair of mappings to obtain coincidence points and common fixed points for a pair of self-mappings satisfying some generalized contraction-type conditions involving a control function in partial metric spaces. Moreover, we provide some examples to analyze and illustrate our main results.

Design/methodology/approach – Theoretical method.

Findings – We establish some coincidence points and common fixed point results in partial metric spaces.

Originality/value – Results of this study are new and interesting.

Keywords Partial metric, Cyclic contraction, 0-Completeness, Coincidence point

Paper type Research paper

1. Introduction

In 1994, Matthews [1] introduced the concept of partial metric spaces as a part of the study of denotational semantics of dataflow networks and proved the well-known Banach contraction principle in this setting. Complete partial metric space is a useful framework to model several complex problems in theory of computation. The works of [2–10] are viable and have opened new avenues for application in different fields of mathematics and applied sciences. It is interesting to note that in partial metric spaces self-distance of an arbitrary point need not be equal to zero. Recently, many authors studied fixed points of cyclic mappings in several spaces. In 2003, Kirk *et al.* [11] introduced the notion of cyclic mappings and proved some fixed-point theorems for these mappings. Some results for cyclic contractions in partial metric spaces have been obtained in [12–16]. In many cases, new results are being obtained by considering contractive conditions that depend on control functions. Indeed, the auxiliary functions which involved in contractive-type conditions are known as control functions. In 2013, Shatanawi *et al.* [17] proved some common fixed-point theorems with the help of control functions, namely, altering distance functions due to Khan *et al.* [18]. After that, several generalized control functions were used to obtain fixed-point results in various spaces. The results of [19–22] have become the source of motivation of this study. In this work, we introduce the concept of cyclic representation of a nonempty set with respect to a pair of mappings and use it to prove a coincidence point and common fixed-point result for a pair of self-mappings satisfying some generalized contraction-type conditions involving a control

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function in partial metric spaces. We also prove another common fixed point result for a pair of self-mappings satisfying a new contraction condition in this framework. Our results extend and unify several existing results in the literature. Finally, we give some examples to justify the validity of our results.

2. Some basic concepts

In this section, we present some basic facts and properties of partial metric spaces.

Definition 2.1. [1] A partial metric on a nonempty set X is a function $p : X \times X \rightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

$$(p_1) \ p(x, x) = p(y, y) = p(x, y) \Leftrightarrow x = y,$$

$$(p_2) \ p(x, x) \leq p(x, y),$$

$$(p_3) \ p(x, y) = p(y, x),$$

$$(p_4) \ p(x, y) \leq p(x, z) + p(z, y) - p(z, z).$$

The pair (X, p) is called a partial metric space.

It is obvious that if $p(x, y) = 0$, then from (p_1) and (p_2) , it follows that $x = y$. However, $x = y$ does not imply $p(x, y) = 0$.

Example 2.2. [1] Let $X = [0, \infty)$ and let $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. Then (X, p) is a partial metric space.

Example 2.3. [1] Let $X = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ and let $p([a, b], [c, d]) = (\max\{b, d\} - \min\{a, c\})$. Then (X, p) is a partial metric space.

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family of open p -balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : p(x, y) < p(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.

Theorem 2.4. If $U \in \tau_p$ and $x \in U$, then there exists $r > 0$ such that $B_p(x, r) \subseteq U$.

Proof. Since U is an open set containing x , there exists an open p -ball, say $B_p(y, \epsilon)$ such that $x \in B_p(y, \epsilon) \subseteq U$. Then $p(x, y) < p(y, y) + \epsilon$. Let us choose $0 < r < p(y, y) - p(x, y) + \epsilon$ and consider the open p -ball $B_p(x, r)$. Then it is easy to verify that $B_p(x, r) \subseteq B_p(y, \epsilon) \subseteq U$.

Remark 2.5. Let (X, p) be a partial metric space, (x_n) be a sequence in X and $x \in X$. Then (x_n) converges to x with respect to (w.r.t.) τ_p if and only if $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$.

Let $x_n \rightarrow x$ w.r.t. τ_p and $\epsilon > 0$. Then there exists a natural number n_0 such that $x_n \in B_p(x, \epsilon)$ for all $n \geq n_0$. This gives that $p(x_n, x) - p(x, x) < \epsilon$ for all $n \geq n_0$. Since $p(x_n, x) - p(x, x) \geq 0$, it follows that $|p(x_n, x) - p(x, x)| < \epsilon$ for all $n \geq n_0$. This proves that $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$.

Conversely, suppose that $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$. We shall show that $x_n \rightarrow x$ w.r.t. τ_p . Let $U \in \tau_p$ and $x \in U$. Then there exists $\epsilon > 0$ such that $x \in B_p(x, \epsilon) \subseteq U$. By hypotheses, it follows that

$$\lim_{n \rightarrow \infty} (p(x_n, x) - p(x, x)) = 0.$$

So, there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x) - p(x, x) < \epsilon$ for all $n \geq n_0$. This ensures that $x_n \in B_p(x, \epsilon)$ for all $n \geq n_0$, and hence $x_n \in U$ for all $n \geq n_0$. Therefore, (x_n) converges to x w.r.t. τ_p on X .

Definition 2.6. [1] Let (X, p) be a partial metric space, and let (x_n) be a sequence in X . Then

- (1) (x_n) converges to a point $x \in X$ if $\lim_{n \rightarrow \infty} p(x_n, x) = p(x, x)$. This will be denoted as $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x (n \rightarrow \infty)$.
- (2) (x_n) is called a Cauchy sequence if $\lim_{n, m \rightarrow \infty} p(x_n, x_m)$ exists and is finite, say l , that is, corresponding to every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|p(x_n, x_m) - l| < \epsilon$, $\forall n, m \geq n_0$.
- (3) (X, p) is said to be complete if every Cauchy sequence (x_n) in X converges to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow \infty} p(x_n, x_m)$.

Definition 2.7. [23] A sequence (x_n) in (X, p) is called 0-Cauchy if

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0.$$

The space (X, p) is said to be 0-complete if every 0-Cauchy sequence in X converges to a point $x \in X$ such that $p(x, x) = 0$.

It is easy to verify that every closed subset of a 0-complete partial metric space is 0-complete.

Lemma 2.8. Let (X, p) be a partial metric space.

- (1) (see [24, 25]) If $p(x_n, z) \rightarrow p(z, z) = 0$ as $n \rightarrow \infty$, then $p(x_n, y) \rightarrow p(z, y)$ as $n \rightarrow \infty$ for each $y \in X$.
- (2) (see [23]) If (X, p) is complete, then it is 0-complete.

The converse assertion of (b) may not hold, in general. The following example supports the above remark.

Example 2.9. [23] The space $X = [0, \infty) \cap \mathbb{Q}$ with the partial metric $p(x, y) = \max\{x, y\}$ is 0-complete, but it is not complete. Moreover, the sequence (x_n) with $x_n = 1$ for each $n \in \mathbb{N}$ is a Cauchy sequence in (X, p) , but it is not a 0-Cauchy sequence.

Definition 2.10. [26] Let f and g be self-mappings of a set X . If $y = fx = gx$ for some x in X , then x is called a coincidence point of f and g and y is called a point of coincidence of f and g .

Definition 2.11. [25] The mappings $f, g : X \rightarrow X$ are weakly compatible, if for every $x \in X$, the following holds:

$$f(gx) = g(fx) \text{ whenever } gx = fx.$$

Proposition 2.12. [26] Let f and g be weakly compatible self-maps of a nonempty set X . If f and g have a unique point of coincidence $y = fx = gx$, then y is the unique common fixed point of f and g .

Let Ψ be a class of functions $\psi : [0, \infty) \rightarrow [0, \infty)$ satisfying the following conditions:

- (ψ_1) ψ is a nondecreasing function;
- (ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for each $t > 0$, where ψ^n is the n th iterate of ψ .

Let us notice that the class Ψ is nonempty. Indeed, the functions $\psi(t) = kt$ belong to Ψ whenever $k \in [0, 1)$.

Remark 2.13. [27] For each $\psi \in \Psi$, we see that the following assertions hold:

- (1) $\lim_{n \rightarrow \infty} \psi^n(t) = 0$, for all $t > 0$;
- (2) $\psi(t) < t$ for each $t > 0$;
- (3) $\psi(0) = 0$.

Lemma 2.14. Let (X, p) be a partial metric space. Let $(x_n)_{n=0}^\infty \subseteq X$ be a sequence and $\psi \in \Psi$ be such that

- (1) $p(x_0, x_1) > 0$;
- (2) $p(x_n, x_{n+1}) \leq \psi(p(x_{n-1}, x_n))$, for each $n \in \mathbb{N}$.

Then $(x_n)_{n=0}^\infty$ is a 0-Cauchy sequence.

Proof. By hypothesis (2), we have

$$p(x_n, x_{n+1}) \leq \psi(p(x_{n-1}, x_n)), \text{ for each } n \in \mathbb{N}. \quad (2.1)$$

By repeated use of condition (2.1) and (ψ_1) , we get

$$p(x_n, x_{n+1}) \leq \psi^n(p(x_0, x_1)), \text{ for each } n \in \mathbb{N}.$$

For $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} p(x_n, x_m) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \dots + p(x_{m-2}, x_{m-1}) \\ &\quad + p(x_{m-1}, x_m) \leq \psi^n(p(x_0, x_1)) + \psi^{n+1}(p(x_0, x_1)) + \dots + \psi^{m-2}(p(x_0, x_1)) \\ &\quad + \psi^{m-1}(p(x_0, x_1)) \\ &= \sum_{i=n}^{m-1} \psi^i(p(x_0, x_1)). \end{aligned}$$

Since $\sum_{n=1}^\infty \psi^n(t) < \infty$ for each $t > 0$ and $p(x_0, x_1) > 0$, it follows that

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0.$$

This proves that $(x_n)_{n=0}^\infty$ is a 0-Cauchy sequence in X .

3. Main results

In this section, we prove our new results. Throughout the paper, we use the following notation:

Let (X, p) be a partial metric space, and $f, g : X \rightarrow X$ be self mappings. Then,

$$M(gx, gy) = \max \left\{ p(gx, gy), p(gx, fx), p(gy, fy), \frac{p(gx, fy) + p(gy, fx)}{2} \right\}.$$

We begin with the following definition.

Definition 3.1. Let X be a nonempty set, $q \in \mathbb{N}$, and let $f, g : X \rightarrow X$ be self-mappings. Then $X = \cup_{i=1}^q A_i$ is a cyclic representation of X with respect to the pair (f, g) if

- (1) $A_i, i = 1, 2, \dots, q$ are nonempty subsets of X ;
- (2) $f(A_i) \subseteq g(A_{i+1})$ for $i = 1, 2, \dots, q$, where $A_{q+1} = A_1$.

Theorem 3.2. Let (X, p) be a 0-complete partial metric space, $q \in \mathbb{N}$, and let $A_1, A_2, \dots, A_q$ be nonempty subsets of X , and $X = \bigcup_{i=1}^q A_i$. Suppose $f, g : X \rightarrow X$ are self-mappings, $g(A_1), g(A_2), \dots, g(A_q)$ are closed subsets of (X, p) and $X = \bigcup_{i=1}^q A_i$ is a cyclic representation of X with respect to the pair (f, g) . If there exists $\psi \in \Psi$ such that

$$p(fx, fy) \leq \psi(M(gx, gy)) \quad (3.1)$$

for all $x, y \in X$ with $(gx, gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \dots, q$, where $A_{q+1} = A_1$, then f and g have a unique point of coincidence u in $\bigcap_{i=1}^q g(A_i)$ with $p(u, u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\bigcap_{i=1}^q g(A_i)$.

Proof. Let x_0 be an arbitrary element of X . Then there exists $i_0 \in \{1, 2, \dots, q\}$ such that $x_0 \in A_{i_0}$. Since $f(A_{i_0}) \subseteq g(A_{i_0+1})$, there exists $x_1 \in A_{i_0+1}$ such that $gx_1 = fx_0$. Continuing this process, we can construct a sequence (x_n) such that $gx_n = fx_{n-1}$, $n = 1, 2, 3, \dots$, where $x_n \in A_{i_0+n}$ and $A_{q+k} = A_k$. If $p(gx_n, gx_{n+1}) = 0$ for some $n \in \mathbb{N}$, then $gx_n = gx_{n+1} = fx_n$ and hence gx_{n+1} is a point of coincidence of f and g .

Without loss of generality, we may assume that

$$p(gx_n, gx_{n+1}) > 0, \forall n \in \mathbb{N}.$$

We note that for all $n \in \mathbb{N}$ there exists $i \in \{1, 2, \dots, q\}$ such that $(x_n, x_{n+1}) \in A_i \times A_{i+1}$ and so, $(gx_n, gx_{n+1}) \in g(A_i) \times g(A_{i+1})$. We first compute $M(gx_{n-1}, gx_n)$. We have,

$$\begin{aligned} M(gx_{n-1}, gx_n) &= \max \left\{ \begin{aligned} &p(gx_{n-1}, gx_n), p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1}), \\ &\frac{p(gx_{n-1}, gx_{n+1}) + p(gx_n, gx_n)}{2} \end{aligned} \right\} \\ &\leq \max \left\{ \begin{aligned} &p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1}), \\ &\frac{p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1})}{2} \end{aligned} \right\} \\ &= \max \{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\}. \end{aligned}$$

By (ψ_1) , it follows that

$$\psi(M(gx_{n-1}, gx_n)) \leq \psi(\max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\}). \quad (3.2)$$

For any natural number n , we have by applying conditions (3.1) and (3.2) that

$$\begin{aligned} p(gx_n, gx_{n+1}) &= p(fx_{n-1}, fx_n) \\ &\leq \psi(M(gx_{n-1}, gx_n)) \\ &\leq \psi(\max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\}). \end{aligned} \quad (3.3)$$

We shall show that (gx_n) is a Cauchy sequence in $g(X)$.

If $\max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\} = p(gx_n, gx_{n+1})$, then from condition (3.3) and using $\psi(t) < t$ for each $t > 0$, we obtain

$$p(gx_n, gx_{n+1}) \leq \psi(p(gx_n, gx_{n+1})) < p(gx_n, gx_{n+1}),$$

which is a contradiction. Therefore,

$$\max\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\} = p(gx_{n-1}, gx_n).$$

Thus, we obtain from condition (3.3) that

$$p(gx_n, gx_{n+1}) \leq \psi(p(gx_{n-1}, gx_n)), \text{ for all } n \in \mathbb{N}.$$

By using Lemma 2.14, it follows that (gx_n) is a 0-Cauchy sequence in $g(X)$. Since $g(X) = \bigcup_{i=1}^q g(A_i)$, it follows that $g(X)$ is a closed subset of the 0-complete partial metric space (X, p) , and hence $g(X)$ is 0-complete. So, (gx_n) converges to some point $u \in g(X)$ such that $p(u, u) = 0$. Therefore,

$$\lim_{n \rightarrow \infty} p(gx_n, u) = p(u, u) = 0. \quad (3.4)$$

We shall prove that $u \in \bigcap_{i=1}^q g(A_i)$.

As $x_0 \in A_{i_0}$, it follows that the sequence $(gx_{nq})_{n \geq 0} \subseteq g(A_{i_0})$. Since $g(A_{i_0})$ is closed, condition (3.4) ensures that $u \in g(A_{i_0})$. Again, we get $(gx_{nq+1})_{n \geq 0} \subseteq g(A_{i_0+1})$, where $A_{q+k} = A_k$. Proceeding as above, we obtain that $u \in g(A_{i_0+1})$. Continuing in this way, we get

$$u \in \bigcap_{i=1}^q g(A_i). \quad (3.5)$$

Now we shall show that u is a point of coincidence of f and g .

Indeed, since $u \in g(X)$, there exists $t \in X$ such that $gt = u$. Now, if $x_n \in A_i$, then $(gx_n, gt) = (gx_n, u) \in g(A_i) \times g(A_{i+1})$ because $u \in \bigcap_{i=1}^q g(A_i)$. Therefore $M(gx_n, gt)$ is well defined, and by applying (3.1), we obtain that for all $n \in \mathbb{N}$,

$$p(gx_{n+1}, ft) = p(fx_n, ft) \leq \psi(M(gx_n, gt)), \quad (3.6)$$

where

$$\begin{aligned} M(gx_n, gt) &= \max \left\{ p(gx_n, gt), p(gx_n, fx_n), p(gt, ft), \right. \\ &\quad \left. \frac{p(gx_n, ft) + p(gt, fx_n)}{2} \right\} \\ &= \max \left\{ p(gx_n, gt), p(gx_n, gx_{n+1}), p(gt, ft), \right. \\ &\quad \left. \frac{p(gx_n, ft) + p(gt, gx_{n+1})}{2} \right\}. \end{aligned}$$

Suppose that $p(gt, ft) \neq 0$. Let $\epsilon = \frac{p(gt, ft)}{2} > 0$. Since $\lim_{n \rightarrow \infty} p(gx_n, gt) = 0$, there exists $k \in \mathbb{N}$ such that

$$p(gx_n, gt) < \epsilon, \text{ for each } n \geq k. \quad (3.7)$$

Then, for each $n \geq k$

$$\begin{aligned} p(gx_n, ft) &\leq p(gx_n, gt) + p(gt, ft) - p(gt, gt) \\ &\leq p(gx_n, gt) + p(gt, ft) \\ &< 3\epsilon. \end{aligned}$$

For $n \geq k$, we have

$$\frac{p(gx_n, ft) + p(gt, gx_{n+1})}{2} < \frac{1}{2}(3\epsilon + \epsilon) = 2\epsilon. \quad (3.8)$$

Moreover, for $n \geq k$, we have

$$p(gx_n, gx_{n+1}) \leq p(gx_n, gt) + p(gt, gx_{n+1}) < 2\epsilon. \quad (3.9)$$

Thus, for $n \geq k$, it follows from conditions (3.7), (3.8) and (3.9) that

$$\max \left\{ \begin{array}{l} p(gx_n, gt), p(gx_n, gx_{n+1}), p(gt, ft), \\ \frac{p(gx_n, ft) + p(gt, gx_{n+1})}{2} \end{array} \right\} = 2\epsilon = p(gt, ft).$$

Therefore, we obtain from (3.6) that

$$p(gx_{n+1}, ft) \leq \psi(p(gt, ft)), \text{ for each } n \geq k. \quad (3.10)$$

By using condition (3.10), for $n \geq k$, we have

$$\begin{aligned} p(gt, ft) &\leq p(gt, gx_{n+1}) + p(gx_{n+1}, ft) - p(gx_{n+1}, gx_{n+1}) \\ &\leq p(gt, gx_{n+1}) + \psi(p(gt, ft)). \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$, we get

$$p(gt, ft) \leq \psi(p(gt, ft)),$$

which is a contradiction since $\psi(t) < t$ for each $t > 0$. Therefore, $p(gt, ft) = 0$ and hence $ft = gt = u$. Therefore, u is a point of coincidence of f and g such that $u \in \cap_{i=1}^q g(A_i)$ and $p(u, u) = 0$.

For uniqueness, we assume that there is another point of coincidence v of f and g such that $v \in \cap_{i=1}^q g(A_i)$ and $p(v, v) = 0$. By supposition, there exists $x \in X$ satisfying $v = gx = fx$. Since $u, v \in \cap_{i=1}^q g(A_i)$ and $gx = v, gt = u$, $M(gx, gy)$ is well defined, therefore applying (3.1), we have

$$\begin{aligned} p(u, v) &= p(ft, fx) \\ &\leq \psi \left(\max \left\{ p(gt, gx), p(gt, ft), p(gx, fx), \frac{p(gt, fx) + p(gx, ft)}{2} \right\} \right) \\ &= \psi \left(\max \left\{ p(u, v), p(u, u), p(v, v), \frac{p(u, v) + p(v, u)}{2} \right\} \right) \\ &= \psi(p(u, v)). \end{aligned} \quad (3.11)$$

If $p(u, v) > 0$, then from condition (3.11), we get

$$0 < p(u, v) \leq \psi(p(u, v)),$$

which is a contradiction since $\psi(t) < t$ for each $t > 0$. So, it must be the case that $p(u, v) = 0$, and hence $u = v$. Thus, f and g have a unique point of coincidence $u \in \cap_{i=1}^q g(A_i)$ and $p(u, u) = 0$.

If f and g are weakly compatible, then by proposition 2.12, f and g have a unique common fixed point in $\cap_{i=1}^q g(A_i)$.

Corollary 3.3. Let (X, p) be a 0-complete partial metric space, and let $f, g : X \rightarrow X$ be such that $f(X) \subseteq g(X)$ and $g(X)$ a closed subset of (X, p) . If there exists $\psi \in \Psi$ such that

$$p(fx, fy) \leq \psi(M(gx, gy))$$

for all $x, y \in X$, then f and g have a unique point of coincidence u in $g(X)$ such that $p(u, u) = 0$.

Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $g(X)$.

Proof. The proof follows from [Theorem 3.2](#) by taking $A_1 = A_2 = \dots = A_q = X$.

Corollary 3.4. Let (X, p) be a 0-complete partial metric space, and let $f : X \rightarrow X$ be a mapping. Suppose there exists $\psi \in \Psi$ such that

$$p(fx, fy) \leq \psi \left(\max \left\{ p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy) + p(y, fx)}{2} \right\} \right)$$

for all $x, y \in X$. Then f has a unique fixed point u in X such that $p(u, u) = 0$.

Proof. Since the maps f and I , the identity map on X are weakly compatible, the proof follows from [Theorem 3.2](#) by taking $A_1 = A_2 = \dots = A_q = X$ and $g = I$.

Corollary 3.5. Let (X, p) be a 0-complete partial metric space, $q \in \mathbb{N}$, and let $A_1, A_2, \dots, A_q$ be nonempty subsets of X , and $X = \bigcup_{i=1}^q A_i$. Suppose $f, g : X \rightarrow X$ are self-mappings, $g(A_1), g(A_2), \dots, g(A_q)$ are closed subsets of (X, p) and $X = \bigcup_{i=1}^q A_i$ is a cyclic representation of X with respect to the pair (f, g) . If there exists $k \in [0, 1)$ such that

$$p(fx, fy) \leq k \max \left\{ p(gx, gy), p(gx, fx), p(gy, fy), \frac{p(gx, fy) + p(gy, fx)}{2} \right\}$$

for all $x, y \in X$ with $(gx, gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \dots, q$, where $A_{q+1} = A_1$, then f and g have a unique point of coincidence u in $\bigcap_{i=1}^q g(A_i)$ with $p(u, u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\bigcap_{i=1}^q g(A_i)$.

Proof. The proof follows from [Theorem 3.2](#) by taking $\psi(t) = kt$ for each $t \geq 0$, where $k \in [0, 1)$ is a fixed number.

Corollary 3.6. Let (X, p) be a 0-complete partial metric space and $f : X \rightarrow X$ be a mapping. If there exists $k \in [0, 1)$ such that

$$p(fx, fy) \leq k \max \left\{ p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy) + p(y, fx)}{2} \right\}$$

for all $x, y \in X$, then f has a unique fixed point u in X with $p(u, u) = 0$.

Proof. The result follows from [Theorem 3.2](#) by taking $A_1 = A_2 = \dots = A_q = X$, $g = I$ and $\psi(t) = kt$ for each $t \geq 0$, where $k \in [0, 1)$ is a fixed number.

Corollary 3.7. Let (X, p) be a 0-complete partial metric space, $q \in \mathbb{N}$, and let $A_1, A_2, \dots, A_q$ be nonempty subsets of X , $X = \bigcup_{i=1}^q A_i$. Suppose $f, g : X \rightarrow X$ are self-mappings, $g(A_1), g(A_2), \dots, g(A_q)$ are closed subsets of (X, p) and $X = \bigcup_{i=1}^q A_i$ is a cyclic representation of X with respect to the pair (f, g) . If there exists $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma + 2\delta < 1$ such that

$$p(fx, fy) \leq \alpha p(gx, gy) + \beta p(gx, fx) + \gamma p(gy, fy) + \delta (p(gx, fy) + p(gy, fx)) \quad (3.12)$$

for any $(gx, gy) \in g(A_i) \times g(A_{i+1})$, $i = 1, 2, \dots, q$ with $A_{q+1} = A_1$, then f and g have a unique point of coincidence u in $\bigcap_{i=1}^q g(A_i)$ with $p(u, u) = 0$. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in $\bigcap_{i=1}^q g(A_i)$.

Proof. From condition (3.12), we obtain

$$\begin{aligned} p(fx, fy) &\leq \alpha p(gx, gy) + \beta p(gx, fx) + \gamma p(gy, fy) + \delta (p(gx, fy) + p(gy, fx)) \\ &\leq (\alpha + \beta + \gamma + 2\delta) M(gx, gy) \\ &= k M(gx, gy), \end{aligned}$$

where $k = (\alpha + \beta + \gamma + 2\delta) \in [0, 1)$. Now, corollary 3.5 can be applied to obtain the desired result.

Remark 3.8. It is worth mentioning that theorem 3.8 [28] can be obtained as a particular case of Theorem 3.2. Moreover, we obtain various important fixed-point results in partial metric spaces including Matthews version of Banach contraction theorem [1] as a special case of corollary 3.7.

We now present our second main theorem

Theorem 3.9. Let (X, p) be a 0-complete partial metric space, and let $f, T : X \rightarrow X$ be mappings. Suppose there exists $\psi \in \Psi$ such that

$$p(fx, Ty) \leq \psi(N(x, y)) \quad (3.13)$$

for all $x, y \in X$, where $N(x, y) = \max \left\{ p(x, y), p(x, fx), p(y, Ty), \frac{p(x, Ty) + p(y, fx)}{2} \right\}$. Then f and T have a unique common fixed point u in X with $p(u, u) = 0$.

Proof. We first prove that u is a fixed point of T if and only if u is a fixed point of f .

Suppose that u is a fixed point of T , that is, $Tu = u$. Then, by using condition (3.13), we obtain

$$\begin{aligned} p(fu, u) &= p(fu, Tu) \\ &\leq \psi(N(u, u)), \end{aligned}$$

where

$$\begin{aligned} N(u, u) &= \max \left\{ p(u, u), p(u, fu), p(u, Tu), \frac{p(u, Tu) + p(u, fu)}{2} \right\} \\ &= \max \left\{ p(u, u), p(u, fu), \frac{p(u, u) + p(u, fu)}{2} \right\} \\ &= \max \{ p(u, u), p(u, fu) \} \\ &= p(u, fu). \end{aligned}$$

Therefore,

$$p(fu, u) \leq \psi(p(u, fu)).$$

If $p(u, fu) > 0$, then $\psi(p(u, fu)) < p(u, fu)$, a contradiction. This gives that $p(u, fu) = 0$ and hence $fu = u$.

Proceeding similarly, we can show that if u is a fixed point of f , then u is also a fixed point of T .

Let $x_0 \in X$ be arbitrary. We can construct a sequence (x_n) in X such that

$$x_n = \begin{cases} fx_{n-1}, & \text{if } n \text{ is odd,} \\ Tx_{n-1}, & \text{if } n \text{ is even.} \end{cases}$$

We assume that $x_n \neq x_{n-1}$ for every $n \in \mathbb{N}$. If $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x_{2n} = fx_{2n}$ and hence x_{2n} is a fixed point of f . By our previous discussion, it follows that x_{2n} is also a fixed point of T . So, x_{2n} becomes a common fixed point of f and T . The case $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N} \cup \{0\}$ can be treated similarly to achieve our goal. Therefore, $p(x_n, x_{n-1}) > 0, \forall n \in \mathbb{N}$.

By using condition (3.13), we obtain

$$p(x_{2n+1}, x_{2n+2}) = p(fx_{2n}, Tx_{2n+1}) \leq \psi(N(x_{2n}, x_{2n+1})), \quad (3.14)$$

where

$$\begin{aligned} N(x_{2n}, x_{2n+1}) &= \max \left\{ p(x_{2n}, x_{2n+1}), p(x_{2n}, fx_{2n}), p(x_{2n+1}, Tx_{2n+1}), \right. \\ &\quad \left. \frac{p(x_{2n}, Tx_{2n+1}) + p(x_{2n+1}, fx_{2n})}{2} \right\} \\ &= \max \left\{ p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2}), \right. \\ &\quad \left. \frac{p(x_{2n}, x_{2n+2}) + p(x_{2n+1}, x_{2n+1})}{2} \right\} \\ &\leq \max \left\{ p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2}), \right. \\ &\quad \left. \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2})}{2} \right\} \\ &= \max \{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

By using (ψ_1) , it follows from (3.14) that

$$p(x_{2n+1}, x_{2n+2}) \leq \psi(\max \{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\}). \quad (3.15)$$

If $\max \{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\} = p(x_{2n+1}, x_{2n+2})$, then by using $\psi(t) < t$ for each $t > 0$, we obtain from condition (3.15) that

$$p(x_{2n+1}, x_{2n+2}) \leq \psi(p(x_{2n+1}, x_{2n+2})) < p(x_{2n+1}, x_{2n+2}),$$

which is a contradiction. Therefore,

$$\max \{p(x_{2n}, x_{2n+1}), p(x_{2n+1}, x_{2n+2})\} = p(x_{2n}, x_{2n+1}).$$

Thus, condition (3.15) becomes

$$p(x_{2n+1}, x_{2n+2}) \leq \psi(p(x_{2n}, x_{2n+1})), \forall n \in \mathbb{N}. \quad (3.16)$$

Similarly, we can show that

$$p(x_{2n}, x_{2n+1}) \leq \psi(p(x_{2n-1}, x_{2n})), \forall n \in \mathbb{N}. \quad (3.17)$$

Combining conditions (3.16) and (3.17), we get

$$p(x_n, x_{n+1}) \leq \psi(p(x_{n-1}, x_n)), \forall n \in \mathbb{N}. \quad (3.18)$$

By using Lemma 2.14, it follows that (x_n) is a 0-Cauchy sequence in X . As (X, p) is 0-complete, there exists $u \in X$ such that $x_n \rightarrow u$ with $p(u, u) = 0$, that is, $\lim_{n \rightarrow \infty} p(x_n, u) = p(u, u) = 0$. This ensures that $\lim_{n \rightarrow \infty} p(x_{2n}, u) = p(u, u) = 0$ and $\lim_{n \rightarrow \infty} p(x_{2n+1}, u) = p(u, u) = 0$.

By using condition (3.13), we obtain

$$p(x_{2n+1}, Tu) = p(fx_{2n}, Tu) \leq \psi(N(x_{2n}, u)), \quad (3.19)$$

where

$$\begin{aligned} N(x_{2n}, u) &= \max \left\{ p(x_{2n}, u), p(x_{2n}, fx_{2n}), p(u, Tu), \right. \\ &\quad \left. \frac{p(x_{2n}, Tu) + p(u, fx_{2n})}{2} \right\} \\ &= \max \left\{ p(x_{2n}, u), p(x_{2n}, x_{2n+1}), p(u, Tu), \right. \\ &\quad \left. \frac{p(x_{2n}, Tu) + p(u, x_{2n+1})}{2} \right\}. \end{aligned}$$

Suppose that $p(u, Tu) \neq 0$. Let $\epsilon = \frac{p(u, Tu)}{2} > 0$. Since $\lim_{n \rightarrow \infty} p(x_{2n}, u) = 0$, there exists $k_1 \in \mathbb{N}$ such that

$$p(x_{2n}, u) < \epsilon, \text{ for each } n \geq k_1. \quad (3.20)$$

Then, for each $n \geq k_1$

$$\begin{aligned} p(x_{2n}, Tu) &\leq p(x_{2n}, u) + p(u, Tu) - p(u, u) \\ &= p(x_{2n}, u) + p(u, Tu) \\ &< 3\epsilon. \end{aligned}$$

As $\lim_{n \rightarrow \infty} p(x_{2n+1}, u) = 0$, there exists $k_2 \in \mathbb{N}$ such that

$$p(x_{2n+1}, u) < \epsilon, \text{ for each } n \geq k_2.$$

Put $k = \max\{k_1, k_2\}$. Then, for $n \geq k$, we have

$$\frac{p(x_{2n}, Tu) + p(u, x_{2n+1})}{2} < \frac{1}{2}(3\epsilon + \epsilon) = 2\epsilon. \quad (3.21)$$

Moreover, for $n \geq k$, we have

$$p(x_{2n}, x_{2n+1}) \leq p(x_{2n}, u) + p(u, x_{2n+1}) < 2\epsilon. \quad (3.22)$$

Thus, for $n \geq k$, it follows from conditions (3.20), (3.21) and (3.22) that

$$\max \left\{ p(x_{2n}, u), p(x_{2n}, x_{2n+1}), p(u, Tu), \right. \\ \left. \frac{p(x_{2n}, Tu) + p(u, x_{2n+1})}{2} \right\} = 2\epsilon = p(u, Tu).$$

Therefore, we obtain from (3.19) that

$$p(x_{2n+1}, Tu) \leq \psi(p(u, Tu)), \text{ for each } n \geq k. \quad (3.23)$$

By using condition (3.23), for $n \geq k$, we have

$$\begin{aligned} p(u, Tu) &\leq p(u, x_{2n+1}) + p(x_{2n+1}, Tu) - p(x_{2n+1}, x_{2n+1}) \\ &\leq p(u, x_{2n+1}) + \psi(p(u, Tu)). \end{aligned}$$

Passing to the limit as $n \rightarrow \infty$ and using Lemma 2.8 (a), we get

$$p(u, Tu) \leq \psi(p(u, Tu)),$$

which is a contradiction, since $\psi(t) < t$ for each $t > 0$. Therefore, $p(u, Tu) = 0$ and hence $Tu = u$. This proves that u is a fixed point of T . By our previous discussion, u is also a fixed point of f . Thus, u is a common fixed point of f and T in X with $p(u, u) = 0$.

For uniqueness, let v be another common fixed point of f and T in X with $p(v, v) = 0$. By applying condition (3.13), we get

$$p(u, v) = p(fu, Tv) \leq \psi(N(u, v)), \quad (3.24)$$

where

$$\begin{aligned} N(u, v) &= \max \left\{ p(u, v), p(u, fu), p(v, Tv), \frac{p(u, Tv) + p(v, fu)}{2} \right\} \\ &= \max \{ p(u, v), 0, 0, p(u, v) \} \\ &= p(u, v). \end{aligned}$$

Thus, condition (3.24) becomes

$$p(u, v) \leq \psi(p(u, v)).$$

If $p(u, v) > 0$, then

$$0 < p(u, v) \leq \psi(p(u, v)),$$

which is a contradiction since $\psi(t) < t$ for each $t > 0$. So, it must be the case that $p(u, v) = 0$ and hence $u = v$. Therefore, u is a unique common fixed point of f and T in X with $p(u, u) = 0$.

Corollary 3.10. Let (X, p) be a 0-complete partial metric space, and let the mappings $f, T : X \rightarrow X$ be such that

$$p(fx, Ty) \leq k \max \left\{ p(x, y), p(x, fx), p(y, Ty), \frac{p(x, Ty) + p(y, fx)}{2} \right\}$$

for all $x, y \in X$, where $k \in [0, 1)$ is a constant. Then f and T have a unique common fixed point u in X with $p(u, u) = 0$.

Proof. The proof follows from Theorem 3.9 by taking $\psi(t) = kt$ for each $t \geq 0$, where $k \in [0, 1)$ is a fixed number.

Corollary 3.11. Let (X, p) be a 0-complete partial metric space, and let $f : X \rightarrow X$ be a mapping. Suppose there exists $\psi \in \Psi$ such that

$$p(fx, fy) \leq \psi(N'(x, y))$$

for all $x, y \in X$, where $N'(x, y) = \max \left\{ p(x, y), p(x, fx), p(y, fy), \frac{p(x, fy) + p(y, fx)}{2} \right\}$. Then f has a unique fixed point u in X with $p(u, u) = 0$.

Proof. The proof follows from Theorem 3.9 by considering $T = f$.

Corollary 3.12. Let (X, p) be a 0-complete partial metric space, and let the mappings $f, T : X \rightarrow X$ be such that

$$p(fx, Ty) \leq \alpha p(x, y) + \beta p(x, fx) + \gamma p(y, Ty) + \delta (p(x, Ty) + p(y, fx)) \quad (3.25)$$

for all $x, y \in X$, where $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + \gamma + 2\delta < 1$. Then f and T have a unique common fixed point u in X with $p(u, u) = 0$.

Proof. From condition (3.25), we obtain

$$\begin{aligned} p(fx, Ty) &\leq \alpha p(x, y) + \beta p(x, fx) + \gamma p(y, Ty) + \delta (p(x, Ty) + p(y, fx)) \\ &\leq (\alpha + \beta + \gamma + 2\delta) N(x, y) \\ &= k N(x, y) \\ &= \psi(N(x, y)), \end{aligned}$$

where $k = (\alpha + \beta + \gamma + 2\delta) \in [0, 1)$ and $\psi(t) = kt$ for each $t \geq 0$. Now applying Theorem 3.9, we can obtain the desired result.

Remark 3.13. The results of this study are obtained under the weaker assumption that the underlying partial metric space is 0-complete. However, they also valid if the space is complete.

Finally we conclude this section by providing two applications of our main results.

Example 3.14. Let $X = \{[3 - 5^{-n}, 3] : n \in \mathbb{N}\} \cup \{[3, 3 + 5^{-n}] : n \in \mathbb{N}\} \cup \{\{3\}\}$, where $\{3\} = [3, 3]$. We define $p : X \times X \rightarrow \mathbb{R}^+$ by $p([a, b], [c, d]) = \max\{b, d\} - \min\{a, c\}$. Then (X, p) is a 0-complete partial metric space. Let $A_1 = \{[3 - 5^{-n}, 3] : n \in \mathbb{N}\} \cup \{\{3\}\}$ and $A_2 = \{[3, 3 + 5^{-n}] : n \in \mathbb{N}\} \cup \{\{3\}\}$. Obviously, $X = A_1 \cup A_2$. Define mappings $f, g : X \rightarrow X$ by

$$fx = \begin{cases} [3, 3 + 5^{-(n+2)}], & \text{if } x = [3 - 5^{-n}, 3], \\ [3 - 5^{-(n+2)}, 3], & \text{if } x = [3, 3 + 5^{-n}], \\ \{3\}, & \text{if } x = \{3\} \end{cases}$$

and

$$gx = \begin{cases} [3 - 5^{-(n+1)}, 3], & \text{if } x = [3 - 5^{-n}, 3], \\ [3, 3 + 5^{-(n+1)}], & \text{if } x = [3, 3 + 5^{-n}], \\ \{3\}, & \text{if } x = \{3\}. \end{cases}$$

Then, $f(A_1) \subseteq g(A_2)$, $f(A_2) \subseteq g(A_1)$ and so $X = A_1 \cup A_2$ is a cyclic representation of X with respect to the pair (f, g) . Moreover, $g(A_1)$, $g(A_2)$ are closed subsets of (X, p) . We now verify condition (3.1) with the control function $\psi : [0, \infty) \rightarrow [0, \infty)$ given by $\psi(t) = \frac{t}{5}$. We now consider the following cases:

Case-I. $x = [3 - 5^{-n}, 3] \in A_1$, $y = [3, 3 + 5^{-k}] \in A_2$, $n, k \in \mathbb{N}$ with $n < k$.

In this case, we have $5^{-k} < 5^{-n}$ and $5^{-k} \leq 5^{-(n+1)}$. Then,

$$p(fx, fy) = p([3, 3 + 5^{-(n+2)}], [3 - 5^{-(k+2)}, 3]) = \frac{1}{25} (5^{-n} + 5^{-k}) < \frac{2}{25} \cdot 5^{-n},$$

$$p(gx, gy) = p([3 - 5^{-(n+1)}, 3], [3, 3 + 5^{-(k+1)}]) = 5^{-(k+1)} + 5^{-(n+1)}$$

$$= \frac{1}{5} \cdot 5^{-k} + 5^{-(n+1)} \leq \left(\frac{1}{5} + 1\right) 5^{-(n+1)} = \frac{6}{25} \cdot 5^{-n},$$

$$p(gx, fx) = p([3 - 5^{-(n+1)}, 3], [3, 3 + 5^{-(n+2)}]) = 5^{-(n+2)} + 5^{-(n+1)} = \frac{6}{25} \cdot 5^{-n},$$

$$p(gy, fy) = p([3, 3 + 5^{-(k+1)}], [3 - 5^{-(k+2)}, 3]) = 5^{-(k+1)} + 5^{-(k+2)} < \frac{6}{25} \cdot 5^{-n},$$

$$p(gx, fy) = p([3 - 5^{-(n+1)}, 3], [3 - 5^{-(k+2)}, 3]) = 5^{-(n+1)} = \frac{1}{5} \cdot 5^{-n},$$

$$p(fx, gy) = p([3, 3 + 5^{-(n+2)}], [3, 3 + 5^{-(k+1)}]) = 5^{-(n+2)} = \frac{1}{25} \cdot 5^{-n}.$$

Now,
$$\frac{p(gx, fy) + p(fx, gy)}{2} = \frac{1}{2} \left(\frac{1}{5} \cdot 5^{-n} + \frac{1}{25} \cdot 5^{-n} \right) = \frac{3}{25} \cdot 5^{-n} < \frac{6}{25} \cdot 5^{-n}.$$

Thus, $M(gx, gy) = \frac{6}{25} \cdot 5^{-n}$. Therefore,

$$p(fx, fy) < \frac{2}{25} \cdot 5^{-n} = \psi(M(gx, gy)).$$

Case-II. $x = [3 - 5^{-n}, 3] \in A_1, y = [3, 3 + 5^{-k}] \in A_2, n, k \in \mathbb{N}$ with $n > k$.

In this case, we have $5^{-k} > 5^{-n}$ and $5^{-n} \leq 5^{-(k+1)}$. Then, $p(fx, fy) < \frac{2}{25} \cdot 5^{-k}, p(gx, gy) = \frac{1}{5} (5^{-k} + 5^{-n}) \leq \frac{6}{25} \cdot 5^{-k}, p(gx, fx) = \frac{6}{25} \cdot 5^{-n}, p(gy, fy) = \frac{6}{25} \cdot 5^{-k}$ and $p(gx, fy) = 5^{-(k+2)} = \frac{1}{25} \cdot 5^{-k}, p(fx, gy) = 5^{-(k+1)} = \frac{1}{5} \cdot 5^{-k}$. So, $\frac{p(gx, fy) + p(fx, gy)}{2} = \frac{6}{25} \cdot 5^{-k}$.

Thus, $M(gx, gy) = \frac{6}{25} \cdot 5^{-k}$. Therefore,

$$p(fx, fy) < \frac{2}{25} \cdot 5^{-k} = \psi(M(gx, gy)).$$

Case-III. $x = [3 - 5^{-n}, 3] \in A_1, y = [3, 3 + 5^{-k}] \in A_2, n, k \in \mathbb{N}$ with $n = k$.

Then, $p(fx, fy) = \frac{2}{25} \cdot 5^{-n}, p(gx, gy) = \frac{2}{5} \cdot 5^{-n}, p(gx, fx) = \frac{6}{25} \cdot 5^{-n}, p(gy, fy) = \frac{6}{25} \cdot 5^{-n}$ and $p(gx, fy) = \frac{1}{5} \cdot 5^{-n}, p(fx, gy) = \frac{1}{5} \cdot 5^{-n}$. So, $\frac{p(gx, fy) + p(fx, gy)}{2} = \frac{1}{5} \cdot 5^{-n}$.

Thus, $M(gx, gy) = \frac{6}{25} \cdot 5^{-n}$. Therefore,

$$p(fx, fy) = \frac{2}{25} \cdot 5^{-n} = \psi(M(gx, gy)).$$

Case-IV. $x = [3 - 5^{-n}, 3] \in A_1, n \in \mathbb{N}, y = \{3\} \in A_2$.

Then,

$$p(fx, fy) = p([3, 3 + 5^{-(n+2)}], \{3\}) = 5^{-(n+2)} = \frac{1}{25} \cdot 5^{-n},$$

$$p(gx, gy) = p([3 - 5^{-(n+1)}, 3], \{3\}) = 5^{-(n+1)} = \frac{1}{5} \cdot 5^{-n},$$

$$p(gx, fx) = p([3 - 5^{-(n+1)}, 3], [3, 3 + 5^{-(n+2)}]) = 5^{-(n+2)} + 5^{-(n+1)} = \frac{6}{25} \cdot 5^{-n},$$

$$p(gy, fy) = p(\{3\}, \{3\}) = 0, p(gx, fy) = p([3 - 5^{-(n+1)}, 3], \{3\}) = \frac{1}{5} \cdot 5^{-n},$$

$$p(fx, gy) = p([3, 3 + 5^{-(n+2)}], \{3\}) = \frac{1}{25} \cdot 5^{-n}. \text{ Thus, } M(gx, gy) = \frac{6}{25} \cdot 5^{-n}.$$

Therefore,
$$p(fx, fy) = \frac{1}{25} \cdot 5^{-n} < \frac{2}{25} \cdot 5^{-n} = \psi(M(gx, gy)).$$

Case-V. $x = \{3\} \in A_1$, $y = [3, 3 + 5^{-n}] \in A_2$, $n \in \mathbb{N}$.

In this case, we have

$$p(fx, fy) = \frac{1}{25} \cdot 5^{-n}, \quad M(gx, gy) = \frac{6}{25} \cdot 5^{-n}.$$

Therefore,

$$p(fx, fy) = \frac{1}{25} \cdot 5^{-n} < \frac{2}{25} \cdot 5^{-n} = \psi(M(gx, gy)).$$

Case-VI. $x = y = \{3\}$ is trivial.

The other possibility is treated similarly. Moreover, f and g are weakly compatible. Thus, all the conditions of [Theorem 3.2](#) are fulfilled, and $\{3\}$ is the unique common fixed point of f and g in $g(A_1) \cap g(A_2)$ with $p(\{3\}, \{3\}) = 0$.

Even if in the following example it is not hard to prove that $\mathbf{0}$ is the only common fixed point, it can provide an alternative proof and can inspire other application of [Theorem 3.9](#).

Example 3.15. Let $\beta \in (0, 1)$ be fixed, and let $X = A \cup \{\mathbf{0}\} \subseteq \ell^1$, where $\mathbf{0} = (0)_{n=1}^\infty$ and the subset A of ℓ^1 defined by $x^q = (x_n^q)_{n=1}^\infty \in A$ if

$$x_n^q = \begin{cases} 0, & \text{if } n < 2q, \text{ or } n = 2k - 1, k \in \mathbb{N}, \\ \beta^n, & \text{if } n = 2k \geq 2q, \end{cases}$$

for $q = 1, 2, 3, \dots$. Define $p : X \times X \rightarrow \mathbb{R}^+$ by $p((x_n), (y_n)) = \sum_{n=1}^\infty \max\{x_n, y_n\}$ for all $(x_n), (y_n) \in X$. Then (X, p) is a 0-complete partial metric space. Let $T, f : X \rightarrow X$ be defined by

$$T\left(\underbrace{0, \dots, 0}_{2q-1}, \beta^{2q}, 0, \beta^{2q+2}, 0, \dots\right) = \left(\underbrace{0, \dots, 0}_{2q+1}, \beta^{2q+2}, 0, \beta^{2q+4}, 0, \dots\right), T\mathbf{0} = \mathbf{0}$$

and

$$f\left(\underbrace{0, \dots, 0}_{2q-1}, \beta^{2q}, 0, \beta^{2q+2}, 0, \dots\right) = \left(\underbrace{0, \dots, 0}_{2q+3}, \beta^{2q+4}, 0, \beta^{2q+6}, 0, \dots\right), f\mathbf{0} = \mathbf{0}.$$

Define $\psi : [0, \infty) \rightarrow [0, \infty)$ by $\psi(t) = \beta t$

We now verify condition [\(3.13\)](#) for all $x, y \in X$.

Case-I. Take $x = \left(\underbrace{0, \dots, 0}_{2q-1}, \beta^{2q}, 0, \beta^{2q+2}, 0, \dots\right) \in A$ and

$$y = \left(\underbrace{0, \dots, 0}_{2r-1}, \beta^{2r}, 0, \beta^{2r+2}, 0, \dots\right) \in A.$$

Then

$$fx = \left(0, \underbrace{\dots, 0}_{2q+3}, \beta^{2q+4}, 0, \beta^{2q+6}, 0, \dots \right),$$

$$Ty = \left(0, \underbrace{\dots, 0}_{2r+1}, \beta^{2r+2}, 0, \beta^{2r+4}, 0, \dots \right).$$

(1) If $r \leq q$, then

$$\begin{aligned} p(fx, Ty) &= \beta^{2r+2} + \beta^{2r+4} + \dots + \beta^{2q+2} + \beta^{2q+4} + \beta^{2q+6} + \dots \\ &= \beta^2 [\beta^{2r} + \beta^{2r+2} + \dots + \beta^{2q-2} + \beta^{2q}] + \frac{\beta^{2q+4}}{1 - \beta^2}, \\ p(x, y) &= \beta^{2r} + \beta^{2r+2} + \dots + \beta^{2q-2} + \beta^{2q} + \beta^{2q+2} + \beta^{2q+4} + \dots \\ &= \beta^{2r} + \beta^{2r+2} + \dots + \beta^{2q-2} + \beta^{2q} + \frac{\beta^{2q+2}}{1 - \beta^2}. \end{aligned}$$

Thus,

$$p(fx, Ty) = \beta^2 p(x, y) < \beta p(x, y) \leq \beta N(x, y) = \psi(N(x, y)).$$

(2) If $r > q$, then $2r + 2 > 2q + 2 \Rightarrow 2r + 2 \geq 2q + 4$ and

$$\begin{aligned} p(fx, Ty) &= \beta^{2q+4} + \beta^{2q+6} + \dots + \beta^{2r} + \beta^{2r+2} + \beta^{2r+4} + \dots \\ &= \beta^{2q+4} + \beta^{2q+6} + \dots + \beta^{2r-2} + \beta^{2r} + \frac{\beta^{2r+2}}{1 - \beta^2}, \\ p(x, y) &= \beta^{2q} + \beta^{2q+2} + \dots + \beta^{2r-2} + \beta^{2r} + \beta^{2r+2} + \beta^{2r+4} + \dots \\ &= \beta^{2q} + \beta^{2q+2} + \dots + \beta^{2r-4} + \beta^{2r-2} + \frac{\beta^{2r}}{1 - \beta^2}, \end{aligned}$$

which implies that

$$p(fx, Ty) < \beta^2 p(x, y) < \beta p(x, y) \leq \beta N(x, y) = \psi(N(x, y)).$$

Case-II. Take $x = \left(0, \underbrace{\dots, 0}_{2q-1}, \beta^{2q}, 0, \beta^{2q+2}, 0, \dots \right) \in A$ and $y = \mathbf{0}$.

Then

$$fx = \left(0, \underbrace{\dots, 0}_{2q+3}, \beta^{2q+4}, 0, \beta^{2q+6}, 0, \dots \right), \quad Ty = \mathbf{0},$$

$$p(fx, Ty) = \beta^{2q+4} + \beta^{2q+6} + \dots = \frac{\beta^{2q+4}}{1 - \beta^2},$$

$$p(x, y) = \beta^{2q} + \beta^{2q+2} + \dots = \frac{\beta^{2q}}{1 - \beta^2}.$$

Thus,

$$p(fx, Ty) = \beta^4 p(x, y) < \beta p(x, y) \leq \beta N(x, y) = \psi(N(x, y)).$$

The case $x = \mathbf{0}$, $y \in A$ can be treated similarly, and the case $x = y = \mathbf{0}$ is trivial.

Thus, we have all the conditions of [Theorem 3.9](#), and $\mathbf{0}$ is the unique common fixed point of f and T in X with $p(\mathbf{0}, \mathbf{0}) = 0$.

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