Operators on the vanishing moments subspace and Stieltjes classes for M-indeterminate probability distributions

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Abstract

Purpose – In this work the author gathers several methods and techniques to construct systematically Stieltjes classes for densities defined on R+.

Design/methodology/approach – The author uses complex integration to obtain integrable functions with vanishing moments sequence, and then the author considers some operators defined on the vanishing moments subspace.

Findings – The author gather several methods and techniques to construct systematically Stieltjes classes for densities defined on $\mathbb{R}^+$. The author constructs explicitly Stieltjes classes with center at well-known probability densities. The author gives a lot of examples, including old cases and new ones.

Originality/value – The author computes the Hilbert transform of powers of $|\ln x|$ to construct Stieltjes classes by using a recent result connecting the Krein condition and the Hilbert transform.

Keywords M-indeterminate density, Stieltjes class, Vanishing moments subspace, Hilbert transform

Paper type Research paper

1. Introduction

Consider the subspace $\mathcal{M}$ of all functions $f \in L^1(\mathbb{R}^+)$ with finite moment sequence, i.e.

$$\int_0^\infty x^n |f(x)| dx < \infty \quad \text{for all } n \in \mathbb{N}_0 = \{0, 1, \ldots\}.$$

The vanishing moments subspace $\mathcal{M}_0$ is given as follows

$$\mathcal{M}_0 = \left\{ f \in \mathcal{M} : \int_0^\infty x^n f(x) dx = 0 \quad \text{for all } n \in \mathbb{N}_0 \right\}.$$

We also consider the subspace $\bar{\mathcal{M}}$ of all functions $f \in L^1(\mathbb{R}^+)$ with strong finite moment sequence, i.e.

$$\int_0^\infty x^n |f(x)| dx < \infty \quad \text{for all } n \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\},$$

and the corresponding strong vanishing moments subspace $\bar{\mathcal{M}}_0$.

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First, we introduce a method to get functions in $\mathcal{M}_0$ or $\bar{\mathcal{M}}_0$: assume that g is an analytic function on a region containing the sector

$$S_\alpha = \{z \in \mathbb{C}^* : 0 \le \arg z \le \pi\alpha\}, \quad 0 < \alpha < 1, \ \mathbb{C}^* = \mathbb{C} \setminus \{0\}.$$

We use complex integration to show that

$$\Im\left(g(e^{i\pi\alpha}x^\alpha)\right) \in \mathcal{M}_0 \text{ or } \bar{\mathcal{M}}_0, \tag{1}$$

provided that g satisfies suitable conditions on the boundary ∂S_α of S_α. As usual, $\Re z$, $\Im z$ denote the real and imaginary parts of $z \in \mathbb{C}$.

Then we introduce some operators mapping the subspace $\mathcal{M}_0$ into itself. For instance, we prove that $\mathcal{M}_0$ is invariant under the operator

$$g \mapsto \mu * g,$$

provided that μ is a positive bounded measure on $(\mathbb{R}^+, \mathcal{B}^+)$ with finite moments sequence, see (18). Here $\mu * g$ is the convolution of the measure μ and the function g on $\mathbb{R}^+$ given by

$$\mu * g(x) = \int_0^x g(x-s)d\mu(s), \quad x > 0.$$

Suppose now that f is a probability density function (we use further just density) of a random variable X such that all moments are finite, i.e., $m_k := E[X^k] = \int_0^\infty x^k f(x)dx < \infty$ for all $k \in \mathbb{N}_0$, hence $m_0 = 1$. This means that $f \in \mathcal{M}$. It is well-known that the moment sequence $\{m_k\}_{k=1}^\infty$ either determines X and f uniquely, and we say that X, and also f, is M-determinate, or that f is M-indeterminate. In the latter case there are infinitely many continuous and infinitely many discrete distributions all sharing the same moments as $X \sim f$. This is a fundamental qualitative result, see [1, 2].

In the survey [3] the author revisited recent developments on the checkable moment-(in) determinacy criteria including Cramér's condition, Carleman's condition, Hardy's condition, Krein's condition and the growth rate of moments. In this survey the author analyzes Hamburger and Stieltjes cases.

In this work we only focus in the Stieltjes case, i.e we consider distributions supported on $\mathbb{R}^+$. Recall that in [4] was introduced the concept of Stieltjes class for M-indeterminate absolutely continuous distribution function. Let f be a density in $\mathcal{M}$. Assume that there exists a function $h \in L^\infty(\mathbb{R}^+)$ such that $\|h\|_\infty = 1$, $fh \in \mathcal{M}_0$ and fh is not identically zero. Then the Stieltjes class $S(f, h)$ with center at f and perturbation h is given by

$$S(f,h) = \{f(x)[1 + \varepsilon h(x)] : x \in \mathbb{R}^+, \ \varepsilon \in [-1, 1]\}.$$

Clearly, $S(f, h)$ is a family of densities all having the same moment sequence as f.

If $X \sim f$ is M-determinate, then the perturbation $h = 0$, and the Stieltjes class consists of a single element, the center f.

The main aim of this work is to find perturbations for Stieltjes classes with center at a density $f > 0$. To do this, the basic idea is take a function $g \in \mathcal{M}_0$ such that $h = g/f$ is bounded on $\mathbb{R}^+$, therefore h will be a perturbation (up to scaling by a constant) for a Stieltjes class with center at f. Thus, in this paper all the densities f are M-indeterminate.

When $X \sim f$ with a density f in $\bar{\mathcal{M}}$, we make the obvious changes to define the strong Stieltjes class with center at f.

In [5, Theorem 1.2] the author proved that if f is a density in $\mathcal{M}$ satisfying the Krein condition

$$\int_0^\infty -\frac{\ln f(x^2)}{1+x^2}dx < \infty, \tag{2}$$

then $S(f, \sin(\mathcal{H}_e \ln f))$ is a Stieltjes class, where $\mathcal{H}_e \ln f$ is the Hilbert transform of $u = \ln f$:

$$\mathcal{H}_e u(t) = \frac{2t^{1/2}}{\pi} P \int_0^\infty \frac{u(x^2)}{t - x^2} dx, \quad t > 0. \tag{3}$$

In particular, here we compute $\mathcal{H}_e(|\ln x|^m)$, $m \in \mathbb{N}$, to obtain new Stieltjes classes corresponding to M-indeterminate generalized log-normal random variables.

In order to test our approach we apply the developed methods to the generalized gamma (GG) distribution (see Examples 4, 8, 11 and 12), powers of the generalized inverse gaussian (GIG) distribution (see Examples 3, 14 and 16), powers of the half-logistic distribution (see Example 7) and to the generalized lognormal (GLN) distribution (see Examples 5, 6, 19 and 21).

This work is organized as follows. In Section 2 we give the precise conditions on g to prove (1), and we apply this result to get functions in $\mathcal{M}_0$. In Section 3 we introduce some operators defined on $\mathcal{M}_0$ and we use the functions obtained in Section 2 to get new perturbations in Examples 5, 6, 8, 11, 19 and 21, hence we give new Stieltjes classes. In the last section we compute $\mathcal{H}_e(|\ln x|^m)$, $m \in \mathbb{N} = \{1, 2, \ldots\}$.

2. Functions with vanishing moments

In this section we use complex integration to obtain functions in $\mathcal{M}_0$ or $\bar{\mathcal{M}}_0$. We follow the technique introduced in [6], also in [7], where a similar result appears. In fact, in [6] the author asks the condition $g(x) \geq A \exp(-ax^\alpha)$ for some $A > 0$, $a > 0$ and some $\alpha \in (0, 1/2)$, which is replaced with our integrability conditions (4), (5) and (6) below.

Let $S \subset \mathbb{C}$, then $hol(S)$ denotes the space of analytic functions on a region containing S.

Lemma 1. *Let* $0 < \alpha < 1$, $\gamma \in \mathbb{C}$. *Suppose that* $g \in hol(S_\alpha)$ *satisfies the following conditions*

$$t^{-1+\Re\gamma} |g(t)| \in L^1(\mathbb{R}^+), \tag{4}$$

$$\lim_{A \to \infty} A^{\Re\gamma} \int_0^{\pi\alpha} |g(Ae^{it})| dt = 0, \tag{5}$$

$$\lim_{\varepsilon \to 0^+} \varepsilon^{\Re\gamma} \int_0^{\pi\alpha} |g(\varepsilon e^{it})| dt = 0, \tag{6}$$

Then

$$\int_0^\infty t^{\gamma-1} g(te^{i\pi\alpha}) dt = e^{-i\pi\alpha\gamma} \int_0^\infty t^{\gamma-1} g(t) dt. \tag{7}$$

Proof. We pick $0 < \varepsilon < A < \infty$, Cauchy's theorem implies that

$$\oint_{C_{\varepsilon,A}} z^{\gamma-1} g(z) dz = 0,$$

where the contour $C_{\varepsilon,A}$ consists of the real axis from ε to A, the arc of the circle $z = Ae^{it}$ from $t = 0$ to $t = \pi\alpha$, the straight line from $Ae^{i\pi\alpha}$ to $\varepsilon e^{i\pi\alpha}$ and the arc of the circle $z = \varepsilon e^{i(\pi\alpha - t)}$ from $t = 0$ to $t = \pi\alpha$. Thus,

$$0 = I_1 + I_2 + I_3 + I_4 = \int_{\varepsilon}^{A} t^{\gamma-1} g(t)dt + i\int_{0}^{\pi\alpha} (Ae^{it})^{\gamma} g(Ae^{it})dt$$
$$- e^{i\pi\alpha}\int_{-A}^{-\varepsilon} (-te^{i\pi\alpha})^{\gamma-1} g(-te^{i\pi\alpha})dt - i\int_{0}^{\pi\alpha} (\varepsilon e^{i(\pi\alpha-t)})^{\gamma} g(\varepsilon e^{i(\pi\alpha-t)})dt.$$

Since $|z^{\lambda}| = |z|^{\Re\lambda}\exp(-\arg z\,\Im\lambda)$ for all $\lambda\in\mathbb{C}$, $z\in\mathbb{C}^*$, conditions (5) and (6) imply that $\lim_{A\to\infty} I_2 = \lim_{\varepsilon\to 0^+} I_4 = 0$. Therefore from condition (4) we get

$$\lim_{\varepsilon\to 0^+, A\to\infty} (I_1 + I_3) = 0, \qquad \square$$

and the result follows.

Theorem 2. *Let* $0 < \alpha < 1$. *Suppose that* $g \in hol(S_\alpha)$ *satisfies conditions* (5), (6) *with* $\gamma = (n+1)/\alpha$, $g(x)\in\mathbb{R}$ *for all* $x > 0$ *and*

$$x^n|g(x^\alpha)| \in L^1(\mathbb{R}^+) \quad \text{for all } n\in\mathbb{N}_0 \text{ or } n\in\mathbb{Z}, \tag{8}$$

then relation (1) *holds.*

Proof. By setting $\gamma = (n+1)/\alpha$ for all $n\in\mathbb{N}_0$ or $n\in\mathbb{Z}$, $t = x^\alpha$, $t^{-1}dt = \alpha x^{-1}dx$ in (7) and taking the imaginary part, the result follows. $\square$

We recall an inequality that will be useful to get our estimates: since $e^x \geq x$ for all $x > 0$ we have

$$e^{-x} \leq s^s x^{-s} \quad \text{for all } x, s > 0. \tag{9}$$

Throughout this work the constant K will be a normalizing constant to produce a density function in each case.

Example 3. *For all* $b_1, b_2 > 0$, $0 < c_1, c_2 < 1/2$ *and* $a\in\mathbb{R}$ *we have*

$$x^{a-1}\sin(\pi a + b_2\tan(\pi c_2)x^{-c_2} - b_1\tan(\pi c_1)x^{c_1})\exp(-b_1x^{c_1} - b_2x^{-c_2}) \in \mathcal{M}_0^s. \tag{10}$$

Indeed, we just apply Theorem 2 with $g(z) = z^\beta\exp(-\rho_1 z^\lambda - \rho_2 z^{-1})$ for any $\beta\in\mathbb{R}$, $\lambda, \rho_1, \rho_2 > 0$. Clearly g is an analytic function on $\{z\in\mathbb{C}^* : |\arg z| < \pi\}$.

From (9) we have that g satisfies condition (8) for all $0 < \alpha < 1$, $n\in\mathbb{Z}$. Assume that $0 < \alpha, \alpha\lambda < 1/2$, then the inequalities

$$0 < \cos(\pi\alpha) \leq \cos t \leq 1, \; 0 < \cos(\pi\alpha\lambda) \leq \cos(\lambda t) \leq 1 \quad \text{for all } t\in[0, \pi\alpha], \tag{11}$$

together the inequality in (9) imply that

$$\lim_{A\to\infty} A^{(n+1)/\alpha+\beta}\int_0^{\pi\alpha} \exp\left(-A^\lambda\rho_1\cos(\lambda t) - A^{-1}\rho_2\cos t\right)dt = 0 \quad \text{for all } n\in\mathbb{Z},$$

and, by making $A = 1/\varepsilon$, the last case implies

$$\lim_{\varepsilon\to 0^+} \varepsilon^{(n+1)/\alpha+\beta}\int_0^{\pi\alpha} \exp\left(-\varepsilon^\lambda\rho_1\cos(\lambda t) - \varepsilon^{-1}\rho_2\cos t\right)dt = 0 \quad \text{for all } n\in\mathbb{Z}.$$

From Theorem 2 we have

$$x^{\alpha\beta}\Im\left(\exp(i\pi\alpha\beta - \rho_1 e^{i\pi\alpha\lambda}x^{\alpha\lambda} - \rho_2 e^{-i\pi\alpha}x^{-\alpha})\right) \in \mathcal{M}_0^s,$$

and the result follows by setting $\alpha = c_2, \lambda = c_1/c_2, \beta = (a-1)/c_2, \rho_i = b_i/\cos(\pi c_i), i = 1, 2$. Thus $h(x) = \sin(\pi a + b_2 \tan(\pi c_2)x^{-c_2} - b_1 \tan(\pi c_1)x^{c_1})$ is a perturbation for the strong Stieltjes class with center at $f(x) = Kx^{a-1}\exp(-b_1 x^{c_1} - b_2 x^{-c_2})$, $x > 0$. This Stieltjes class was also founded in [6].

Example 4. *For all* $0 < \alpha < 1/2$, $a, b > 0$ *we have*

$$x^{a-1}\sin(\pi a - b\tan(\pi\alpha)x^{\alpha})\exp(-bx^{\alpha}) \in \mathcal{M}_0. \tag{12}$$

Indeed, consider $g(z) = z^{\beta}\exp(-\rho z)$ for any $\beta > -1/\alpha, \rho > 0$. From (9) we have that g satisfies condition (8) for all $n \in \mathbb{N}_0$. The inequality in (11) implies condition (5) holds for all $\mu = (n+1)/\alpha$, $n \in \mathbb{N}_0$. Since

$$\lim_{\varepsilon\to 0^+} \varepsilon^{(n+1)/\alpha+\beta}\int_0^{\pi\alpha}\exp(-\varepsilon\rho\cos t)dt \leq \pi\alpha \lim_{\varepsilon\to 0^+}\varepsilon^{(n+1)/\alpha+\beta} = 0 \quad \text{for all } n \in \mathbb{N}_0,$$

Theorem 2 implies that

$$x^{\alpha\beta}\Im\left(e^{i\pi\alpha\beta}\exp(-\rho e^{i\pi\alpha}x^{\alpha})\right) \in \mathcal{M}_0,$$

and the result follows by setting $\beta = (a-1)/\alpha$ and $\rho = b/\cos(\pi\alpha)$. Thus $h(x) = \sin(\pi a - b\tan(\pi\alpha)x^{\alpha})$ is a perturbation for the Stieltjes class with center at $f(x) = Kx^{a-1}\exp(-bx^{\alpha})$, $x > 0$. This Stieltjes class was also founded in [7, Example 3.2].

Recall that $\lfloor\cdot\rfloor$ is the floor function and $\lceil\cdot\rceil$ is the ceiling function. For $x, y \in \mathbb{R}$, $m \in \mathbb{N}$, we have

$$(x+iy)^m = \sum_{j=0}^{\lfloor m/2\rfloor}\binom{m}{2j}(-1)^j x^{m-2j}y^{2j} + i\sum_{j=0}^{\lceil m/2\rceil-1}\binom{m}{2j+1}(-1)^j x^{m-2j-1}y^{2j+1}. \tag{13}$$

Example 5. *For all* $0 < \alpha < 1/2$, $b > 0$, $m \in \mathbb{N}$, *we have*

$$e^{-bx^{\alpha}(\theta_m(x)\cos(\pi\alpha)-\psi_m(x)\sin(\pi\alpha))}\sin(bx^{\alpha}(\psi_m(x)\cos(\pi\alpha)+\theta_m(x)\sin(\pi\alpha))) \in \mathcal{M}_0,$$

where

$$\begin{aligned}\theta_m(x) &= \sum_{j=0}^{\lfloor m/2\rfloor}\binom{m}{2j}(-1)^j\pi^{2j}(\ln x)^{m-2j},\\ \psi_m(x) &= \sum_{j=0}^{\lceil m/2\rceil-1}\binom{m}{2j+1}(-1)^j\pi^{2j+1}(\ln x)^{m-2j-1}.\end{aligned} \tag{14}$$

To see this, consider $g(z) = \exp(-\rho z(\operatorname{Log} z)^m)$ for any $\rho > 0$, here $\operatorname{Log} z$ stands for the principal branch of the logarithm function. For $n \in \mathbb{N}_0$ we write

$$\int_0^{\infty} x^n g(x^{\alpha})dx = \left(\int_0^1 + \int_1^e + \int_e^{\infty}\right)x^n g(x^{\alpha})dx = I_1 + I_2 + I_3.$$

Clearly $I_2 < \infty$, and (9) implies that

$$I_3 \leq \int_e^{\infty}\frac{Cx^n}{(\rho\alpha^m x^{\alpha})^s}dx < \infty \quad \text{for } s > 0 \text{ big enough.}$$

Clearly $I_1 < \infty$ when m is even. Assume that m is odd, thus

$$I_1 = \int_1^\infty y^{-n-2}\exp(\rho\alpha^m y^{-\alpha}(\ln y)^m)dy \le C\int_1^\infty \frac{dy}{y^{n+2}} < \infty.$$

Hence g satisfies condition (8) for all $n \in \mathbb{N}_0$, $m \in \mathbb{N}$.

Since the real part is an additive function, we have for $A > \mathrm{e}$ and $t \in [0, \pi\alpha]$ that

$$\begin{aligned}\Re\left(\mathrm{e}^{it}(\ln A + it)^m\right) &= (\ln A)^m\cos t + \sum_{j=0}^{m-1}\binom{m}{j}(\ln A)^j\Re(\mathrm{e}^{it}(it)^{m-j}) \\ &\ge (\ln A)^m\cos(\pi\alpha) - C(\ln A)^{m-1}\end{aligned} \tag{15}$$

for some constant $C > 0$. Therefore,

$$\lim_{A\to\infty} A^{(n+1)/\alpha}\int_0^{\pi\alpha}|g(A\mathrm{e}^{it})|dt \le \pi\alpha\lim_{A\to\infty}A^{(n+1)/\alpha}\exp\left(-\cos(\pi\alpha)\rho A(\ln A)^m + C\rho A(\ln A)^{m-1}\right) = 0$$

for all $n \in \mathbb{N}_0$.

On the other hand, there is $C > 0$ such that $-\Re\left(\mathrm{e}^{it}(\ln\varepsilon + it)^m\right) \le C(|\ln\varepsilon|^m + 1)$ for all $t \in [0, \pi\alpha]$, thus

$$\lim_{\varepsilon\to 0^+}\varepsilon^{(n+1)/\alpha}\int_0^{\pi\alpha}|g(\varepsilon\mathrm{e}^{it})|dt \le \pi\alpha\lim_{\varepsilon\to 0^+}\varepsilon^{(n+1)/\alpha}\exp(C\varepsilon|\ln\varepsilon|^m) = 0 \quad \text{for all } n \in \mathbb{N}_0.$$

Theorem 2 implies that

$$\Im\left(\exp(-\rho\alpha^m\mathrm{e}^{i\pi\alpha}x^\alpha(i\pi + \ln x)^m)\right) \in \mathcal{M}_0,$$

the result follows by setting $\rho = b/\alpha^m$ and using (13). As before, we can consider the corresponding Stieltjes class for the density

$$f(x) = K\mathrm{e}^{-bx^\alpha(\theta_m(x)\cos(\pi\alpha) - \psi_m(x)\sin(\pi\alpha))}, \quad x > 0.$$

Example 6. *For all $a \in \mathbb{R}$, $b > 0$, $m \in \mathbb{N}$, we have*

$$x^a\mathrm{e}^{-b\theta_{2m}(x)}\sin(\pi a - b\psi_{2m}(x)) \in \bar{\mathcal{M}}_0.$$

Consider $g(z) = z^\beta\exp(-\rho(\mathrm{Log}\, z)^{2m})$ for any $\beta \in \mathbb{R}$, $\rho > 0$. We make the change of variable $y = \ln x$ to get

$$\int_0^\infty x^n g(x^\alpha)dx = \int_{-\infty}^\infty \exp\left(-\rho\alpha^{2m}y^{2m} + (n + \alpha\beta + 1)y\right)dy < \infty \quad \text{for all } n \in \mathbb{Z}.$$

As in (15) and using (13) we can see that

$$\lim_{A\to\infty}A^{(n+1)/\alpha}\int_0^{\pi\alpha}|g(A\mathrm{e}^{it})|dt \le \pi\alpha\lim_{A\to\infty}A^{(n+1)/\alpha+\beta}\exp\left(-\rho(\ln A)^{2m} + C\rho(\ln A)^{2m-2}\right) = 0$$

for all $n \in \mathbb{Z}$. The function g also satisfies condition (6) for $\mu = (n+1)/\alpha$, $n \in \mathbb{Z}$, we just set $A = 1/\varepsilon$ and apply the last case.

Theorem 2 implies that

$$x^{\alpha\beta}\Im\left(\mathrm{e}^{i\pi\alpha\beta}\exp\left(-\alpha^{2m}\rho(\ln x+i\pi)^{2m}\right)\right)\in\bar{\mathcal{M}}_0,$$

and the result follows by setting $\rho = b/\alpha^{2m}$, $\beta = a/\alpha$ and using (13). Thus $h(x) = \sin(\pi a - b\psi_{2m}(x))$ is a perturbation for the strong Stieltjes class with center at $f(x) = Kx^a\mathrm{e}^{-b\theta_{2m}(x)}$.

Example 7. *For all $a > 0$, $0 < \alpha < 1/2$ we have*

$$x^{a-1}\mathrm{e}^{-\theta(x)}\frac{\sin(\pi a-\psi(x))+2\mathrm{e}^{-\theta(x)}\sin(\pi a)+\mathrm{e}^{-2\theta(x)}\sin(\pi a+\psi(x))}{\left(1+2\cos(\psi(x))\mathrm{e}^{-\theta(x)}+\mathrm{e}^{-2\theta(x)}\right)^2}\in\mathcal{M}_0,$$

where $\theta(x) = x^\alpha\cos(\pi\alpha)$, $\psi(x) = x^\alpha\sin(\pi\alpha)$.

Consider $g(z) = z^\beta(1+\mathrm{e}^{-z})^{-2}\mathrm{e}^{-z}$ for arbitrary $\beta > -1/\alpha$. For all $n\in\mathbb{N}_0$ we have that

$$\int_0^\infty x^n g(x^\alpha)dx \le \int_0^\infty x^{n+\alpha\beta}\mathrm{e}^{-x^\alpha}dx < \infty.$$

When $\Re z > 0$ we have that $|1+\mathrm{e}^{-z}| \ge 1-\mathrm{e}^{-\Re z}$, therefore

$$\lim_{A\to\infty}A^{(n+1)/\alpha}\int_0^{\pi\alpha}|g(A\mathrm{e}^{it})|dt \le \pi\alpha\lim_{A\to\infty}A^{(n+1)/\alpha+\beta}\frac{\mathrm{e}^{-A\cos(\pi\alpha)}}{\left(1-\mathrm{e}^{-A\cos(\pi\alpha)}\right)^2}=0 \quad \text{for all } n\in\mathbb{N}_0.$$

For $0 < \varepsilon < 1$ we have

$$\int_0^{\pi\alpha}\frac{dt}{|1+\mathrm{e}^{-\varepsilon it}|^2}=\int_0^{\pi\alpha}\frac{dt}{2+2\cos(\varepsilon t)}\le\frac{\pi\alpha}{2},$$

Hence

$$\lim_{\varepsilon\to 0^+}\varepsilon^{(n+1)/\alpha}\int_0^{\pi\alpha}|g(\varepsilon\mathrm{e}^{it})|dt\le\frac{\pi\alpha}{2}\lim_{\varepsilon\to 0^+}\varepsilon^{(n+1)/\alpha+\beta}\mathrm{e}^{-\varepsilon\cos(\pi\alpha)}=0.$$

Theorem 2 implies that

$$x^{\alpha\beta}\Im\left(\mathrm{e}^{i\pi\alpha\beta}(1+\exp(-x^\alpha\mathrm{e}^{i\pi\alpha}))^{-2}\exp(-x^\alpha\mathrm{e}^{i\pi\alpha})\right)\in\mathcal{M}_0,$$

and the result follows by setting $\beta = (a-1)/\alpha$.

Notice that

$$\tilde{h}(x)=\frac{\sin(\pi a-\psi(x))+2\mathrm{e}^{-\theta(x)}\sin(\pi a)+\mathrm{e}^{-2\theta(x)}\sin(\pi a+\psi(x))}{\left(1+\mathrm{e}^{-\theta(x)}\right)^{-2}\left(1+2\cos(\psi(x))\mathrm{e}^{-\theta(x)}+\mathrm{e}^{-2\theta(x)}\right)^2}$$

is a bounded continuous function on $\mathbb{R}^+$, and it can be used to construct a Stieltjes class with center at $\tilde{f}(x) = Kx^{a-1}(1+\mathrm{e}^{-\theta(x)})^{-2}\mathrm{e}^{-\theta(x)}$. Now we set $\delta = (\cos(\pi\alpha))^{-1/\alpha}$, and a change of variable implies that $h(x) = \tilde{h}(\delta x)$ can be used to get a Stieltjes class with center at

$$f(x)=\delta\tilde{f}(\delta x)=K'x^{a-1}\frac{\mathrm{e}^{-x^\alpha}}{(1+\mathrm{e}^{-x^\alpha})^2}.$$

For $0 < \alpha < 1/2$ the last densities are the densities of M-indeterminate powers of random variables following a half-logistic distribution, see [8, Section 6].

3. Operators on the vanishing moment subspace

For $m\in\mathbb{N}$, $s>0$ we introduce the operator $T_{m,s}$ as follows

$$T_{m,s}g(x)=x^{1/m-1}g(x^{1/m}-s)\chi_{(s^m,\infty)}(x).$$

The binomial formula implies that $T_{m,s}\mathcal{M}_0\subset\mathcal{M}_0$:

$$\begin{aligned}\int_{s^m}^{\infty}x^n g(x^{1/m}-s)\frac{dx}{x^{1-1/m}}&=m\int_0^{\infty}(x+s)^{mn}g(x)dx\\&=m\sum_{j=0}^{mn}\binom{mn}{j}s^j\int_0^{\infty}x^{mn-j}g(x)dx=0.\end{aligned}\tag{16}$$

The case $m=1$ was considered in [8, Lemma 1].

For $a,b>0$ and $0<\alpha<1$ we have

$$(a+b)^{\alpha}\le a^{\alpha}+b^{\alpha}.\tag{17}$$

Example 8. *Let $m\in\mathbb{N}, s>0, 0<\alpha<1/2$ fixed. From (12) we have*

$$x^{1/m-1}\sin(\tan(\pi\alpha)((x^{1/m}-s)^{\alpha}))\mathrm{e}^{-(x^{1/m}-s)^{\alpha}}\chi_{(s^m,\infty)}(x)\in\mathcal{M}_0,$$

thus (17) implies that

$$h(x)=\sin(\tan(\pi\alpha)((x^{1/m}-s)^{\alpha}))\mathrm{e}^{(x^{\alpha/m}-(x^{1/m}-s)^{\alpha})}\chi_{(s^m,\infty)}(x)$$

is a bounded continuous function on $\mathbb{R}^+$ that can be used to obtain a Stieltjess *class with center at $f(x)=Kx^{1/m-1}\exp(-x^{\alpha/m})$, $x>0$.*

If $g_1,\dots g_m\in\mathcal{M}_0$ and $a_1,\dots a_m\in\mathbb{R}$ then $\sum_i a_i g_i\in\mathcal{M}_0$, hence the following result is a generalization of the last observation.

Proposition 9. *Let (J,μ) be a measure space. Assume that $\mathcal{G}:\mathbb{R}^+\times J\to\mathbb{R}$ is a measurable function such that $x^n\mathcal{G}(x,\omega)\in L^1(\mathbb{R}^+\times J, dx\otimes d\mu)$ for all $n\in\mathbb{N}_0$ or $n\in\mathbb{Z}$ and*

$$\mathcal{G}(\cdot,\omega)\in\mathcal{M}_0\ \text{or}\ \bar{\mathcal{M}}_0\quad\text{for all}\ \omega\in\Omega,$$

therefore $\int_{\Omega}\mathcal{G}(\cdot,\omega)d\mu(\omega)\in\mathcal{M}_0$ or $\bar{\mathcal{M}}_0$.

Proof. Fubini's theorem implies that

$$\int_0^{\infty}x^n\int_{\Omega}\mathcal{G}(x,\omega)d\mu(\omega)dx=\int_{\Omega}\int_0^{\infty}x^n\mathcal{G}(x,\omega)dxd\mu(\omega)=0\quad\text{for all}\ n\in\mathbb{N}_0\ \text{or}\ n\in\mathbb{Z}.$$

□

Corollary 10. *Let μ be a positive bounded measure on $(\mathbb{R}^+,\mathcal{B}^+)$ such that*

$$\int_0^{\infty}x^n d\mu<\infty\quad\text{for all}\ n\in\mathbb{N}_0.\tag{18}$$

If $g\in\mathcal{M}_0$, then $\mu*g\in\mathcal{M}_0$.

Proof. We consider the function $\mathcal{G}:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ given by

$$\mathcal{G}(x,s)=g(x-s)\chi_{(s,\infty)}(x).$$

Clearly $\mathcal{G}$ is a measurable function and satisfies

$$\int_0^\infty \int_0^\infty x^n |\mathcal{G}(x,s)| dx \otimes d\mu(s) = \int_0^\infty \int_0^\infty (x+s)^n |g(x)| dx d\mu(s)$$
$$= \sum_{j=0}^{n} \binom{n}{j} \int_0^\infty x^j |g(x)| dx \int_0^\infty s^{n-j} d\mu(s) < \infty$$

for all $n \in \mathbb{N}_0$. Since $g \in \mathcal{M}_0$ we have that $\mathcal{G}(\cdot, s) \in \mathcal{M}_0$ for all $s > 0$, and the last result implies that $\int_0^\infty \mathcal{G}(\cdot, s) ds \in \mathcal{M}_0$. On the other hand, we have

$$\int_0^\infty \mathcal{G}(x,s) ds = \int_0^\infty g(x-s) \chi_{(0,x)}(s) d\mu(s) = \mu * g(x), \quad x > 0.$$

We apply the last result to obtain new Stieltjes classes with center at $f(x) = K \exp(-x^\alpha)$, $x > 0$, as follows.

Example 11. *By (12) we have* $g(x) = \sin(\tan(\pi\alpha)x^\alpha)\exp(-x^\alpha) \in \mathcal{M}_0$, *with* $0 < \alpha < 1/2$ *fixed. Consider the measures* $d\mu_1(s) = \chi_{(0,1)} ds$ *and* $d\mu_2(s) = \mathrm{e}^{-s} ds$ *on* $\mathbb{R}^+$. *Thus,*

$$\mu_1 * g(x) = \int_0^{x \wedge 1} \sin(\tan(\pi\alpha)(x-s)^\alpha) \mathrm{e}^{-(x-s)^\alpha} ds \in \mathcal{M}_0,$$

where $x \wedge 1 = \min\{x, 1\}$ *and*

$$\mu_2 * g(x) = \int_0^{x} \sin(\tan(\pi\alpha)(x-s)^\alpha) \mathrm{e}^{-(x-s)^\alpha - s} ds \in \mathcal{M}_0.$$

From (17) we get

$$|\mu_1 * g(x)| \leq \int_0^1 \mathrm{e}^{s^\alpha} ds \ \mathrm{e}^{-x^\alpha} \quad \text{for all } x > 0,$$

hence the bounded function $h(x) = \mu_1 * g(x) \exp(x^\alpha)$ can be used to construct a Stieltjes class with center at $f(x) = K \exp(-x^\alpha)$, $x > 0$.

Since $(x/\mathrm{e})^\alpha \leq x/\mathrm{e}$ for all $x \geq \mathrm{e}$, there exist a constant $0 < C < 1$ such that $x^\alpha - x \leq -Cx$ for all $x \geq \mathrm{e}$, thus

$$|\mu_2 * g(x)| \leq \int_0^x \mathrm{e}^{s^\alpha - s} ds \ \mathrm{e}^{-x^\alpha} \leq \left(\widetilde{C} + \int_0^\infty \mathrm{e}^{-Cs} ds \right) \mathrm{e}^{-x^\alpha} \quad \text{for all } x > 0,$$

and we proceed as before to construct the corresponding Stieltjes class.

Now, let p be a polynomial with real coefficients, with $p(0) = 0$ and $p' > 0$ on $\mathbb{R}^+$. We introduce the operator R_p as follows

$$R_p g(x) = \frac{g(p^{-1}(x))}{p'(p^{-1}(x))}, \quad x > 0,$$

where p^{-1} is the inverse function of p on $\mathbb{R}^+$. As in (16), a change of variable and the binomial formula implies that $R_p \mathcal{M}_0 \subset \mathcal{M}_0$.

Example 12. *Let* $0 < \alpha < 1/2$, $a, b > 0$ *be fixed and* $1 \le n < (2\alpha)^{-1}$, $n \in \mathbb{N}$. *From* (12) *we have that* $g(x) = x^{na-1}\sin(\pi na - b\tan(n\pi\alpha)x^{n\alpha})\exp(-bx^{n\alpha}) \in \mathcal{M}_0$. *We set* $p_n(x) = x^n$, $x \ge 0$, *to get that*

$$R_{p_n}g(x) = n^{-1}x^{a-1}\sin(\pi na - b\tan(\pi n\alpha)x^{\alpha})\exp(-bx^{\alpha}) \in \mathcal{M}_0,$$

therefore $h_n(x) = \sin(\pi na - b\tan(n\pi\alpha)x^{\alpha})$, $x > 0$, *is a perturbation for a Stieltjes class with center at* $f(x) = Kx^{a-1}\exp(-bx^{\alpha})$, $x > 0$. *As far as we know, these are new Stieltjes classes when* $2 \le n < (2\alpha)^{-1}$, $n \in \mathbb{N}$.

Remark 13. *Let* $\Lambda \ne \varnothing$. *Assume that* $\{f_\lambda\}_{\lambda\in\Lambda} \subset \mathcal{M}_0$. *If* $x^{-1}f_{\lambda_0}(x^{-1}) \in \{f_\lambda\}_{\lambda\in\Lambda}$, *then* $f_{\lambda_0} \in \bar{\mathcal{M}}_0$.

Example 14. *Let* $0 < c_1, c_2 < 1/2$, $b_1, b_2 > 0$, $a \in \mathbb{R}$ *be fixed and* $1 \le n < 2^{-1}(c_1^{-1} \wedge c_2^{-1})$, $n \in \mathbb{N}$. *From* (10) *we have that*

$$x^{na-1}\sin(\pi na + b_2\tan(\pi nc_2)x^{-nc_2} - b_1\tan(\pi nc_1)x^{nc_1})\exp(-b_1x^{nc_1} - b_2x^{-nc_2}) \in \bar{\mathcal{M}}_0,$$

we proceed as in Example 12 and use Remark 13 to get that

$$x^{a-1}\sin(\pi na + b_2\tan(\pi nc_2)x^{-c_2} - b_1\tan(\pi nc_1)x^{c_1})\exp(-b_1x^{c_1} - b_2x^{-c_2}) \in \bar{\mathcal{M}}_0.$$

Once again, we obtain new perturbations for strong Stieltjes classes with center at $f(x) = Kx^{a-1}\exp(-b_1x^{c_1} - b_2x^{-c_2})$, $x > 0$.

4. Krein criterion and the Hilbert transform

In this section we use a different technique to construct Stieltjes classes. This method involves the computation of the Hilbert transform of $\ln f$, where f is a density $f \in \mathcal{M}$ satisfying the Krein criterion (2). In [5, Theorem 1.2] was proved that $S(f, \sin(\mathcal{H}_e \ln f))$ is a Stieltjes class with center at f, where the Hilbert transform $\mathcal{H}_e$ is defined in (3). The following result can be found in [5, Remark 2.1, Lemmas 2.2 and 2.3] and provides the computation of the Hilbert transform of power functions, constant functions and the logarithm function.

Proposition 15. *a) For any constant* $c \in \mathbb{R}$ *we have* $\mathcal{H}_e(c) \equiv 0$. *b) Let* $0 < |\gamma| < 1$. *Then*

$$\mathcal{H}_e(x^\gamma)(t) = -\tan(\gamma\pi)t^\gamma, \quad t > 0.$$

c) $\mathcal{H}_e(\ln x) \equiv -\pi$.

As a consequence we obtain a Stieltjes class with center at a generalized inverse Gaussian density.

Example 16. *Let* $a \in \mathbb{R}$, $b_1, b_2 > 0$, $0 < c_1, c_2 < 1/2$. *Consider the density* $f(x) = Kx^{a-1}\exp(-b_1x^{c_1} - b_2x^{-c_2})$, $x > 0$. *Proposition 15 implies that*

$$\mathcal{H}_e(\ln f)(t) = -\pi(a-1) - b_2\tan(\pi c_2)t^{-c_2} + b_1\tan(\pi c_1)t^{c_1}, \quad t > 0.$$

Thus $h(x) = \sin(\pi a + b_2\tan(\pi c_2)x^{-c_2} - b_1\tan(\pi c_1)x^{c_1})$, $x > 0$, *is a perturbation for the Stieltjes class with center at* f. *This is the case* $n = 1$ *in Example 14.*

Remark 17. *As before, we can see that* $h(x) = \sin(\pi a - b\tan(\pi\alpha)x^{\alpha})$, $x > 0$, *is a perturbation for the Stieltjes class with center at the density* $f(x) = Kx^{a-1}\exp(-bx^{\alpha})$, $x > 0$, *provided that* $0 < \alpha < 1/2$, $a, b > 0$. *This is the case* $n = 1$ *in Example 12.*

Finally, in the last examples we get two Stieltjes classes that we could not obtain by the method of complex integration given in Section 2. The densities involved are special cases of generalized log-normal densities, see [9]. In order to construct these examples we need to find out the Hilbert transform of $|\ln x|^n, x>0, n\in\mathbb{N}$. Thus, we need to compute the principal value of the singular integral in (3) with $u = |\ln x|^n, n\in\mathbb{N}$.

For all $k, n\in\mathbb{N}_0$ we have the identity, see [10, p. 69, eq. 4.1.51],

$$\int x^n(\ln x)^k dx = x^{n+1}\sum_{j=0}^{k}\frac{(-1)^j k!}{(n+1)^{j+1}(k-j)!}(\ln x)^{k-j}. \tag{19}$$

We introduce the following constants

$$\gamma_j := \sum_{n=0}^{\infty}\frac{1}{(2n+1)^j} = (1-2^{-j})\zeta(j), \quad j>1, j\in\mathbb{N},$$

where $\zeta(z)$ is the zeta function.

First we compute the Hilbert transform of even powers of $|\ln x|$.

Lemma 18. *For $m\in\mathbb{N}$ we have*

$$\mathcal{H}_e(|\ln x|^{2m})(t) = -\frac{2(2m)!}{\pi}\sum_{\ell=1}^{m}\frac{2^{2\ell}\gamma_{2\ell}}{(2m-2\ell+1)!}(\ln t)^{2m-2\ell+1}, \quad t>0. \tag{20}$$

Proof. By (3) it follows that

$$\mathcal{H}_e(|\ln x|^{2m})(t^2) = \frac{2^{2m+1}t}{\pi}P\int_0^\infty \frac{|\ln x|^{2m}}{t^2-x^2}dx, \quad t>0.$$

Let $t>0$ fixed and $\varepsilon>0$ small enough. Since the geometric series with ratio $r = x^2/t^2$ converges uniformly for $x\in[0, t-\varepsilon]$, and by using (19), we get that

$$\begin{aligned}\frac{1}{t^2}\int_0^{t-\varepsilon}\frac{|\ln x|^{2m}}{1-(x/t)^2}dx &= \sum_{n=0}^{\infty}\frac{1}{t^{2n+2}}\int_0^{t-\varepsilon}x^{2n}|\ln x|^{2m}dx\\ &= \sum_{j=0}^{2m}\frac{(-1)^j(2m)!}{(2m-j)!}\sum_{n=0}^{\infty}\frac{1}{t^{2n+2}}\left[\frac{x^{2n+1}(\ln x)^{2m-j}}{(2n+1)^{j+1}}\Bigg|_{x=0}^{x=t-\varepsilon}\right]\\ &= \sum_{j=0}^{2m}\frac{(-1)^j(2m)!}{(2m-j)!}(\ln(t-\varepsilon))^{2m-j}\sum_{n=0}^{\infty}\frac{(t-\varepsilon)^{2n+1}}{(2n+1)^{j+1}t^{2n+2}}.\end{aligned}$$

Multiplying the last equality by t and using that $\operatorname{arctanh}(x) = \sum_{n=0}^{\infty}x^{2n+1}/(2n+1)$ for $|x|<1$, we have

$$\begin{aligned}t\int_0^{t-\varepsilon}\frac{|\ln x|^{2m}}{t^2-x^2}dx &= \sum_{n=0}^{\infty}\frac{(2m)!(t-\varepsilon)^{2n+1}}{(2n+1)^{2m+1}t^{2n+1}} + (\ln(t-\varepsilon))^{2m}\operatorname{arctanh}\left(\frac{t-\varepsilon}{t}\right)\\ &\quad + \sum_{j=1}^{2m-1}\frac{(-1)^j(2m)!}{(2m-j)!}(\ln(t-\varepsilon))^{2m-j}\sum_{n=0}^{\infty}\frac{(t-\varepsilon)^{2n+1}}{(2n+1)^{j+1}t^{2n+1}} \qquad (21)\\ &= I_1(\varepsilon)+I_2(\varepsilon)+I_3(\varepsilon).\end{aligned}$$

Similarly, we can obtain that

$$-\int_{t+\varepsilon}^{\infty}\frac{1}{x^2}\frac{|\ln x|^{2m}}{1-(t/x)^2}dx=-\sum_{j=0}^{2m}\frac{(2m)!}{(2m-j)!}(\ln(t+\varepsilon))^{2m-j}\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n+1)^{j+1}(t+\varepsilon)^{2n+1}}.$$

As before, we multiply the last equality by t to get

$$\begin{aligned} t\int_{t+\varepsilon}^{\infty}\frac{|\ln x|^{2m}}{t^2-x^2}dx &= -\sum_{n=0}^{\infty}\frac{(2m)!t^{2n+1}}{(2n+1)^{2m+1}(t+\varepsilon)^{2n+1}}-(\ln(t+\varepsilon))^{2m}\text{arctanh}\left(\frac{t}{t+\varepsilon}\right)\\ &\quad -\sum_{j=1}^{2m-1}\frac{(2m)!}{(2m-j)!}(\ln(t+\varepsilon))^{2m-j}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)^{j+1}(t+\varepsilon)^{2n+1}}\\ &= J_1(\varepsilon)+J_2(\varepsilon)+J_3(\varepsilon). \end{aligned} \tag{22}$$

By the other hand,

$$\lim_{\varepsilon\to 0^+}I_1(\varepsilon)+J_1(\varepsilon)=0,$$

and we apply the Weierstrass M-test to the third terms in (21) and (22), considering $\varepsilon\in[0,\varepsilon_0)$ with ε_0 small enough, to obtain

$$\lim_{\varepsilon\to 0^+}I_3(\varepsilon)+J_3(\varepsilon)=-\sum_{\ell=1}^{m}\frac{2(2m)!\gamma_{2\ell}}{(2m-2\ell+1)!}(\ln t)^{2m-2\ell+1}.$$

Finally, we use that $arctanh(x)=2^{-1}\ln\frac{1+x}{1-x}$, $|x|<1$, to obtain

$$\begin{aligned} \mathcal{H}_e(|\ln x|^{2m})(t^2) &= -\frac{2^{2m+2}}{\pi}\sum_{\ell=1}^{m}\frac{(2m)!\gamma_{2\ell}}{(2m-2\ell+1)!}(\ln t)^{2m-2\ell+1}\\ &\quad +\frac{2^m}{\pi}\lim_{\varepsilon\to 0^+}\left[(\ln(t-\varepsilon))^{2m}\ln\frac{2t-\varepsilon}{\varepsilon}-(\ln(t+\varepsilon))^{2m}\ln\frac{2t+\varepsilon}{\varepsilon}\right]. \end{aligned}$$

L'Hôpital's rule implies that the last limit is equal to zero, and the result follows. □

Similar to (9), now we give a basic estimate for the logarithm function: since $x^s\le\exp(x^s)$ for all $x, s>0$, we have

$$\ln x\le\frac{x^s}{s}\quad \text{for all } x,s>0. \tag{23}$$

Example 19. *For $m\in\mathbb{N}$ consider the density $f(x)=K\exp(-|\ln x|^{2m})$, $x>0$. Clearly*

$$\int_1^{\infty}x^n f(x)dx=K\int_0^{\infty}e^{-x^{2m}+(n+1)x}dx<\infty,$$

for all $n\in\mathbb{Z}$, then $f\in\bar{\mathcal{M}}$. From (23) we get

$$\int_0^{\infty}\frac{|\ln x|^{2m}}{1+x^2}dx=2\int_1^{\infty}\frac{|\ln x|^{2m}}{1+x^2}dx<\infty,$$

thus f satisfies (2), *therefore* $\sin(\mathcal{H}_e(|\ln x|^{2m})(t))$ *is a perturbation for the* Stieltjes *class with center at f, where* $\mathcal{H}_e(|\ln x|^{2m})$ *is given in* (20).

Finally, we compute the Hilbert transform of odd powers of $|\ln x|$. The computations are very similar to those in the proof of Lemma 18.

Lemma 20. *For* $m \in \mathbb{N}$ *we have*

$$\mathcal{H}_e(|\ln x|^{2m-1})(t) = \frac{2(2m-1)!}{\pi}\left[\sum_{\ell=1}^{m}\frac{2^{2\ell}\gamma_{2\ell}}{(2m-2\ell)!}(\ln t)^{2m-2\ell} - \sum_{n=0}^{\infty}\frac{2^{2m}t^{n+\frac{1}{2}}}{(2n+1)^{2m}}\right],\quad 0<t<1. \tag{24}$$

For $t>1$ *we have* $\mathcal{H}_e(|\ln x|^{2m-1})(t) = \mathcal{H}_e(|\ln x|^{2m-1})(t^{-1})$.

Proof. We just make a sketch of the proof. Let $t \in (0, 1)$ fixed. We have the following equalities

$$-\int_0^{t-\varepsilon}\frac{(\ln x)^{2m-1}}{t^2-x^2}dx = -\sum_{j=0}^{2m-1}\frac{(-1)^j(2m-1)!}{(2m-1-j)!}(\ln(t-\varepsilon))^{2m-1-j}\sum_{n=0}^{\infty}\frac{(t-\varepsilon)^{2n+1}}{(2n+1)^{j+1}t^{n+2}},$$

$$-\int_{t+\varepsilon}^{1}\frac{(\ln x)^{2m-1}}{t^2-x^2}dx = -\sum_{n=0}^{\infty}\frac{(2m-1)!t^{2n}}{(2n+1)^{2m}} + \sum_{j=0}^{2m-1}\frac{(2m-1)!}{(2m-1-j)!}\sum_{n=0}^{\infty}\frac{(\ln(t+\varepsilon))^{2m-1-j}t^{2n}}{(2n+1)^{j+1}(t+\varepsilon)^{2n+1}},$$

$$\text{and}\quad \int_1^{\infty}\frac{(\ln x)^{2m-1}}{t^2-x^2}dx = -(2m-1)!\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n+1)^{2m}}.$$

Therefore we obtain

$$\begin{aligned}\frac{\pi}{2^{2m+1}}\mathcal{H}_e(|\ln x|^{2m-1})(t^2) &= -\sum_{n=0}^{\infty}\frac{(2m-1)!t^{2n+1}}{(2n+1)^{2m}} + \sum_{\ell=1}^{m}\frac{(2m-1)!\gamma_{2\ell}}{(2m-2\ell)!}(\ln t)^{2m-2\ell}\\ &\quad +\frac{1}{4}\lim_{\varepsilon\to 0^+}(\ln(t+\varepsilon))^{2m-1}\ln\frac{2t+\varepsilon}{\varepsilon} - (\ln(t-\varepsilon))^{2m-1}\ln\frac{2t-\varepsilon}{\varepsilon}.\end{aligned}$$

The last limit is equal to zero and the result follows. When $t>1$ a change of variables shows that $\mathcal{H}_e(|\ln x|^{2m-1})(t^2) = \mathcal{H}_e(|\ln x|^{2m-1})(t^{-2})$. □

Example 21. *For* $m \in \mathbb{N}$ *consider the density* $f(x) = K\exp(-|\ln x|^{2m-1})$, $x>0$. *Then* $\sin(\mathcal{H}_e(|\ln x|^{2m-1})(t))$ *is a perturbation for the* Stieltjes *class with center at f, where* $\mathcal{H}_e(|\ln x|^{2m-1})$ *is given in* (24).

In this setting, we also can use the functions in $\mathcal{M}_0$ obtained in Examples 5 and 6 to construct perturbations for Stieltjes classes with center at generalized log-normal densities.

5. Conclusion

We gather several methods and techniques to construct systematically Stieltjes classes for M-indeterminate probability densities defined on $\mathbb{R}^+$. We construct explicitly Stieltjes classes with centers at densities of M-indeterminate powers of generalized log-normal random variables.

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