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OF ROAD PRICING
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MATHEMATICAL AND ECONOMIC THEORY OF ROAD PRICING

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United Kingdom – North America – Japan
India – Malaysia – China
## Acronyms

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<thead>
<tr>
<th>Acronym</th>
<th>Abbreviation</th>
<th>Description</th>
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<tbody>
<tr>
<td>AC</td>
<td>Average Cost</td>
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<tr>
<td>BLPP</td>
<td>Bi-Level Programming Problem</td>
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<tr>
<td>BOT</td>
<td>Build-Operate-Transfer</td>
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<td>BPR</td>
<td>Bureau of Public Roads</td>
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<tr>
<td>BLVI</td>
<td>Bi-Level Variational Inequality</td>
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<tr>
<td>CN</td>
<td>Cournot-Nash</td>
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<tr>
<td>DUE</td>
<td>Deterministic User Equilibrium</td>
<td></td>
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<tr>
<td>EB</td>
<td>Economic Benefit</td>
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<tr>
<td>EPEC</td>
<td>Equilibrium Problem with Equilibrium Constraints</td>
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<tr>
<td>KKT</td>
<td>Karush-Kuhn-Tucker</td>
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<tr>
<td>LP</td>
<td>Linear Program</td>
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<tr>
<td>MC</td>
<td>Marginal Cost</td>
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<tr>
<td>MPEC</td>
<td>Mathematical Program with Equilibrium Constraints</td>
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<tr>
<td>OD</td>
<td>Origin-Destination</td>
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<tr>
<td>SC</td>
<td>Social Cost</td>
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<tr>
<td>SO</td>
<td>System Optimum</td>
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<tr>
<td>SW</td>
<td>Social Welfare</td>
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<tr>
<td>STEN</td>
<td>Space-Time Expanded Network</td>
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<tr>
<td>SUE</td>
<td>Stochastic User Equilibrium</td>
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<tr>
<td>UB</td>
<td>User Benefit</td>
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</tr>
<tr>
<td>UE</td>
<td>User Equilibrium</td>
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</tr>
<tr>
<td>VI</td>
<td>Variational Inequality</td>
<td></td>
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<tr>
<td>VOT</td>
<td>Value Of Time</td>
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</tbody>
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Glossary of Notation

\(a\)  a link \(a \in A\)
\(A\)  the set of all links
\(\overline{A}\)  the subset of links subject to a toll charge
\(\tilde{A}\)  the subset of links with observed flows
\(B_w(d_w)\)  the benefit function between OD pair \(w \in W\)
\(B^m_w(d^m_w)\)  the benefit function of user class \(m \in M\) between OD pair \(w \in W\)
\(c_{rw}\)  the travel cost on path \(r\) connecting OD pair \(w \in W\)
\(c\)  the vector of path travel costs
\(c^m_{rw}\)  the travel cost on path \(r\) between OD pair \(w\) by users of class \(m \in M\)
\(C_a\)  the capacity of link \(a \in A\)
\(d_w\)  the travel demand between OD pair \(w \in W\)
\(d\)  the vector of OD demands
\(\hat{D}_w\)  the potential demand or the upper-bound of demand between OD pair \(w \in W\)
\(d^m_w\)  the travel demand of user class \(m\) between OD pair \(w \in W\)
\(\hat{D}^m_w\)  the potential demand of user class \(m\) between OD pair \(w \in W\)
\(D_w(\mu_w)\)  the demand function between OD pair \(w \in W\)
\(D^-_{w}(d^-_w)\)  the inversed demand function between OD pair \(w \in W\)
\(E^v_w\)  Elasticity of \(v\) to \(u\), \(E^v_w = (u/v)dv(u)/du\)
\(f_{rw}\)  the flow on path \(r \in R\)
\(f\)  the vector of path flows
\(f^m_{rw}\)  the flow of user class \(m \in M\) on path \(r \in R_w\) between OD pair \(w \in W\)
\(G\)  \((N, A)\) be a network with node set \(N\) and link set \(A\)
\(K\)  the set of Cournot-Nash (CN) players or the set of vehicle types
\(I_a(y_a)\)  the construction cost of link \(a\) as a function of link capacity \(y_a\)
\(m\)  a user class \(m \in M\)
\(M\)  the set of all user classes with different values of time
\(N\)  the set of all nodes
\(r\)  a path or a route \(r \in R\)
\(R_w\)  the set of all paths connecting OD pair \(w \in W\)
the free-flow travel time (moving time on link \( a \in A \))

the travel time on link \( a \in A \) as a function of link flow \( v_a \)

the marginal social travel time on link \( a \in A \), including the congestion externality, \( \tilde{t}_a(v_a) = t_a(v_a) + v_a \frac{dt_a(v_a)}{dv_a} \)

the link travel time as a function of both flow and capacity variables

the vector of link travel times

toll charge on link \( a \in A \)

the vector of link tolls

the flow on link \( a \in A \)

the vector of link flows

the flow of user class \( m \) on link \( a \in A \)

the link flow of vehicle type \( k \in K \) or CN player \( k \in K \)

an OD pair \( w \in W \)

the set of all OD pairs

the set of OD pairs of which users are controlled by CN player \( k \in K \)

= \( \bigcup_{k \in K} W^k \)

the set of OD pairs of which users are controlled by UE class \( m \in M \)

= \( \bigcup_{m \in M} W^m \)

the capacity of a new link \( a \) or capacity increase of an existing link \( a \)

the vector of link capacity variables

the users’ value of time

the average value of time of user class \( m \in M \)

1, if link \( a \in A \) is on path \( r \in R \) and 0, otherwise

the link/path incidence matrix, \( \Delta = [\delta_{ar}] \)

a predetermined tolerance for stopping iterative process

1 if path \( r \in R_{w} \) and 0, otherwise

the OD/path incidence matrix, \( \Lambda = [\Lambda_{rw}] \)

the Lagrange multiplier associated with the demand conservation constraint of OD pair \( w \in W \), or the generalized travel cost between OD pair \( w \in W \) at equilibrium

the generalized travel cost of user class \( m \in M \) between OD pair \( w \in W \)

the vector of generalized OD travel costs
Glossary of Notation

Ω the feasible region of link flows and OD flows

Ωf the feasible region of path flows

Ωr the feasible region of link flows

Ωk the feasible region of link flows of CN player \( k \in K \)

Ωm the feasible region of link flows of a UE class \( m \in M \)

ΩU the feasible region of link flows of UE player

\( \nabla, t(v) \) the Jacobian matrix of link travel time function \( t \) with respect to \( v \)

\( \in \) element membership

\( \subset \) proper set inclusion

\( \cup \) union of sets

\( \infty \) infinity

\( | \cdot | \) the cardinality of a finite set

\( \| \| \) the Euclidean norm

\( \text{arg min}_x F(x) \) the set of \( x \) attaining the minimum of the function \( F(x) \)
PREFACE

Road pricing as an effective means of both managing road traffic demand and raising additional revenue for road construction has been studied extensively by both transportation researchers and economists. Practical implementation has been progressing rapidly and electronic road pricing schemes have been proposed and tested worldwide. It is likely that over the next few years, with increasing public acceptability, there will be greater use of road pricing. The incontestable fact is that there is a great need for the development of efficient road-use pricing models.

Despite a few monographs and journal special issues in the literature that have been devoted to the topic in recent years, most studies have focused on empirical studies, policy experience, environmental issue of road congestion and road pricing. There is still scope for methodological development of advanced road pricing systems, such as dynamic pricing, integrated road and transit pricing, as well as practical toll charging schemes in general road networks. This book is intended to deal with a number of timely topics including: fundamentals of user-equilibrium problems; principle of marginal-cost pricing applied to road pricing; existence of optimal link tolls for system optimum under multi-class, multi-criteria, multiple equilibrium behaviors; social and spatial equity issues in road pricing; optimal design of private toll roads; simultaneous determination of optimal toll levels and locations; trial-and-error implementation of marginal-cost pricing on networks with unknown demand functions; dynamic road pricing.

This book would appear to be the first book devoted exclusively to the mathematical and economic investigation of road-use pricing in general congested networks, which aims at alleviating traffic congestion, improving transport conditions and enhancing social welfare. It constitutes an update of the state of the art of the latest research, mainly by the authors and their colleagues. The book is targeted at students, professionals and scientists who are studying and working in relevant transportation fields.

We are most thankful to a number of individuals for their help during our work in this field. First and foremost we would like to acknowledge the contribution and continuing encouragement of Prof. M.G.H. Bell of Imperial College London (ICL). In fact, the manuscript was finalized during the first author’s sabbatical leave at ICL in early 2005, and he is particularly grateful to Prof. Bell for his hospitality during a very fruitful and enjoyable stay in ICL. Thanks are also due to Prof. R. Lindsey of the University of Alberta, Prof. E.
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Hai-Jun Huang

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