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On the primeness of near-rings

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Abstract
In this paper, we study the different kinds of the primeness on the class of near-rings and we give new characterizations for them. For that purpose, we introduce new concepts called set-divisors, ideal-divisors, etc. and we give equivalent statements for 3-primeness which make 3-primeness looks like the forms of the other kinds of primeness. Also, we introduce a new different kind of primeness in near-rings called K-primeness which lies between 3-primeness and e-primeness. After that, we study different kinds of prime ideals in near-rings and find a connection between them and new concepts called set-attractors, ideal-attractors, etc. to make new characterizations for them. Also, we introduce a new different kind of prime ideals in near-rings called K-prime ideals.

Keywords Near-rings, Rings, Primeness, Prime ideals
Paper type Original Article

1. Introduction

We say that R is a right (left) near-ring if $(R,+)$ is a group, $(R,\cdot)$ is a semigroup and R satisfies the right (left) distributive law. Throughout this paper, R will be a left near-ring. We say that R is an abelian near-ring if $x+y=y+x$ for all $x,y\in R$ and we say that R is a commutative near-ring if $xy=yx$ for all $x,y\in R$. A zero-symmetric element is an element $x\in R$ satisfying $0x=0$. A near-ring R is called a zero-symmetric near-ring, if $0x=0$ for all $x\in R$. A constant element is an element $y\in R$ satisfying $zy=y$ for all $z\in R$. An element $x\in R$ is called a right (left) zero divisor in R if there exists a non-zero element $y\in R$ such that $yx=0$ $(xy=0)$. A zero divisor is either a right or a left zero divisor. By a near-ring without zero divisors, we mean a near-ring without non-zero divisors of zero. If A and B are two non-empty subsets of R, then the product AB means the set $\{ab|a\in A, b\in B\}$. We say that U is a right (left) R-subgroup of R, if U is a subgroup of $(R,+)$ satisfies $UR\subseteq U$ $(RU\subseteq U)$. We say that U is a two-sided R-subgroup of R, if U is both a right and a left R-subgroup of R. We say that I is a right (left) ideal of R, if I is a normal subgroup of $(R,+)$ satisfies $(r+i)s-rs\in I$ for all $i\in I, r,s\in R$ $(RI\subseteq I)$. We say that I is an ideal of R if it is both a right and a left ideal of R. We say that U is a semigroup right (left) ideal of R, if U is a non-empty subset of R satisfies $UR\subseteq U$ $(RU\subseteq U)$. We say that U is a semigroup ideal of R if it is both a semigroup right and

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left ideal of R (some authors call U a right (left, two-sided) R-subset of R [8]). For any group $(G,+)$, $M(G)$ denotes the near-ring of all maps from G to G with the two operations of addition and composition of maps. $M_o(G)$ is the zero-symmetric subnear-ring of $M(G)$ consisting of all zero preserving maps from G to itself (and to make them left near-rings we should write $f(g)$ by gf, where $f \in M(G)$ or $M_o(G)$ and $g \in G$). A trivial zero-symmetric near-ring R is a zero-symmetric near-ring such that the multiplication on the group $(R,+)$ is defined by $xy = y$ and $0y = 0$ for all $x \in R - \{0\}, y \in R$. A near-field N is a near-ring in which $(N - \{0\}, \cdot)$ is a group. For further information about near-rings, see [8] and [9].

In near-rings, there are five well-known kinds of primeness. We say that: R is 0-prime (the usual primeness) if, for every two ideals I and J of R, $IJ = \{0\}$ implies $I = \{0\}$ or $J = \{0\}$, R is 1-prime if, for every two right ideals K and L of R, $KL = \{0\}$ implies $K = \{0\}$ or $L = \{0\}$. R is 2-prime if, for every two right R-subgroups A and B of R, $AB = \{0\}$ implies $A = \{0\}$ or $B = \{0\}$. R is 3-prime if, for all $x, y \in R$, $xRy = \{0\}$ implies $x = 0$ or $y = 0$ and R is equiprime (e-prime) if, for any $0 \neq a, x, y \in R$, $xca = yca$ for all $c \in R$ implies $x = y$. These five kinds of primeness are **equivalent** in the class of **rings**. But in the class of **near-rings**, we have: (1) R is equiprime implies that R is zero-symmetric 3-prime, (2) R is 3-prime implies that R is 2-prime, (3) R is zero-symmetric 2-prime implies that R is 1-prime and (4) R is 1-prime implies that R is 0-prime. For details about these kinds and their examples and relationships see [1–3,5–7] and [10]. A near-ring (a ring) R is called 3-semiprime (semiprime) if, for all $x \in R$, $xRx = \{0\}$ implies $x = 0$. An ideal P of R is: (i) a 0-prime ideal of R if for every two ideals A and B of R, $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$, (ii) a 1-prime ideal of R if for every two right ideals A and B of R, $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$, (iii) a 2-prime ideal of R if for every two right R-subgroups A and B of R, $AB \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$, (iv) a 3-prime ideal of R if for $a, b \in R$, $aRb \subseteq P$ implies that $a \in P$ or $b \in P$, (v) an e-prime (equiprime) ideal of R if for every $a \in R - P$ and $x, y \in R$, $xca - yca \in P$ for all $c \in R$ implies that $x - y \in P$. Clearly that any near-ring is a υ-prime ideal of itself, where $\upsilon \in \{0, 1, 2, 3, e\}$. It is well-known that (ii) implies (i) and (iv) implies (iii). Also, for zero-symmetric near-rings we have (iii) implies (ii). An ideal I of R is called completely prime if, for $a, b \in R$, $ab \in I$ implies that $a \in I$ or $b \in I$. If the zero ideal is completely prime, then we say that R is completely prime. Then R is completely prime if and only if R is without zero divisors. For more details about prime ideals, see [2,4,5] and [10].

In [1], the authors gave us a short historical view about the primeness of near-rings. We will use it and add some information to it.

Several different generalizations of primeness for rings have been introduced for near-rings. In [6], Holcombe studied three different concepts of primeness, which he called 0-prime, 1-prime and 2-prime. In [5], Groenewald obtained further results for these and introduced further notion which he called 3-primeness. In [2], Booth, Groenewald and Veldsman gave another definition, called equiprimeness, or e-primeness. In [10], Veldsman made more studies on equiprime near-rings. In [1], Booth and Groenewald gave an element-wise characterization of the radical associated with ν-primeness for $\nu = 1, 2, 3, e$.

In this paper we extend the idea of primeness that they did and give some new results for the primeness of near-rings. Firstly, we introduce new concepts called set-divisors, ideal divisors, etc. These concepts are generalizations of the concept of zero divisors and give another characterization of different kinds of the primeness in near-rings and hence in rings. Also, we study the 3-primeness and give new characterizations of 3-prime (3-semiprime) near-rings and hence for prime (semiprime) rings. These characterizations make 3-primeness looks like the forms of the other kinds of primeness. In fact, we show that a near-ring (a ring) is 3-prime (prime) if and only if $UV = \{0\}$ implies $U = \{0\}$ or $V = \{0\}$, where U and V are semigroup left ideals of R. Hence, a ring is prime if and only if it is without zero-semigroup right (left) ideal divisors. A similar result is made for 3-semiprime near-rings (semiprime rings) and we conclude that: for a near-ring R, if $r^2 \neq 0$ for all $r \in R - \{0\}$, then R is 3-semiprime. We show that some kinds of near-rings are 3-prime if and only if they are

2-prime. Also, we introduce a new kind of primeness in near-rings (the sixth one) called K-primeness and we show that it is totally different from the other kinds of primeness and it lies between 3-primeness and e-primeness. Depending on that, we give two chains of primeness in the class of zero-symmetric near-rings for comparison. In the last part of the paper, we study different kinds of prime ideals. We introduce a new kind of prime ideals called K-prime ideals and we show that they are different from the other kinds of prime ideals. they lie between 3-prime ideal and e-prime ideals. Also, we give a new characterization of 3-prime ideals and show that P is a 3-prime ideal of R if and only if $UV \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$, where U and V are semigroup left ideals of R. We introduce new concepts called set-attractors, ideal-attractors, etc. which are generalizations of the new concepts above (set-divisors, etc.). We make a connection between these concepts and different kinds of prime ideals in near-rings to give a new characterization of these prime ideals. Finally, we use these concepts to show that: P is a completely prime ideal of R if and only if R is without external P set-attractors.

2. On prime near-rings

Let R be a near-ring. It is clear that R is without zero divisors if and only if $AB = \{0\}$ implies $A = \{0\}$ or $B = \{0\}$, where A and B are non-empty subsets of R. This observation gives us a hint of a new definition.

Definition 2.1. Let R be a non-zero near-ring.

(1) Let A be a non-empty subset of R. We say that A is a left zero-set divisor (a right zero-set divisor) of R if there exists a non-empty non-zero subset B of R such that $AB = \{0\}$ ($BA = \{0\}$). We say that A is a zero-set divisor of R if A is a left or a right zero-set divisor of R.

(2) Let A be an ideal of R. We say that A is a left zero-ideal divisor (a right zero-ideal divisor) of R if there exists a non-zero ideal B of R such that $AB = \{0\}$ ($BA = \{0\}$). We say that A is a zero-ideal divisor of R if A is a left or a right zero-ideal divisor of R.

We can do same definitions if A is a left (right) ideal, a left (right) R -subgroup, a two-sided R-subgroup, a semigroup left (right) ideal or a semigroup ideal.

Definition 2.1 generalizes the concept of zero divisors in rings and near-rings. So, we have the following remark.

Remark 2.1. From Definition 2.1, we can rewrite the definitions of different kinds of the primeness as follows:

Let R be a near-ring. Then

(1) R is completely prime if and only if R is without zero divisors if and only if R is without zero-set divisors.

(2) R is 0-prime if and only if R is without zero-ideal divisors.

(3) R is 1-prime if and only if R is without zero-right ideal divisors.

(4) R is 2-prime if and only if R is without zero-right R-subgroup divisors.

Remark 2.1 enhances a question: Can we get a definition of 3-primeness like that mentioned in Remark 2.1? The following result answers this question.

Theorem 2.1. *Let R be a near-ring. Then the following statements are equivalent:*

(i) R is 3-prime.

(ii) $aU = \{0\}$ *implies* $a = 0$ *or* $U = \{0\}$, *where* $a \in R$ *and U is a semigroup left ideal of R.*

(iii) $AU = \{0\}$ *implies* $A = \{0\}$ *or* $U = \{0\}$*, where A is a non-empty subset of R and U is a semigroup left ideal of R.*

(iv) $UV = \{0\}$ *implies* $U = \{0\}$ *or* $V = \{0\}$*, where U and V are semigroup left ideals of R.*

Proof. (i) implies (ii), (ii) implies (iii) and (iii) implies (iv) are clear.

To prove that (iv) implies (i), we will use the contradiction. For that purpose, suppose R is not 3-prime. So there exist non-zero elements $x, y \in R$ such that $xRy = \{0\}$. Thus, $RxRy = \{0\}$. But Rx and Ry are semigroup left ideals of R, so $Rx = \{0\}$ or $Ry = \{0\}$ by (iv). Hence, $R\{0, x\} = \{0\}$ or $R\{0, y\} = \{0\}$ and either $\{0, x\}$ or $\{0, y\}$ is a semigroup left ideal of R. But R is also a semigroup left ideal of R. Thus, $\{0, x\} = 0$, $\{0, y\} = \{0\}$ or $R = \{0\}$ by (iv), a contradiction with that x, y, R are all non-zero. So R is 3-prime and (iv) implies (i). ■

For zero-symmetric near-rings, we have the following extra result.

Theorem 2.2. *Let R be a zero-symmetric near-ring. Then the following statements are equivalent:*

(i) R is 3-prime.

(ii) $Ua = \{0\}$ implies $a = 0$ or $U = \{0\}$, where $a \in R$ and U is a semigroup right ideal of R.

(iii) $UA = \{0\}$ implies $U = \{0\}$ or, $A = \{0\}$ where U is a semigroup right ideal of R and A is a non-empty subset of R.

(iv) $UV = \{0\}$ implies $U = \{0\}$ or $V = \{0\}$, where U and V are semigroup right ideals of R.

(v) $UV = \{0\}$ implies $U = \{0\}$ or $V = \{0\}$, where U is a semigroup right ideal of R and V is a semigroup left ideal of R.

Now, we can add (5) to Remark 2.1:

(5) R is 3-prime if and only if $UV = \{0\}$ implies $U = \{0\}$ or $V = \{0\}$, where U and V are semigroup left ideals of R if and only if R is without zero-semigroup left ideal divisors.

Since any ring is a zero-symmetric near-ring, we have the following result:

Corollary 2.3. *A ring is prime if and only if it is without zero-semigroup right (left) ideal divisors.*

Using the same idea, the following result gives us a result for 3-semiprime zero-symmetric near-rings.

Theorem 2.4. *Let R be a zero-symmetric near-ring. Then the following statements are equivalent:*

(i) R is 3-semiprime.

(ii) $aU = \{0\}$ *implies* $a = 0$*, where* $a \in U$ *and U is a semigroup left ideal of R.*

(iii) $Ua = \{0\}$ *implies* $a = 0$*, where* $a \in U$ *and U is a semigroup right ideal of R.*

(iv) $U^2 = \{0\}$ *implies* $U = \{0\}$*, where U is a semigroup left ideal of R.*

(v) $U^2 = \{0\}$ *implies* $U = \{0\}$*, where U is a semigroup right ideal of R.*

Proof. (i) implies (ii). Suppose (i) holds. Let U be a semigroup left ideal of R such that $aU = \{0\}$, where $a \in U$. Then for all $v \in U$, we have $aRv = \{0\}$. Thus, $aRa = \{0\}$ and $a = 0$ by (i).

(i) implies (iii) can be proved by the same way.

(ii) implies (iv) and (iii) implies (v) are clear.

(iv) implies (v). Suppose that (iv) holds and $U^2 = \{0\}$, where U is a semigroup right ideal of R. So $uRu = \{0\}$ for all $u \in U$ and hence $RuRu = \{0\}$. But Ru is a semigroup left ideal of R. So $Ru = \{0\}$ for all $u \in U$ by (iv). So $\{0, u\}$ is a semigroup left ideal of R and $\{0, u\}\{0, u\} = \{0\}$ for all $u \in U$. So $u = 0$ by (iv) and hence $U = \{0\}$.

(v) implies (i). Suppose that (v) holds and that $xRx = \{0\}$ for some $x \in R$. Thus, $xRxR = \{0\}$. But xR is a semigroup right ideal of R, so $xR = \{0\}$ by (v). Hence, $\{0, x\}\{0, x\} = \{0\}$. But $\{0, x\}$ is a semigroup right ideal of R. Thus, $\{0, x\} = \{0\}$ by (v) and hence $x = 0$. So R is 3-semiprime and (v) implies (i). ■

Corollary 2.5. *A ring R is semiprime if and only if* $U^2 = \{0\}$ *implies* $U = \{0\}$, *where U is a semigroup right (left) ideal of R.*

But in the general case of 3-semiprime near-rings, we have only the following result.

Theorem 2.6. *Let R be a near-ring. Then the following statements are equivalent:*

(i) R is 3-semiprime.

(ii) $aU = \{0\}$ *implies* $a = 0$, *where* $a \in U$ *and U is a semigroup left ideal of R.*

(iii) $U^2 = \{0\}$ *implies* $U = \{0\}$, *where U is a semigroup left ideal of R.*

Unfortunately, we cannot remove the word "zero-symmetric" in Theorems 2.2 and 2.4. The following example is the near-ring in [9, Appendix, E, 22] and it shows that the condition "zero-symmetric" in Theorems 2.2 and 2.4 is not redundant.

Example 1. Let $(R, +)$ be the Klein's four group $\{0, a, b, c\}$. Then it is an abelian group such that $x + x = 0$ for all $x \in R$ and $x + y = z$ for all different non-zero elements $x, y, z \in R$. Define the multiplication on R as follows:

$\cdot$	0	a	b	c
0	0	a	0	a
a	0	a	0	a
b	0	a	0	a
c	0	a	b	c

Clearly R is an abelian non-zero-symmetric near-ring. The only semigroup right ideals of R are R, $\{0, a\}$ and $\{0, a, b\}$. So R satisfies the conditions "$UV = \{0\}$ implies $U = \{0\}$ or $V = \{0\}$, where U and V are semigroup right ideals of R" and "$U^2 = \{0\}$ implies $U = \{0\}$, where U is a semigroup right ideal of R". But R is not 3-semiprime as $bRb = \{0\}$. From Theorem 2.6, we can deduce that there is a non-zero semigroup left ideal V of R such that $V^2 = \{0\}$ and $vV = \{0\}$, where $v \in V - \{0\}$. It is easy to find out that $V = \{0, b\}$ and $v = b$.

From the above example, observe that

$$\{0, a, b\}b = \{0, a, b\}\{b\} = \{0\}.$$

So, we cannot use this example for (ii) or (iii) in Theorem 2.2 and for (iii) in Theorem 2.4. In fact, removing "zero-symmetric" from those parts is an open problem.

Corollary 2.7. *Let R be a near-ring. If* $r^2 \neq 0$ *for all* $r \in R - \{0\}$, *then R is 3-semiprime.*

Proof. Suppose there exists a non-zero semigroup left ideal U of R such that $aU = \{0\}$, where $a \in U$. That means $a^2 = 0$. By hypothesis, $a = 0$ and hence R is 3-semiprime. ■

Example 2. Let $R = \mathbb{Z}_6$. Then R is semiprime since $r^2 \neq 0$ for all $r \in R - \{0\}$.

Example 3. Let $R = \{0, 2, 4, 6, 8, 10, 12\}$ the subring of $\mathbb{Z}_{14}$. Then R is semiprime since $r^2 \neq 0$ for all $r \in R - \{0\}$.

The converse of Corollary 2.7 is not true as the following example shows.

Example 4. Let $R = M_2(\mathbb{Z}_2)$. Then R is a prime ring and hence semiprime, but

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For commutative near-rings, we have the converse and we get the following result.

Corollary 2.8. *Let R be a commutative near-ring. Then $r^2 \neq 0$ for all $r \in R - \{0\}$ if and only if R is 3-semiprime.*

We conclude this section by the following results about the relation between 2-primeness and 3-primeness. The fact that R is 3-prime implies R is 2-prime is well-known. The following results have the converse.

Theorem 2.9. *Let R be a zero-symmetric near-ring such that $2R = \{0\}$. Then R is 3-prime if and only if R is 2-prime.*

Proof. Suppose that $xRy = \{0\}$. Thus, $xRyR = \{0\}$. But xR and yR are right R-subgroups of R. So $xR = \{0\}$ or $yR = \{0\}$ as R is 2-prime. Hence, $\{0, x\}R = \{0\}$ or $\{0, y\}R = \{0\}$ and then either $\{0, x\}$ or $\{0, y\}$ is a right R-subgroup of R. But R is also a right R-subgroup of R. Thus, $\{0, x\} = 0$, $\{0, y\} = \{0\}$ or $R = \{0\}$. Hence, $x = 0$ or $y = 0$ and R is 3-prime. ■

Theorem 2.10 *Any distributive near-ring R is 3-prime if and only if it is 2-prime.*

Proof. Suppose that R is 2-prime and $xRy = \{0\}$ for some $x, y \in R$. So $xRyR = \{0\}$ and hence $xR = \{0\}$ or $yR = \{0\}$. So $AR = \{0\}$ or $BR = \{0\}$, where $A = \{nx | n \in \mathbb{Z}\}$ and $B = \{ny | n \in \mathbb{Z}\}$. So A and B are right R-subgroups of R and hence $A = \{0\}$ or $B = \{0\}$. Therefore, $x = 0$ or $y = 0$ and R is 3-prime. ■

3. K-prime near-rings

In this section, we will introduce a new kind of primeness of near-rings called K-primeness. Firstly, we will begin with the following result.

Theorem 3.1. *Let R be a* ***ring****. Then the following statements are equivalent:*

(i) R is prime.

(ii) for any $0 \neq a, x, y \in R$, $xsa = yra$ for all $s, r \in R - \{0\}$ implies $x = y$.

Proof. A ring R is prime if and only if it is equiprime, so we will use the definition of equiprimeness, i.e. for any $0 \neq a, x, y \in R$, $xca = yca$ for all $c \in R$ implies $x = y$.

(i) implies (ii) is clear.

(ii) implies (i). Suppose (ii) holds. If for all $c \in R$, $xca = yca$ for $0 \neq a, x, y \in R$, then $(x - y)ca = 0 = 0ra$ for all $c, r \in R$. So $x = y$ by (ii). ■

Part (ii) enhances the following definition for near-rings.

Definition 3.1. Let R be a near-ring. We say that R is K-prime if, for any $0 \neq a, x, y \in R$, $xsa = yra$ for all $s, r \in R - \{0\}$ implies $x = y$.

As we mentioned before for rings, a ring is prime if and only if it is equiprime. So we have the following result.

Corollary 3.2. *A ring R is prime if and only if it is K-prime.*

The following result shows that every K-prime near-ring is zero-symmetric 3-prime.

Theorem 3.3. *Let R be a K-prime near-ring. Then R is zero-symmetric 3-prime.*

Proof. Firstly, we will show that R is zero-symmetric. If R is not zero-symmetric, then it has at least one non-zero constant element c (see [8, Theorem 1.15). For different elements x, y of R, we have that $xsc = yrc = c$ for all $s, r \in R - \{0\}$, a contradiction with the hypothesis. So R is zero-symmetric. Now, suppose $xRy = \{0\}$ for some $x, y \in R$. So $xcy = 0$ for all $c \in R$. If $y \neq 0$, then $xcy = 0ry$ for all $c, r \in R$. So $x = 0$ from the hypothesis and hence R is 3-prime. ■

In the case of near-rings, we have only that e-primeness implies K-primeness as shown in the proof of Theorem 3.1 (since an e-prime near-ring is zero-symmetric [10]). But the converse is not true as we will show in the next example. We will use the near-ring mentioned in [9, Appendix, F, 7] in the next example.

Example 5. Let $(R, +)$ be the cyclic group $\mathbb{Z}_5$ and define the multiplication on R as follows:

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	4	3	2	1
3	0	1	2	3	4
4	0	4	3	2	1

So R is an abelian near-ring which is not a ring (as $(1+1)2 = 3 \neq 4 = 2+2 = (1)2 + (1)2$). Clearly that R is without zero divisors. Hence, R is 3-prime. R is not equiprime. Indeed, $1c1 = 3c1 = c1$ for all $c \in R$. But if $0 \neq a, x, y \in R$ such that $xsa = yra$ for all $s, r \in R - \{0\}$, then $x = y$. Clearly that is true if x or y is equal to zero, since R is without zero divisors. That is the only possible case. In fact, if $xsa = yra$ for all $s, r \in R - \{0\}$ and x, y, a are all non-zero, then from the table we can choose $s_o, r_o \in R - \{0\}$ to satisfy that $xs_o = 1$ and $yr_o = 2$. Hence, $a = 2a$ which implies that $a = 0$ (from the table), a contradiction with $0 \neq a$. Therefore, K-primeness does not imply e-primeness.

Also, we can find zero-symmetric 3-prime near-rings which are not K-prime, as the following example shows.

Example 6. Let R be a trivial zero-symmetric near-ring of order greater than 2. Clearly R is 3-prime. Taking two non-zero elements x and y such that $x \neq y$, we have $xsx = yrx = x$ for all $s, r \in R - \{0\}$. So R is not K-prime.

Theorem 3.1, Theorem 3.3 and the examples after them show that K-primeness is a new kind of primeness.

Observe that K-primeness lies between 3-primeness and e-primeness (equiprimeness). So we have the following chain of primeness in the class of zero-symmetric near-rings:

The class of e-prime near-rings
$\subseteq$ The class of K-prime near-rings
$\subseteq$ The class of 3-prime near-rings
$\subseteq$ The class of 2-prime near-rings
$\subseteq$ The class of 1-prime near-rings
$\subseteq$ The class of 0-prime near-rings

Remark 3.1. Observe that:

(i) It is well-known that $M_o(G)$ is e-prime (see [10]) and hence K-prime. Observe that it has zero divisors.

(ii) Since $M(G)$ is not zero-symmetric, so it is not K-prime (and hence not e-prime), but it has zero divisors.

(iii) Let N be any near-field. Then N is e-prime and hence K-prime. Indeed, for any $0 \neq a, x, y \in R$ such that $xca = yca$ for all $c \in R$, we have that $x = y$ by choosing $c = a^{-1}$. Observe that N is without zero divisors.

(iv) Example 6 shows a 3-prime near-ring without zero divisors which is not K-prime (and hence not e-prime).

From the above parts in Remark 3.1, there is no relation between e-primeness (K-primeness) and the existence of zero divisors in near-rings. So, we have another chain of the primeness in the class of zero-symmetric near-rings:

$$\begin{aligned} &\text{The class of completely prime near-rings} \\ \subseteq\ &\text{The class of 3-prime near-rings} \\ \subseteq\ &\text{The class of 2-prime near-rings} \\ \subseteq\ &\text{The class of 1-prime near-rings} \\ \subseteq\ &\text{The class of 0-prime near-rings} \end{aligned}$$

4. On prime ideals

The next definition introduces K-prime ideals.

Definition 4.1. Let R be a near-ring and P an ideal of R. Then P is a K-prime ideal of R if for every $a \in R - P$ and $x, y \in R$, $xra - ysa \in P$ for all $r, s \in R - P$ implies $x - y \in P$.

Clearly R is K-prime if and only if $\{0\}$ is a K-prime ideal of R.

The relationship between K-prime ideals and other kinds of prime ideals is stated in the following result.

Theorem 4.1. *Let R be a near-ring with an ideal P.*

(i) If P is a K-prime ideal of R, then P is a 3-prime ideal of R.

(ii) If P is an e-prime ideal of R, then P is a K-prime ideal of R.

Proof. (i) Firstly, we will show that P contains all the constant elements of R. Let c be a constant element in R. If $c \in R - P$, then

$$xrc - ysc = c - c = 0 \in P$$

for all $x, y \in R$ and $r, s \in R - P$. So $x - y \in P$ and hence $x - 0 = x \in P$ for all $x \in R$. Thus, $P = R$, a contradiction with $c \notin P$. So $c \in P$.

Now, suppose $aRb \subseteq P$ for some $a, b \in R$ and $b \notin P$. From above, any element $s \in R - P$ is a zero-symmetric element. So $0sb = 0 \in P$ for all $s \in R - P$. So $arb - 0sb \in P$ for all $r, s \in R - P$. Thus, $a \in P$ by the hypothesis and P is 3-prime.

(ii) Firstly, observe that if $r \in P$ and $s \in R$ is a zero-symmetric element, then

$$rs = (r + 0)s - 0s \in P.$$

Suppose $xra - ysa \in P$ for all $r, s \in R - P$, where $a \in R - P$ and $x, y \in R$. So $xca - yca \in P$ for all $c \in R - P$. Now, suppose $c \in P$. As $a \notin P$, we have that a is a zero-symmetric element (see [10]). So $ca \in P$ and hence $xca - yca \in P$. But P is e-prime. So $x - y \in P$ and P is a K-prime ideal of R. ■

The next result generalizes Theorem 2.1 for 3-prime ideals.

Theorem 4.2. *Let R be a near-ring and P an ideal of R. Then the following statements are equivalent:*

(i) P is a 3-prime ideal of R.

(ii) $BU \subseteq P$ implies $B \subseteq P$ or $U \subseteq P$, where B is a non-empty subset of R and U is a semigroup left ideal of R.

(iii) $UV \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$, where U and V are semigroup left ideals of R.

Proof. (i) implies (ii). Suppose (i) holds. Let U be a semigroup left ideal of R and B be a non-empty subset of R such that $BU \subseteq P$. If $B \nsubseteq P$, then there exists $b \in B - P$ such that $bRu \subseteq P$ for all $u \in U$. Thus, $U \subseteq P$ by (i).

(ii) implies (iii) is clear.

(iii) implies (i). To prove it, we will use the contradiction. Suppose that (iii) holds and P is not a 3-prime ideal. So there exist $x, y\, R - P$ such that $xRy \subseteq P$. Thus, $RxRy \subseteq P$. So $Rx \subseteq P$ or $Ry \subseteq P$ by (iii). Hence, $R(P \cup \{x\}) \subseteq P$ or $R(P \cup \{y\}) \subseteq P$ and then $P \cup \{x\}$ or $P \cup \{y\}$ is a semigroup left ideal of R. But R itself is also a semigroup left ideal of R. Thus, $P \cup \{x\} \subseteq P$, $P \cup \{y\} \subseteq P$ or $R \subseteq P$ by (iii), a contradiction with that $x, y \in R - P$. So P is 3-prime and (iii) implies (i). ■

Remark 4.1. From Theorem 4.2, a new characterization of 3-prime ideals can be written as follows:

(*) P is a 3-prime ideal of R if for every two semigroup left ideals A and B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

Using Theorem 4.2 and its proof, we can prove the following result which generalizes Theorem 2.2 for 3-prime ideals.

Theorem 4.3. *Let R be a zero-symmetric near-ring and P an ideal of R. Then the following statements are equivalent:*

(i) P is a 3-prime ideal of R.

(ii) $UB \subseteq P$ implies $U \subseteq P$ or $B \subseteq P$, where U is a semigroup right ideal of R and B is a non-empty subset of R.

(iii) $UV \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$, where U and V are semigroup right ideals of R.

We cannot eliminate the condition "zero-symmetric" in Theorem 4.3 as the following example shows:

Example 7. Observe that $\{0\}$ is not a 3-prime ideal in Example 1 although it satisfies the condition "If $UV \subseteq \{0\}$, then $U \subseteq \{0\}$ or $V \subseteq \{0\}$, where U and V are semigroup right ideals of R". This shows that "zero-symmetric" in Theorem 4.3 is not redundant.

Now, we would like to generalize Definition 2.1.

Definition 4.2. Let R be a near-ring with an ideal I.

(i) Let A be a non-empty subset of R. We say that A is a left I set-attractor (a right I set-attractor) of R if there exists a non-empty subset B of R and $B \nsubseteq I$ such that $AB \subseteq I$ ($BA \subseteq I$). We say that A is an I set-attractor of R if A is a left or a right I set-attractor of R.

(ii) Let A be an ideal of R. We say that A is a left I ideal-attractor (a right I ideal-attractor) of R if there exists an ideal B of R and $B \nsubseteq I$ such that $AB \subseteq I$ ($BA \subseteq I$). We say that A is an I ideal-attractor of R if A is a left or a right I ideal-attractor of R.

We can do the same definitions if A is a left (right) ideal of R, a left (right, two-sided) R-subgroup of R, a semigroup ideal of R or a semigroup left (right) ideal of R.

Example 8. Let R be a near-ring with an ideal $I \neq R$. Any non-empty subset of I is a right I set-attractor of R and hence an I set-attractor of R. In particular, I is an I set-attractor of R. Also, if there exist an ideal (a left (right) ideal, a left R-subgroup, a semigroup left ideal) B of R such that $B \nsubseteq I$, then I is an I ideal-attractor (I left (right) ideal-attractor, I left R-subgroup-attractor, I semigroup left ideal-attractor) of R.

Definition 4.3. Let R be a near-ring with an ideal P. If A is a P set-attractor (P ideal-attractor, etc.) of R, then we say that A is an internal P set-attractor (P ideal-attractor, etc.) of R

if $A \subseteq P$. If $A \nsubseteq P$, then we say that A is an external P set-attractor (P ideal-attractor, etc.) of R. If R does not have any external P set-attractors (P ideal-attractors, etc.), then we say that R is without external P set-attractors (P ideal-attractors, etc.), i.e. for a P set-attractor (P ideal-attractor, etc.) A of R, we have that $A \subseteq P$

Example 9. (i) Any near-ring R is without external (or internal) R-set attractors.

(ii) Any near-ring without zero divisors is without external $\{0\}$-set attractors.

(iii) Let R be the ring $\mathbb{Z}_4$. Take P to be the ideal $\{0, 2\}$. Then R is without external P set-attractors.

(iv) Let R be the ring $\mathbb{Z}_6$. Take P to be the ideal $\{0\}$. Then $\{2\}$, $\{3\}$ and $\{4\}$ are external P set-attractors and $\{0\}$ is an internal P set-attractor.

Theorem 4.4. Let R be a near-ring with an ideal P. Then the following statements are equivalent:

(i) R is without external P set-attractors.

(ii) P is a completely prime ideal of R.

Proof. (i) implies (ii), Suppose (i) holds and $ab \in P$ for some $a, b \in R$. So $\{a\}\{b\} \subseteq P$. If $a \notin P$, then $b \in P$ by (i) and P is completely prime.

(ii) implies (i). Suppose (ii) holds and A is a P set-attractor of R. So there exists a non-empty subset B of R and $B \nsubseteq P$ such that $AB \subseteq P$ or $BA \subseteq P$. Suppose the case is $AB \subseteq P$. Take $y \in B - P$. So $xy \in P$ for all $x \in A$ and then $A \subseteq P$ by (ii). By the same way we can do for the other case. So R is without external P set-attractors. ■

Remark 4.2. (i) If $I = \{0\}$ in Definition 4.2, then we have Definition 2.1.

(ii) From the above two definitions, Theorem 4.2 and 4.4, we can rewrite the statements of different kinds of prime ideals as follows:

Let R be a near-ring with an ideal P. Then

(1) P is completely prime if and only if R is without external P set-attractors if and only if for every two non-empty subsets A and B of R, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$.

(2) P is 0-prime if and only if R is without external P ideal-attractors.

(3) R is 1-prime if and only if R is without external P right ideal-attractors.

(4) R is 2-prime if and only if R is without external P right R-subgroup-attractors.

(5) R is 3-prime if and only if R is without external P semigroup left ideal-attractors.

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