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# Multivariate Hardy and Littlewood inequalities on time scales



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## Abstract
In the paper we extend some Hardy and Littlewood type inequalities on time scales for the function of $n$ variables. Special cases of obtained results include generalized Wirtinger, Hardy and Littlewood type inequalities.

**Keywords** Hardy and Littlewood inequalities, Wirtinger type inequality, Time scales calculus
**Paper type** Orginal Article

## 1. Introduction
The discrete Hardy inequality [8] was proved and published by Hardy himself. It states that if $(c_n)$ is a sequence of non-negative real numbers which are not identically zero, then for every real number $p > 1$, one has that

$$\sum_{k=1}^{\infty}\left(\frac{c_1+c_2+c_3+\cdots+c_k}{k}\right)^p < \left(\frac{p}{p-1}\right)^p \sum_{k=1}^{\infty} c_k^p.$$

The classical Hardy inequality [9] states that if $f \geq 0$ and integrable over any finite interval $(0, r)$ and $f^d$ is integrable and convergent over $(0, \infty)$ then for $d > 1$,

$$\int_0^{\infty}\left(\frac{1}{r}\int_0^r f(\tau)d\tau\right)^d dr \leq \left(\frac{d}{d-1}\right)^d \int_0^{\infty} f^d(r)dr, \tag{1}$$

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equality holds if and only if $f(r)=0$ almost everywhere. Hardy inequality (1) has been generalized by Hardy himself in [11], where he exposed that, for any integrable function $f(y)>0$ on $(0,\infty)$ and $d>1$, the following hold

$$\int_0^\infty \frac{1}{y^n}\left(\int_y^\infty f(h)dh\right)^d dy \le \left(\frac{d}{1-n}\right)^d \int_0^\infty \frac{1}{y^{n-d}} f^d(y)dy, \quad n<1, \tag{2}$$

$$\int_0^\infty \frac{1}{y^n}\left(\int_0^y f(h)dh\right)^d dy \le \left(\frac{d}{n-1}\right)^d \int_0^\infty \frac{1}{y^{n-d}} f^d(y)dy, \quad n>1. \tag{3}$$

Hardy and Littlewood [10] demonstrate the discrete versions of (2) and (3). In particular they proved that if $d>1$ and $(p_m)$ is a sequence of non-negative terms then

$$\sum_{m=1}^\infty \frac{1}{m^j}\left(\sum_{i=m}^\infty p_i\right)^d \le N\sum_{m=1}^\infty \frac{1}{m^{j-d}}\, p_m^d, \quad j<1,$$

$$\sum_{m=1}^\infty \frac{1}{m^j}\left(\sum_{i=1}^m p_i\right)^d \le N\sum_{m=1}^\infty \frac{1}{m^{j-d}}\, p_m^d, \quad j>1,$$

where $N$ is a non-negative constant. Time scales calculus [12] was introduced in 1988 by the German mathematician Stefan Hilger, which unifies sums and integrals. Some extension of Hardy type inequalities on time scales can be found in [2–4].

S. H. Saker et al. [13] proved some Hardy and Littlewood type inequalities on time scales in the following form:

**Theorem 1.1.** *Let $\mathbb{T}$ be a time scale with $a\in(0,\infty)_{\mathbb{T}}$ and $p,q>0$ such that $p/q\ge 2$ and $\gamma>1$. Furthermore assume that $g$ is a nonnegative and the delta integral $\int_a^\infty t^{\frac{p}{q}-\gamma}g^{p/q}(t)\Delta t$ exists. Let*

$$\Lambda(t)=\int_a^t g(s)\Delta s, \quad \text{for any } t\in[a,\infty)_{\mathbb{T}}. \tag{4}$$

*Then one gets*

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \le \frac{2^{\frac{p}{q}-2}pk^\gamma}{q(\gamma-1)}\left\{\int_a^\infty \frac{1}{t^{\gamma-\frac{p}{q}}}g^{p/q}(t)\Delta t\right\}^{\frac{p}{q}}$$
$$\times\left\{\int_a^\infty \frac{\Lambda^\sigma(t)\Lambda^{p/q}}{t^\gamma}\Delta t\right\}^{\frac{p-q}{p}}$$
$$+\frac{2^{\frac{p}{q}-2}pk^\gamma}{q(\gamma-1)}\int_a^\infty \frac{\mu^{\frac{p}{q}}-1}{t^{\gamma-1}}g^{p/q}(t)\Delta t.$$

**Theorem 1.2.** *Let $\mathbb{T}$ be a time scale with $a\in(0,\infty)_{\mathbb{T}}$ and $p,q>0$ such that $p/q\ge 2$ and $\gamma>1$. Furthermore assume that $g$ is a nonnegative function and the delta integral $\int_a^\infty t^{\frac{p}{q}-\gamma}g^{p/q}(t)\Delta t$ exist. Let $\Lambda(t)$ be as defined in (4). Then*

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \le \left(\frac{2^{\frac{p}{q}-1}pk^\gamma}{q(\gamma-1)}\right)^{p/q}\int_a^\infty \frac{1}{t^{\gamma-\frac{p}{q}}}g^{p/q}(t)\Delta t.$$

**Theorem 1.3.** *Let $\mathbb{T}$ be a time scale with $a \in (0,\infty)_\mathbb{T}$ and $p,q>0$ such that $p/q>1$ and $\gamma>1$. Furthermore assume that $g$ is a nonnegative function and the delta integral $\int_a^\infty t^{\frac{p}{q}-\gamma}g^{p/q}(t)\Delta t$ exists. Let $\Lambda(t)$ be as defined in (4). Then*

$$\int_a^\infty \frac{1}{t^\gamma}(\Lambda^\sigma(t))^{p/q}\Delta t \le \left(\frac{pk^\gamma}{q(\gamma-1)}\right)^{p/q}\int_a^\infty \frac{1}{t^{\gamma-\frac{p}{q}}}g^{p/q}(t)\Delta t.$$

**Theorem 1.4.** *Let $\mathbb{T}$ be a time scale with $a \in (0,\infty)_\mathbb{T}$ and $p,q>0$ such that $p/q>1$ and $\gamma<1$. Furthermore assume that $g$ is a nonnegative and delta integral $\int_a^\infty (\sigma(t))^{\frac{p}{q}-\gamma}g^{p/q}(t)\Delta t$ exists. Let*

$$\Omega(t) = \int_t^\infty g(s)\Delta s, \quad \textit{for any } t \in [a,\infty]_\mathbb{T}.$$

*Then one gets*

$$\int_a^\infty \frac{(\Omega(t))^{p/q}}{\sigma^\gamma(t)} \le \left(\frac{p}{q(1-\gamma)}\right)^{p/q}\int_a^\infty \frac{g^{p/q}(t)}{(\sigma(t))^{\gamma-\frac{p}{q}}}\Delta t.$$

In this paper we extend results of Theorem 1.1 to Theorem 1.4 for the function of $n$ variables.

## 2. Preliminaries

In this section, we recall the following concepts from theory of time scales [5,7]. A time scale is an arbitrary, non empty closed subset of real numbers. Set of integers and Cantor set are examples of time scales, while rational numbers, complex numbers and open interval between 0 and 1 not time scales. Let $\mathbb{T}$ be a time scale, for $t \in \mathbb{T}$, forward and backward jump operators are defined by

$$\sigma(t) := \inf\{a \in \mathbb{T}; \quad a > t\}, \quad \rho(t) := \sup\{a \in \mathbb{T}; \quad a < t\},$$

respectively. The conventions for these operators are $\inf\phi = \sup\mathbb{T}$ and $\sup\phi = \inf\mathbb{T}$. If $\sigma(t) > t$, then $t$ is right-scattered and if $\rho(t) < t$, then $t$ is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated points.

If $\sigma(t) = t$, then $t$ is right-dense and if $\rho(t) = t$, then $t$ is left-dense. Points that are right-dense and left-dense at the same time are called dense points. The functions $\mu : \mathbb{T} \to \mathbb{R}, \nu : \mathbb{T} \to \mathbb{R}$ defined by $\mu(t) = \sigma(t) - t$ and $\nu(t) = t - \rho(t)$ are called forward and backward graininess functions, respectively.

A function $g : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided $g$ is continuous at right-dense points and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. For any function $g : \mathbb{T} \to \mathbb{R}$, the notation $g^\sigma(t)$ denotes $g(\sigma(t))$. The delta derivative (also Hilger derivative) $g^\Delta(t)$ exists if and only if for every $\epsilon > 0$ there exists a neighborhood $U$ of $t$ such that

$$|g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t)-s)| \le |\sigma(t)-s|, \quad \text{for all } s,\ t \text{ in } \ U.$$

Assume that $h : \mathbb{T} \to \mathbb{R}$, if $H^{\Delta}(t) = h(t)$, then the Cauchy (delta) integral of $h$. defined by

$$\int_a^t h(s)\Delta s := H(t) - H(a).$$

Integration by parts formula [7, Theorem1.77]:

If $a, b \in \mathbb{T}$ and $u, v \in C_{rd}(\mathbb{T})$, then

$$\int_a^b u(t)v^{\Delta}(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^{\Delta}(t)v^{\sigma}(t)\Delta t. \tag{5}$$

Chain rule 1 [7, Theorem 1.90]:

Assume that $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f\,og : \mathbb{T} \to \mathbb{R}$ is delta differentiable and

$$(f\,og)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t) \tag{6}$$

holds.

Chain rule 2 [7, Theorem 1.87]:

If $f$ and $g$ satisfy the conditions of Chain rule 1, Then $f\,og : \mathbb{T} \to \mathbb{R}$ is delta differentiable and there exists $c$ in the real interval $[t, \sigma(t)]$ such that

$$(f\,og)^{\Delta}(t) = f'(g(c))g^{\Delta}(t). \tag{7}$$

Hölder's inequality [7, Theorem 6.13]:

For continuous real-valued functions $g : \mathbb{T} \to \mathbb{R}$, $h : \mathbb{T} \to \mathbb{R}$, let $a, b \in \mathbb{T}$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_a^b g(t)h(t)dt = \left( \int_a^b g^p(t)dt \right)^{1/p} \left( \int_a^b h^q(t)dt \right)^{1/q}. \tag{8}$$

Fubini's Theorem on time scales [6]:

Let $(\psi, M, \mu_\Delta)$ and $(\Gamma, N, \lambda_\Delta)$ be two finite dimensional time scales measure spaces. If $\Lambda : \psi \times \Gamma \to \mathbb{R}$ is a $\mu_\Delta \times \lambda_\Delta$-integrable function. The function $\varsigma(t_2) = \int_\psi \Lambda(t_1, t_2)\Delta t_1$ exists for any $t_1 \in \Gamma$ and $\xi(t_1) = \int_\Gamma \Lambda(t_1, t_2)\Delta t_2$ exists for $t_2 \in \psi$, then

$$\int_\psi \Delta t_1 \int_\Gamma \Lambda(t_1, t_2)\Delta t_2 = \int_\Gamma \Delta t_2 \int_\psi \Lambda(t_1, t_2)\Delta t_1. \tag{9}$$

We assume throughout that all the functions are non-negative and the integrals considered exist.

In this paper, we use the following notations. We assume that there exists constant $k_i > 0$ with

$$\frac{s_i}{\sigma_i(s_i)} \geq \frac{1}{k_i} \quad \text{for} \quad s_i \geq a_i, \quad i \in \{1, \ldots, n\}. \tag{10}$$

$$\Lambda_k^{\sigma_1 \cdots \sigma_j}(t_1, \ldots, t_n) \doteq \Lambda_k^{\sigma_1 \cdots \sigma_j} \doteq \Lambda_k(\sigma_1(t_1), \ldots, \sigma_j(t_j), t_{j+1}, \ldots, t_n),\ k, j \in \{1, \ldots, n\}$$

$$\int_{a_1}^{\infty} \ldots \int_{a_n}^{\infty} f(t_1, \ldots, t_n)\,\Delta t_1, \ldots, \Delta t_n \doteq \int_{\prod_{i=1}^n a_i}^{\infty} f(t_1, \ldots, t_n) \prod_{i=1}^n \Delta t_i.$$

## 3. Hardy and Littlewood-type inequalities for $p/q \geq 2$ and $\gamma > 1$

The following inequalities are used to prove next results.

$$a^{\lambda} + b^{\lambda} \leq (a+b)^{\lambda} \leq 2^{\lambda-1}(a^{\lambda} + b^{\lambda}) \quad \text{for} \quad a, b \geq 0, \lambda \geq 1. \tag{11}$$

$$2^{\lambda-1}(a^{\lambda} + b^{\lambda}) \leq (a+b)^{\lambda} \leq a^{\lambda} + b^{\lambda} \quad \text{for} \quad a, b \geq 0, 0 \leq \lambda \leq 1. \tag{12}$$

**Theorem 3.1.** *Assume $i \in \{1, \ldots, n\}$, $\mathbb{T}_i$ is a time scale with $a_i \in (0, \infty)_{\mathbb{T}_i}$ and $\gamma_i > 1$, further assume $g : [a_1, \infty)_{\mathbb{T}_1} \times \cdots \times [a_n, \infty)_{\mathbb{T}_n} \to \mathbb{R}_+$ is such that the delta integrals $\int_{\prod_{i=1}^{n} a_i}^{\infty} \prod_{i=1}^{n} (t_i)^{\frac{p}{q}-\gamma_i} g^{p/q}(t_1, \ldots, t_n) \Delta t_i$ for any $(t_1, \ldots, t_n) \in [a_1, \infty)_{\mathbb{T}_1} \times \cdots \times [a_n, \infty)_{\mathbb{T}_n}$ exist, define*

$$\Lambda_k(t_1, \ldots, t_n) = \int_{\prod_{j=1}^{k} a_j}^{\infty} g(s_1, \ldots, s_n) \prod_{j=1}^{k} \Delta s_j, \; k \in \{1, \ldots, n\}, \tag{13}$$

*then for $p, q > 0$ and $p/q \geq 2$*

$$\begin{aligned} &\int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{(\Lambda_n^{\sigma_1 \ldots \sigma_n})^{p/q}}{\prod_{i=1}^{n} t_i^{\gamma_i}} \prod_{i=1}^{n} \Delta t_i \\ &\leq \sum_{r=1}^{n} \prod_{j=r+1}^{n} c_j \tilde{c}_r \int_{\prod_{j=r+1}^{n} a_j}^{\infty} \prod_{j=r+1}^{n} \frac{(\mu_j(t_j))^{(p/q-1)}}{t_j^{\gamma_j - 1}} \int_{\prod_{i=1}^{r-1} a_i}^{\infty} \prod_{i=1}^{r-1} \frac{1}{t_i^{\gamma_i}} \times \\ &\quad \left\{ \int_{a_r}^{\infty} \frac{(\Lambda_{r-1}^{\sigma_1 \cdots \sigma_{r-1}})^{p/q}}{t_r^{\gamma_r - p/q}} \Delta t_r \right\}^{q/p} \left\{ (\Lambda_r^{\sigma_1 \ldots \sigma_r})(\Lambda_r^{\sigma_1 \ldots \sigma_{r-1}})^{p/q} \Delta t_r \right\}^{\frac{p-q}{p}} \prod_{i=1}^{r-1} \Delta t_i \prod_{j=r+1}^{n} \Delta t_j \\ &\quad + \prod_{i=1}^{n} \tilde{c}_i \int_{\prod_{i=1}^{n} a_i}^{\infty} \prod_{i=1}^{n} \frac{(\mu_i(t_i))^{(p/q-1)}}{t_i^{\gamma_i - 1}} g^{p/q}(t_1, \ldots, t_n) \prod_{i=1}^{n} \Delta t_i \end{aligned} \tag{14}$$

*holds, where $\tilde{c}_r = c_r p/q$, $c_r = \frac{2^{p/q-2} k_r^{\gamma_r}}{\gamma_{r-1}}$.*

**Proof.** To prove the result, we use the principle of mathematical induction. For $n = 1$ the statement is true by Theorem 1.1. Let the statement be true for $1 \leq n \leq k$.

To prove the result for $n = k+1$. The left-hand side of (14) can be written as,

$$\int_{\prod_{i=1}^{k+1} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k+1} t_i^{\gamma_i}} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_{k+1}})^{p/q} \prod_{i=1}^{k+1} \Delta t_i. \tag{15}$$

Denote $\int_{a_{k+1}}^{\infty} \frac{(\Lambda_{k+1}^{\sigma_1 \cdots \sigma_{k+1}})^{p/q}}{t_{k+1}^{\gamma_{k+1}}} \Delta t_{k+1} = I_{k+1}$. Apply (5) with $\frac{\partial}{\Delta t_{k+1}} u(t_{k+1}) = \frac{1}{t_{k+1}^{\gamma_{k+1}}}$ and $v^{\sigma_{k+1}}(t_{k+1}) = (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_{k+1}})^{p/q}$ by keeping fix $(t_1, \ldots, t_k) \in [a_1, \infty)_{\mathbb{T}_1} \times \cdots \times [a_k, \infty)_{\mathbb{T}_k}$.

$$I_{k+1} = \left[ u(t_{k+1})((\Lambda_{k+1}^{\sigma_1 \cdots \sigma_k})^{p/q}) \right] \Big|_{a_{k+1}}^{\infty} \int_{a_{k+1}}^{\infty} -u(t_{k+1}) \frac{\partial}{\Delta t_{k+1}} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_k})^{p/q} \Delta t_{k+1}, \tag{16}$$

where,

$$u(t_{k+1}) = \int_{t_{k+1}}^{\infty} -\frac{1}{s_{k+1}^{\gamma_{k+1}}} \Delta s_{k+1}. \tag{17}$$

Use chain rule (6) and the fact that $\sigma_{k+1}(s_{k+1}) \geq s_{k+1}$ to get

$$\begin{aligned} \frac{\partial}{\Delta s_{k+1}}\left(-\frac{1}{s_{k+1}^{\gamma_{k+1}-1}}\right) &= (\gamma_{k+1}-1)\int_0^1 [h_{k+1}\sigma_{k+1}(s_{k+1}) + (1-h_{k+1})s_{k+1}]^{-\gamma_{k+1}} dh_{k+1} \\ &\geq \frac{(\gamma_{k+1}-1)}{\sigma_{k+1}^{\gamma_{k+1}}(s_{k+1})}. \end{aligned} \tag{18}$$

(10) together with (18) gives

$$\frac{\partial}{\Delta s_{k+1}}\left(-\frac{1}{s_{k+1}^{\gamma_{k+1}-1}}\right) \geq \frac{(\gamma_{k+1}-1)}{k_{k+1}^{\gamma_{k+1}} s_{k+1}^{\gamma_{k+1}}}.$$

Therefore

$$\begin{aligned} &\int_{t_{k+1}}^{\infty} -\frac{1}{s_{k+1}^{\gamma_{k+1}}} \Delta s_{k+1} \\ &\geq \int_{t_{k+1}}^{\infty} -\frac{k_{k+1}^{\gamma_{k+1}}}{\gamma_{k+1}-1} \frac{\partial}{\Delta s_{k+1}}\left(-\frac{1}{s_{k+1}^{\gamma_{k+1}-1}}\right)\Delta s_{k+1} = -\frac{k_{k+1}^{\gamma_{k+1}}}{\gamma_{k+1}-1}\left(\frac{1}{t_{k+1}^{\gamma_{k+1}-1}}\right). \end{aligned} \tag{19}$$

(17) together with (19) gives

$$-u(t_{k+1}) = -\int_{t_{k+1}}^{\infty} -\frac{1}{s_{k+1}^{\gamma_{k+1}}} \Delta s_{k+1} \leq \frac{k_{k+1}^{\gamma_{k+1}}}{\gamma_{k+1}-1}\left(\frac{1}{t_{k+1}^{\gamma_{k+1}-1}}\right). \tag{20}$$

From (13), (16), (17), (20), we have (note that $u_{k+1}(\infty) = 0$ and $\Lambda_{k+1}(t_1, \ldots, t_k, a_{k+1}) = 0$)

$$I_{k+1} = \frac{k_{k+1}^{\gamma_{k+1}}}{\gamma_{k+1}-1} \int_{a_{k+1}}^{\infty} \frac{1}{t_{k+1}^{\gamma_{k+1}-1}} \frac{\partial}{\Delta t_{k+1}} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_k})^{p/q} \Delta t_{k+1}. \tag{21}$$

Apply chain rule 1 (6) on the right-hand side of (21)

$$\begin{aligned} &\frac{\partial}{\Delta t_{k+1}} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_k})^{p/q} \\ &= \frac{p}{q} \frac{\partial}{\Delta t_{k+1}} \Lambda_{k+1}^{\sigma_1 \cdots \sigma_k} \int_0^1 \left[\Lambda_{k+1} + h_{k+1}\mu_{k+1}(t_{k+1}) \frac{\partial}{\Delta t_{k+1}} \Lambda_{k+1}^{\sigma_1 \cdots \sigma_k}\right]^{\frac{p}{q}-1} dh_{k+1}. \end{aligned} \tag{22}$$

Use right part of (11) on the right-hand side of (22),

$$\begin{aligned} &\frac{\partial}{\Delta t_{k+1}} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_k})^{p/q} \\ &\leq \frac{p}{q} 2^{p/q-2} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_k})^{p/q-1} \frac{\partial}{\Delta t_{k+1}} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_k}) \\ &\quad + \frac{p}{q} 2^{p/q-2} (\mu_{k+1}(t_{k+1}))^{p/q-1} \left(\frac{\partial}{\Delta t_{k+1}} \Lambda_{k+1}^{\sigma_1 \cdots \sigma_k}\right)^{p/q}. \end{aligned} \tag{23}$$

Substitute (23) into (21)

$$I_{k+1} \leq \frac{p2^{p/q-2}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \int_{a_{k+1}}^{\infty} \frac{1}{t_{k+1}^{\gamma_{k+1}-1}} (\Lambda_{k+1}^{\sigma_1\cdots\sigma_k})^{p/q-1} \frac{\partial}{\Delta t_{k+1}} \Lambda_{k+1}^{\sigma_1\cdots\sigma_k} \Delta t_{k+1}$$
$$+ \frac{p2^{p/q-2}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \int_{a_{k+1}}^{\infty} \frac{1}{t_{k+1}^{\gamma_{k+1}-1}} (\mu_{k+1}(t_{k+1}))^{p/q-1} \left(\frac{\partial}{\Delta t_{k+1}} \Lambda_{k+1}^{\sigma_1\cdots\sigma_k}\right)^{p/q} \Delta t_{k+1}. \tag{24}$$

Since

$$\frac{\partial}{\Delta t_{k+1}} \Lambda_{k+1}^{\sigma_1\cdots\sigma_k} = \Lambda_k^{\sigma_1\cdots\sigma_k} \geq 0. \tag{25}$$

Use (25) in (24)

$$I_{k+1} \leq \frac{p2^{p/q-2}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \int_{a_{k+1}}^{\infty} \frac{1}{t_{k+1}^{\gamma_{k+1}-1}} (\Lambda_{k+1}^{\sigma_1\cdots\sigma_k})^{p/q-1} \Lambda_k^{\sigma_1\cdots\sigma_k} \Delta t_{k+1}$$
$$+ \frac{p2^{p/q-2}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \int_{a_{k+1}}^{\infty} \frac{1}{t_{k+1}^{\gamma_{k+1}-1}} (\mu_{k+1}(t_{k+1}))^{p/q-1} (\Lambda_k^{\sigma_1\cdots\sigma_k})^{p/q} \Delta t_{k+1}. \tag{26}$$

Substitute (26) in (15)

$$\int_{\prod_{i=1}^{k+1} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k+1} t_i^{\gamma_i}} (\Lambda_{k+1}^{\sigma_1\cdots\sigma_{k+1}})^{p/q} \prod_{i=1}^{k+1} \Delta t_i$$
$$\leq \int_{\prod_{i=1}^{k} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k} t_i^{\gamma_i}} \frac{p2^{p/q-2}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \int_{a_{k+1}}^{\infty} \frac{1}{t_{k+1}^{\gamma_{k+1}-1}} (\Lambda_{k+1}^{\sigma_1\cdots\sigma_k})^{p/q-1} \Lambda_k^{\sigma_1\cdots\sigma_k} \prod_{i=1}^{k} \Delta t_{k+1} \Delta t_i \tag{27}$$
$$+ \int_{\prod_{i=1}^{k} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k} t_i^{\gamma_i}} \frac{p2^{p/q-2}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \int_{a_{k+1}}^{\infty} \frac{(\mu_{k+1}(t_{k+1}))^{p/q-1}}{t_{k+1}^{\gamma_{k+1}-1}} (\Lambda_k^{\sigma_1\cdots\sigma_k})^{p/q} \prod_{i=1}^{k} \Delta t_{k+1} \Delta t_i.$$

Exchange integrals on right-hand side of (27) $k$ -times by using (9)

$$= \frac{p2^{p/q-2}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \int_{a_{k+1}}^{\infty} \frac{1}{t_{k+1}^{\gamma_{k+1}-1}} \int_{\prod_{i=1}^{k} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k} t_i^{\gamma_i}} (\Lambda_{k+1}^{\sigma_1\cdots\sigma_k})^{p/q-1} \Lambda_k^{\sigma_1\cdots\sigma_k} \prod_{i=1}^{k} \Delta t_i \Delta t_{k+1}$$
$$+ \frac{p2^{p/q-2}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} (\mu_{k+1}(t_{k+1}))^{p/q-1} \int_{a_{k+1}}^{\infty} \frac{1}{t_{k+1}^{\gamma_{k+1}-1}} \tag{28}$$
$$\times \int_{\prod_{i=1}^{k} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k} t_i^{\gamma_i}} (\Lambda_k^{\sigma_1\cdots\sigma_k})^{p/q} \prod_{i=1}^{k} \Delta t_i \Delta t_{k+1}.$$

Use the induction hypothesis with $\Lambda_k^{\sigma_1\cdots\sigma_k}$ in (28) for fixed $t_{k+1} \in \mathbb{T}_{k+1}$ and again apply (9) $k$-times to get

$$= \frac{p2^{p/q-2}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)}\int_{a_{k+1}}^{\infty}\frac{1}{t_{k+1}^{\gamma_{k+1}-1}}\int_{\prod_{i=1}^{k}a_i}^{\infty}\frac{1}{\prod_{i=1}^{k}t_i^{\gamma_i}}(\Lambda_{k+1}^{\sigma_1\cdots\sigma_k})^{p/q-1}\Lambda_k^{\sigma_1\cdots\sigma_k}\prod_{i=1}^{k}\Delta t_i\Delta t_{k+1}$$

$$+\frac{p2^{p/q-2}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)}(\mu_{k+1}(t_{k+1}))^{p/q-1}\int_{a_{k+1}}^{\infty}\frac{1}{t_{k+1}^{\gamma_{k+1}-1}}$$

$$\times\sum_{r=1}^{k}\prod_{j=r+1}^{k}c_j\tilde{c}_r\int_{\prod_{j=r+1}^{k}a_j}^{\infty}\prod_{j=r+1}^{k}\frac{(\mu_j(t_j))^{(p/q-1)}}{t_j^{\gamma_j-1}}\times$$

$$\int_{\prod_{i=1}^{r-1}a_j}^{\infty}\prod_{i=1}^{r-1}\frac{1}{t_i^{\gamma_i}}\left\{\int_{a_r}^{\infty}\frac{(\Lambda_{r-1}^{\sigma_1\cdots\sigma_{r-1}})^{p/q}}{t_r^{\gamma_r-p/q}}\Delta t_r\right\}^{q/p}$$

$$\times\left\{(\Lambda_r^{\sigma_1\cdots\sigma_r})(\Lambda_r^{\sigma_1\cdots\sigma_{r-1}})^{p/q}\Delta t_r\right\}^{\frac{p-q}{p}}\prod_{i=1}^{r-1}\Delta t_i\prod_{j=r+1}^{k}\Delta t_j$$

$$+\prod_{i=1}^{k}\tilde{c}_i\int_{\prod_{i=1}^{k}a_j}^{\infty}\prod_{i=1}^{k}\frac{(\mu_i(t_i))^{(p/q-1)}}{t_i^{\gamma_i-1}}g^{p/q}(t_1,\ldots,t_k)\prod_{i=1}^{k}\Delta t_i.$$

Hence

$$\int_{\prod_{i=1}^{k+1}a_i}^{\infty}\frac{(\Lambda_n^{\sigma_1\ldots\sigma_n})^{p/q}}{\prod_{i=1}^{k+1}t_i^{\gamma_i}}\prod_{i=1}^{k+1}\Delta t_i$$

$$\leq\sum_{r=1}^{k+1}\prod_{j=r+1}^{k+1}c_j\tilde{c}_r\int_{\prod_{j=r+1}^{k+1}a_j}^{\infty}\prod_{j=r+1}^{k+1}\frac{(\mu_j(t_j))^{(p/q-1)}}{t_j^{\gamma_j-1}}\int_{\prod_{i=1}^{r-1}a_i}^{\infty}\prod_{i=1}^{r-1}\frac{1}{t_i^{\gamma_i}}\times$$

$$\left\{\int_{a_r}^{\infty}\frac{(\Lambda_{r-1}^{\sigma_1\cdots\sigma_{r-1}})^{p/q}}{t_r^{\gamma_r-p/q}}\Delta t_r\right\}^{q/p}\left\{\Lambda_r^{\sigma_1\ldots\sigma_r}(\Lambda_r^{\sigma_1\ldots\sigma_{r-1}})^{p/q}\Delta t_r\right\}^{\frac{p-q}{p}}\prod_{i=1}^{r-1}\Delta t_i\prod_{j=r+1}^{k+1}\Delta t_j$$

$$+\prod_{i=1}^{k+1}\tilde{c}_i\int_{\prod_{i=1}^{k+1}a_i}^{\infty}\prod_{i=1}^{k+1}\frac{(\mu_i(t_i))^{(p/q-1)}}{t_i^{\gamma_i-1}}g^{p/q}(t_1,\ldots,t_{k+1})\prod_{i=1}^{k+1}\Delta t_i.$$

Hence by induction principle, the statement is true $\forall\, n\in\mathbb{N}$. □

**Theorem 3.2.** *Assume $i\in\{1,\ldots,n\}$, $\mathbb{T}_i$ is a time scale with $a_i\in(0,\infty)_{\mathbb{T}_i}$ and $\gamma_i>1$, further assume $g:[a_1,\infty)_{\mathbb{T}_1}\times\cdots\times[a_n,\infty)_{\mathbb{T}_n}\to\mathbb{R}_+$ is such that the delta integrals $\int_{\prod_{i=1}^{n}a_i}^{\infty}\prod_{i=1}^{n}(t_i)^{\frac{p}{q}-\gamma_i}g^{p/q}(t_1,\ldots,t_n)\prod_{i=1}^{n}\Delta t_i$ exist. Let $\Lambda_k(t_1,\ldots,t_n)$ be defined in (13), then for $p,q>0$ and $p/q\geq 2$*

$$\int_{\prod_{i=1}^{n}a_i}^{\infty}\frac{1}{\prod_{i=1}^{n}t_i^{\gamma_i}}(\Lambda_n^{\sigma_1\cdots\sigma_n})^{p/q}\prod_{i=1}^{n}\Delta t_i$$
$$\leq\left(\frac{p}{q}\right)^{\frac{np}{q}}\prod_{i=1}^{n}\left(\frac{2^{\frac{p}{q}-1}k_i^{\gamma_i}}{(\gamma_i-1)}\right)^{p/q}\int_{\prod_{i=1}^{n}a_i}^{\infty}\prod_{i=1}^{n}\frac{1}{t_i^{\gamma_i-p/q}}g^{p/q}(t_1,\ldots,t_n)\prod_{i=1}^{n}\Delta t_i, \tag{29}$$

*holds.*

***Proof.*** To prove the result, we use the principle of mathematical induction. For $n = 1$ the statement is true by Theorem 1.2. Let the statement be true for $1 \leq n \leq k$.

To prove the result for $n = k + 1$. Proceed it as in the proof of Theorem 3.1 up to (21). Apply chain rule 1 (6) on the right-hand side of (21) yields

$$\begin{aligned}&\frac{\partial}{\Delta t_{k+1}}(\Lambda_{k+1}^{\sigma_1\cdots\sigma_k})^{p/q}\\&\quad=\left(\frac{p}{q}\right)\frac{\partial}{\Delta t_{k+1}}\Lambda_{k+1}^{\sigma_1\cdots\sigma_k}\int_0^1\left[h_{k+1}\Lambda_{k+1}^{\sigma_1\cdots\sigma_{k+1}}+(1-h_{k+1})\Lambda_{k+1}^{\sigma_1\cdots\sigma_k}\right]^{\frac{p}{q}-1}dh_{k+1}.\end{aligned} \tag{30}$$

Use (11) on the right-hand side of (30),

$$\leq\left(\frac{p}{q}\right)2^{\frac{p}{q}-2}(\Lambda_{k+1}^{\sigma_1\cdots\sigma_{k+1}})^{\frac{p}{q}-1}\frac{\partial}{\Delta t_{k+1}}\Lambda_{k+1}^{\sigma_1\cdots\sigma_k}+\left(\frac{p}{q}\right)2^{\frac{p}{q}-2}(\Lambda_{k+1}^{\sigma_1\cdots\sigma_k})^{\frac{p}{q}-1}\frac{\partial}{\Delta t_{k+1}}\Lambda_{k+1}^{\sigma_1\cdots\sigma_k},$$

use the fact $\sigma_{k+1}(t_{k+1}) \geq t_{k+1}$

$$\begin{aligned}&=\left(\frac{p}{q}\right)2^{\frac{p}{q}-2}(\Lambda_{k+1}^{\sigma_1\cdots\sigma_{k+1}})^{\frac{p}{q}-1}\frac{\partial}{\Delta t_{k+1}}\Lambda_{k+1}^{\sigma_1\cdots\sigma_k}+\left(\frac{p}{q}\right)2^{\frac{p}{q}-2}(\Lambda_{k+1}^{\sigma_1\cdots\sigma_{k+1}})^{\frac{p}{q}-1}\frac{\partial}{\Delta t_{k+1}}\Lambda_{k+1}^{\sigma_1\cdots\sigma_k}\\&=\left(\frac{p}{q}\right)2^{\frac{p}{q}-1}(\Lambda_{k+1}^{\sigma_1\cdots\sigma_{k+1}})^{\frac{p}{q}-1}\frac{\partial}{\Delta t_{k+1}}\Lambda_{k+1}^{\sigma_1\cdots\sigma_k}.\end{aligned} \tag{31}$$

Since

$$\frac{\partial}{\Delta t_{k+1}}\Lambda_{k+1}^{\sigma_1\cdots\sigma_k}=\Lambda_k^{\sigma_1\cdots\sigma_k}\geq 0. \tag{32}$$

Use (32) in (31) and substitute in (21) to get

$$I_{k+1}\leq\frac{p2^{\frac{p}{q}-1}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)}\int_{a_{k+1}}^{\infty}\frac{1}{t_{k+1}^{\gamma_{k+1}-1}}(\Lambda_{k+1}^{\sigma_1\cdots\sigma_{k+1}})^{\frac{p}{q}-1}\Lambda_k^{\sigma_1\cdots\sigma_k}\,\Delta t_{k+1}. \tag{33}$$

Apply Hölder's inequality on the right-hand side of (33) with indices $p/q$ and $p/(p-q)$

$$I_{k+1}\leq\frac{p2^{\frac{p}{q}-1}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)}\left\{\int_{a_{k+1}}^{\infty}\left\{\frac{t_{k+1}^{\gamma_{k+1}\left(\frac{p-q}{q}\right)}}{t_{k+1}^{\gamma_{k+1}-1}}\Lambda_k^{\sigma_1\cdots\sigma_k}\right\}^{p/q}\Delta t_{k+1}\right\}^{q/p}\times\{I_{k+1}\}^{\frac{p-q}{p}}.$$

After simplification, we get

$$I_{k+1}\leq\left(\frac{p2^{\frac{p}{q}-1}k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)}\right)^{p/q}\int_{a_{k+1}}^{\infty}\frac{(\Lambda_k^{\sigma_1\cdots\sigma_k})^{p/q}}{t_{k+1}^{-\frac{p}{q}+\gamma_{k+1}}}\Delta t_{k+1}. \tag{34}$$

Substitute (34) into (15)

$$\int_{\prod_{i=1}^{k+1} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k+1} t_i^{\gamma_i}} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_{k+1}})^{p/q} \prod_{i=1}^{k+1} \Delta t_i$$
$$\leq \int_{\prod_{i=1}^{k} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k} t_i^{\gamma_i}} \left( \frac{p 2^{\frac{p}{q}-1} k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \right)^{p/q} \int_{a_{k+1}}^{\infty} \frac{(\Lambda_k^{\sigma_1 \cdots \sigma_k})^{p/q}}{t_{k+1}^{-\frac{p}{q}+\gamma_{k+1}}} \Delta t_{k+1}. \tag{35}$$

Exchange integrals on right-hand side of (35) $k$ -times by using (9)

$$\left( \frac{p 2^{\frac{p}{q}-1} k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \right)^{p/q} \int_{a_{k+1}}^{\infty} \frac{1}{t_{k+1}^{-\frac{p}{q}+\gamma_{k+1}}} \left\{ \int_{\prod_{i=1}^{k} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k} t_i^{\gamma_i}} (\Lambda_k^{\sigma_1 \cdots \sigma_k})^{p/q} \prod_{i=1}^{k} \Delta t_i \right\} \Delta t_{k+1}. \tag{36}$$

Use the induction hypothesis for $\Lambda_k^{\sigma_1 \cdots \sigma_k}$ in (36) for fixed $t_{k+1} \in \mathbb{T}_{k+1}$ and again apply (9) $k$ times to get

$$\int_{\prod_{i=1}^{k+1} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k+1} t_i^{\gamma_i}} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_{k+1}})^{p/q} \prod_{i=1}^{k+1} \Delta t_i$$
$$\leq \left(\frac{p}{q}\right)^{\frac{(k+1)p}{q}} \prod_{i=1}^{k+1} \left( \frac{2^{\frac{p}{q}-1} k_i^{\gamma_i}}{\gamma_i - 1} \right)^{p/q} \int_{\prod_{i=1}^{k+1} a_i}^{\infty} \prod_{i=1}^{k+1} \frac{1}{t_i^{\gamma_i - p/q}} g^{p/q}(t_1, \ldots, t_{k+1}) \prod_{i=1}^{k} \Delta t_i.$$

Hence by induction principle, the statement is true $\forall \quad n \in \mathbb{N}$. ☐

**Corollary 3.3.** *As a special case of Theorem 3.2, when* $\mathbb{T}_1 = \cdots = \mathbb{T}_n = \mathbb{R}$, $p/q = \lambda > 1$ *and* $\gamma_i < 1$, (29) *becomes the following Wirtinger type inequality*

$$\int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} t_i^{\gamma_i}} (G(t_1, \ldots, t_n))^{\lambda} \prod_{i=1}^{n} dt_i$$
$$\leq \prod_{i=1}^{n} \left( \frac{\lambda 2^{\lambda-1}}{1-\gamma_i} \right)^{\lambda} \int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} t_1^{\gamma_i - \lambda}} \left( \frac{\partial^n}{\partial t_1 \cdots \partial t_n} G^{\lambda}(t_1, \ldots, t_n) \right) \prod_{i=1}^{n} dt_i,$$

*where* $G(t_1, \ldots, t_n) \doteq \int_{\prod_{i=1}^{n} a_i}^{t_i} g(s_1, \ldots, s_n) \prod_{i=1}^{n} ds_i$.

*When* $\gamma_1 = \cdots = \gamma_n = \lambda > 1$, *we have another Hardy type inequality for function of n-variables*

$$\int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} t_i} \left( \int_{\prod_{i=1}^{n} a_i}^{t_i} g(s_1, \ldots, s_n) \prod_{i=1}^{n} ds_i \right)^{\lambda} \prod_{i=1}^{n} dt_i$$
$$\leq \left( \frac{\lambda 2^{\lambda-1}}{\lambda - 1} \right)^{\lambda} g^{\lambda}(t_1, \ldots, t_n) \prod_{i=1}^{n} dt_i.$$

**Remark 3.4.** Assume that $\mathbb{T}_1 = \cdots = \mathbb{T}_n = \mathbb{N}$ in Theorem 3.2, $p/q = \lambda > 1$, $a_i > 1$, $\gamma_i > 1$ for $i \in \{1, \ldots, n\}$, further assume that $\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} g^{\lambda}(m_1, \ldots, m_n)$ is convergent. (29) becomes the following discrete Hardy and Littlewood inequality

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{1}{m_1^{\gamma_1} \ldots m_n^{\gamma_n}} \left( \sum_{k_1=1}^{m_1} \cdots \sum_{k_n=1}^{m_n} g(k_1, \ldots, k_n) \right)^{\lambda}$$

$$\leq \prod_{i=1}^{n} \left( \frac{2^{\lambda-1}\lambda}{\gamma_i - 1} \right)^{\lambda} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{1}{\prod_{i=1}^{n} m_i^{\gamma_i - \lambda}} g^{\lambda}(m_1, \ldots, m_n).$$

## 4. Hardy and Littlewood-type inequalities for $p/q \geq 1$ and $\gamma > 1$

**Theorem 4.1.** *Assume $i \in \{1, \ldots, n\}$, $\mathbb{T}_i$ is a time scale with $a_i \in (0, \infty)_{\mathbb{T}_i}$ and $\gamma_i < 1$, further assume $g : [a_1, \infty)_{\mathbb{T}_1} \times \cdots \times [a_n, \infty)_{\mathbb{T}_n} \to \mathbb{R}_+$ is such that the delta integrals $\int_{\prod_{i=1}^{n} a_i}^{\infty} \prod_{i=1}^{n} t_i^{\frac{p}{q} - \gamma_i} g^{p/q}(t_1, \ldots, t_n) \prod_{i=1}^{n} \Delta t_i$ exist. Let $\Lambda_k(t_1, \ldots, t_n)$ be defined in* (13), *then for $p$, $q > 0$ and $p/q > 1$*

$$\int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} t_i^{\gamma_i}} (\Lambda_n^{\sigma_1 \cdots \sigma_n})^{p/q} \prod_{i=1}^{n} \Delta t_i$$
$$\leq \left( \frac{p}{q} \right)^{\frac{np}{q}} \prod_{i=1}^{n} \left( \frac{k_i^{\gamma_i}}{\gamma_i - 1} \right)^{p/q} \int_{\prod_{i=1}^{n} a_i}^{\infty} \prod_{i=1}^{n} \frac{1}{t_i^{\gamma_i - p/q}} g^{p/q}(t_1, \ldots, t_n) \prod_{i=1}^{n} \Delta t_i, \quad (37)$$

*holds, where $n$ is a positive integer.*

**Proof.** To prove the result, we use the principle of mathematical induction. For $n = 1$ the statement is true by Theorem 1.3. Let the statement be true for $1 \leq n \leq k$.

To prove the result for $n = k + 1$. Proceed it as in the proof of Theorem 3.1 up to (21). Apply the chain rule 2 (7) to get

$$\frac{\partial}{\Delta t_{k+1}} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_k})^{p/q} = \frac{p}{q} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_k}(t_1, \ldots, t_k, c_{k+1}))^{\frac{p}{q} - 1} \frac{\partial}{\Delta t_{k+1}} \Lambda_{k+1}^{\sigma_1 \cdots \sigma_k},$$

where $c_{k+1} \in [t_{k+1}, \sigma_{k+1}(t_{k+1})]$. Since

$$\frac{\partial}{\Delta t_{k+1}} \Lambda_{k+1}^{\sigma_1 \cdots \sigma_k} \doteq \Lambda_k^{\sigma_1 \cdots \sigma_k} \geq 0,$$

and $\sigma_{k+1}(t_{k+1}) \geq c_{k+1}$, one has that

$$\frac{\partial}{\Delta t_{k+1}} \Lambda_{k+1}^{\sigma_1 \cdots \sigma_k} \leq \frac{p}{q} (\Lambda_{k+1}^{\sigma_1 \cdots \sigma_{k+1}})^{\frac{p}{q} - 1} \Lambda_k^{\sigma_1 \cdots \sigma_k}. \quad (38)$$

Substitute (38) into (21)

$$I_{k+1} \leq \frac{p k_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1} - 1)} \int_{a_{k+1}}^{\infty} \frac{(\Lambda_{k+1}^{\sigma_1 \cdots \sigma_{k+1}})^{\frac{p}{q} - 1}}{t_{k+1}^{\gamma_{k+1} - 1}} \Lambda_k^{\sigma_1 \cdots \sigma_k} \Delta t_{k+1}. \quad (39)$$

Apply Hölder's inequality on the right-hand side of (39) with indices $p/q$ and $p/(p-q)$

$$I_{k+1} \leq \frac{pk_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \left\{ \int_{a_{k+1}}^{\infty} \left\{ \frac{t_{k+1}^{\gamma_{k+1}\left(\frac{p-q}{p}\right)}}{t_{k+1}^{\gamma_{k+1}-1}} \Lambda_k^{\sigma_1,\ldots,\sigma_k} \right\}^{p/q} \Delta t_{k+1} \right\}^{q/p} \times \{I_{k+1}\}^{\frac{p-q}{p}}.$$

After simplification, we get

$$I_{k+1} \leq \left( \frac{pk_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \right)^{p/q} \int_{a_{k+1}}^{\infty} \frac{\left(\Lambda_k^{\sigma_1,\ldots,\sigma_k}\right)^{p/q}}{t_{k+1}^{-\frac{p}{q}+\gamma_{k+1}}} \Delta t_{k+1}. \tag{40}$$

Substitute (40) into (15)

$$\begin{aligned} &\int_{\prod_{i=1}^{k+1} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k+1} t_i^{\gamma_i}} \left(\Lambda_{k+1}^{\sigma_1\cdots\sigma_{k+1}}\right)^{p/q} \prod_{i=1}^{k+1} \Delta t_i \\ &\quad \leq \int_{\prod_{i=1}^{k} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k} t_i^{\gamma_i}} \left( \frac{pk_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \right)^{p/q} \int_{a_{k+1}}^{\infty} \frac{\left(\Lambda_k^{\sigma_1,\ldots,\sigma_k}\right)^{p/q}}{t_{k+1}^{\gamma_{k+1}-\frac{p}{q}}} \Delta t_{k+1}. \end{aligned} \tag{41}$$

Exchange integrals on right-hand side of (41) $k$ -times by using (9)

$$= \left( \frac{pk_{k+1}^{\gamma_{k+1}}}{q(\gamma_{k+1}-1)} \right)^{p/q} \int_{a_{k+1}}^{\infty} \frac{1}{t_{k+1}^{\gamma_{k+1}-\frac{p}{q}}} \left\{ \int_{\prod_{i=1}^{k} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k} t_i^{\gamma_i}} \left(\Lambda_k^{\sigma_1\cdots\sigma_k}\right)^{p/q} \prod_{i=1}^{k} \Delta t_i \right\} \Delta t_{k+1}. \tag{42}$$

Use the induction hypothesis with $(\Lambda_k^{\sigma_1\cdots\sigma_k})^{p/q}$ in (42) for fixed $t_{k+1} \in \mathbb{T}_{k+1}$ and again apply (9) $k$-times to get

$$\begin{aligned} &\int_{\prod_{i=1}^{k+1} a_i}^{\infty} \frac{1}{\prod_{i=1}^{k+1} t_i^{\gamma_i}} \left(\Lambda_{k+1}^{\sigma_1\cdots\sigma_{k+1}}\right)^{p/q} \prod_{i=1}^{k+1} \Delta t_i \\ &\quad \leq \left(\frac{p}{q}\right)^{\frac{(k+1)p}{q}} \prod_{i=1}^{k+1} \left( \frac{k_i^{\gamma_i}}{\gamma_i - 1} \right)^{p/q} \int_{\prod_{i=1}^{k+1} a_i}^{\infty} \prod_{i=1}^{k+1} \frac{1}{t_i^{\gamma_i - p/q}} g^{p/q}(t_1,\ldots,t_{k+1}) \prod_{i=1}^{k+1} \Delta t_i. \end{aligned}$$

Hence by induction principle, the statement is true $\forall \quad n \in \mathbb{N}$. $\square$

**Corollary 4.2.** *As a special case of Theorem 4.1, when* $\mathbb{T}_1 = \cdots = \mathbb{T}_n = \mathbb{R}$, $p/q = \lambda > 1$ *and* $\gamma_1, \ldots, \gamma_n < 1$, *(37) becomes the following Wirtinger type inequality,*

$$\begin{aligned} &\int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} t_i^{\gamma_i}} G^{\lambda}(t_1,\ldots,t_n) \prod_{i=1}^{n} dt_i \\ &\quad \leq \prod_{i=1}^{n} \left( \frac{\lambda}{1-\gamma_i} \right)^{\lambda} \int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} t_1^{\gamma_i - \lambda}} \left( \frac{\partial^n}{\partial t_1 \cdots \partial t_n} G^{\lambda}(t_1,\ldots,t_n) \right) \prod_{i=1}^{n} dt_i, \end{aligned}$$

*where* $G(t_1,\ldots,t_n) \doteq \int_{\prod_{i=1}^{n} a_i}^{t_i} g(s_1,\ldots,s_n) \prod_{i=1}^{n} \Delta s_i$.

*When $\gamma_1 = \cdots = \gamma_n = \lambda > 1$, we have the classical Hardy type inequality for function of n -variables*

$$\int_{\prod_{i=1}^n a_i}^{\infty} \frac{1}{\prod_{i=1}^n t_i}\left(\int_{\prod_{i=1}^n a_i}^{t_i} g(s_1,\ldots,s_n)\prod_{i=1}^n ds_i\right)^{\lambda}\prod_{i=1}^n dt_i$$
$$\leq \left(\frac{\lambda}{\lambda-1}\right)^{\lambda} g^{\lambda}(t_1,\ldots,t_n)\prod_{i=1}^n dt_i.$$

**Corollary 4.3.** *Assume that $\mathbb{T}_1 = \cdots = \mathbb{T}_n = \mathbb{N}$ in Theorem 4.1, $p/q = \lambda > 1$, $a_i > 1$, $\gamma_i > 1$ for $i \in \{1,\ldots,n\}$, further assume that $\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty} g^{\lambda}(m_1,\ldots,m_n)$ is convergent. Note that in this case $\frac{m_i}{\sigma_i(m_i)} = \frac{m_i}{m_i+1}$ therefore $\frac{1}{2} \leq \frac{m_i}{m_i+1} \leq 1$, and we get following discrete Hardy and Littlewood inequality*

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{1}{m_1^{\gamma_1}\ldots m_n^{\gamma_n}}\left(\sum_{k_1=1}^{m_1}\cdots\sum_{k_n=1}^{m_n} g(k_1,\ldots,k_n)\right)^{\lambda}$$
$$\leq \prod_{i=1}^n\left(\frac{2^{\lambda}\lambda}{\gamma_i-1}\right)^{\lambda}\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{1}{\prod_{i=1}^n m_i^{\gamma_i-\lambda}} g^{\lambda}(m_1,\ldots,m_n).$$

**Remark 4.4.** Assume $i \in \{1,\ldots,n\}$, $\mathbb{T}_i$ is a time scale with $a_i \in (0,\infty)_{\mathbb{T}_i}$ and $\gamma_i < 1$, further assume $g : [a_1,\infty)_{\mathbb{T}_1} \times \cdots \times [a_n,\infty)_{\mathbb{T}_n} \to \mathbb{R}_+$ is such that the delta integrals $\int_{\prod_{i=1}^n a_i}^{\infty} \prod_{i=1}^n \sigma_i(t_i)^{\frac{p}{q}-\gamma_i}\left(\frac{\sigma_i(t_i)}{t_i}\right)^{\frac{p}{q}(\gamma_i-1)} g_n^{p/q}(t_1,\ldots,t_n)\prod_{i=1}^n \Delta t_i$ exist. Let $\Lambda_k(t_1,\ldots,t_n)$ be defined in Theorem 3.1, then for $p,q > 0$ and $p/q > 1$

$$\int_{\prod_{i=1}^n a_i}^{\infty}\frac{(\Lambda_n^{\sigma_1\cdots\sigma_n})^{p/q}}{\prod_{i=1}^n(\sigma_i(t_i))^{\gamma_i}}\prod_{i=1}^n \Delta t_i$$
$$\leq \left(\frac{p}{q}\right)^{\frac{np}{q}}\prod_{i=1}^n\left(\frac{1}{\gamma_i-1}\right)^{p/q}\int_{\prod_{i=1}^n a_i}^{\infty}\frac{g^{p/q}(t_1,\ldots,t_n)}{\prod_{i=1}^n \sigma^{\gamma_i-\frac{p}{q}}(t_i)}\prod_{i=1}^n\left(\frac{\sigma_i(t_i)}{t_i}\right)^{\frac{p}{q}(\gamma_i-1)}\prod_{i=1}^n \Delta t_i,$$

holds.

**Proof.** Replace left-hand side of (37) in Theorem 4.1 by

$$\int_{\prod_{i=1}^n a_i}^{\infty}\frac{(\Lambda_n^{\sigma_1\cdots\sigma_n})^{p/q}}{\prod_{i=1}^n(\sigma_i(t_i))^{\gamma_i}}\prod_{i=1}^n \Delta t_i,$$

and proceed as in the proof of Theorem 4.1. □

## 5. Hardy and Littlewood-type inequalities for *p/q* ≤ 2 and *γ* > 1

**Theorem 5.1.** *Assume $i \in \{1,\ldots,n\}$, $\mathbb{T}_i$ is a time scale with $a_i \in (0,\infty)_{\mathbb{T}_i}$ and $\gamma_i < 1$, further assume $g : [a_1,\infty)_{\mathbb{T}_1} \times \cdots \times [a_n,\infty)_{\mathbb{T}_n} \to \mathbb{R}_+$ is such that the delta integrals*

$\int_{\prod_{i=1}^{n} a_i}^{\infty} \prod_{i=1}^{n} (t_i)^{\frac{p}{q}-\gamma_i} g^{p/q}(t_1, \ldots, t_n) \prod_{i=1}^{n} \Delta t_i$ *exist. Let* $\Lambda_k(t_1, \ldots, t_n)$ *be defined in* (13), *then for* $p, q > 0$ *and* $p/q \leq 2$

$$\int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} t_i^{\gamma_i}} (\Lambda_n^{\sigma_1 \cdots \sigma_n})^{p/q} \prod_{i=1}^{n} \Delta t_i$$
$$\leq \left(\frac{p}{q}\right)^{\frac{np}{q}} \prod_{i=1}^{n} \left(\frac{2k_i^{\gamma_i}}{(\gamma_i - 1)}\right)^{p/q} \int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} t_i^{\gamma_i - \frac{p}{q}}} g^{p/q}(t_1, \ldots, t_n) \prod_{i=1}^{n} \Delta t_i. \tag{43}$$

**Proof.** Proceed as in the proof of Theorem 3.2 and apply inequality (12) in (21) to get (43). □

**Remark 5.2.** As a special case of Theorem 5.1, when $\mathbb{T}_1 = \cdots = \mathbb{T}_n = \mathbb{R}$, $p/q = \lambda > 1$ and $\gamma_1, \ldots, \gamma_n < 1$, we have the following Hardy type inequality

$$\int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} t_i^{\gamma_i}} \left(\int_{\prod_{i=1}^{n} a_i}^{t_i} g(s_1, \ldots, s_n) \prod_{i=1}^{n} ds_i\right)^{\lambda} \prod_{i=1}^{n} dt_i$$
$$\leq \prod_{i=1}^{n} \left(\frac{2\lambda}{1 - \gamma_i}\right)^{\lambda} \int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} t_i^{\gamma_i - \lambda}} g^{\lambda}(t_1, \ldots, t_n) \prod_{i=1}^{n} dt_i.$$

**Remark 5.3.** Assume that $\mathbb{T}_1 = \cdots = \mathbb{T}_n = \mathbb{N}$ in Theorem 5.1, $p/q = \lambda > 1$, $a_i > 1$, $\gamma_i > 1$ for $i \in \{1, \ldots, n\}$, further assume that $\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} g^{\lambda}(m_1, \ldots, m_n)$ is convergent. In this case, (43) becomes the following discrete Hardy and Littlewood inequality

$$\sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{1}{m_1^{\gamma_1} \ldots m_n^{\gamma_n}} \left(\sum_{k_1=1}^{m_1} \cdots \sum_{k_n=1}^{m_n} g(k_1, \ldots, k_n)\right)^{\lambda}$$
$$\leq \prod_{i=1}^{n} \left(\frac{2\lambda}{\gamma_i - 1}\right)^{\lambda} \sum_{m_1=1}^{\infty} \cdots \sum_{m_n=1}^{\infty} \frac{1}{\prod_{i=1}^{n} m_i^{\gamma_i - \lambda}} g^{\lambda}(m_1, \ldots, m_n).$$

## 6. Hardy and Littlewood-type inequalities for $p/q > 1$ and $\gamma < 1$

**Theorem 6.1.** *Assume* $i \in \{1, \ldots, n\}$, $\mathbb{T}_i$ *is a time scale with* $a_i \in (0, \infty)_{\mathbb{T}_i}$ *and* $\gamma_i < 1$, *further assume* $g : [a_1, \infty)_{\mathbb{T}_1} \times \cdots \times [a_n, \infty)_{\mathbb{T}_n} \to \mathbb{R}_+$ *is such that the delta integrals* $\int_{\prod_{i=1}^{n} a_i}^{\infty} \prod_{i=1}^{n} (\sigma_i(t_i))^{\frac{p}{q}-\gamma_i} g^{p/q}(t_1, \ldots, t_n) \prod_{i=1}^{n} \Delta t_i$ *exist, for any* $(t_1, \ldots, t_n) \in [a_1, \infty)_{\mathbb{T}_1} \times \cdots \times [a_n, \infty)_{\mathbb{T}_n}$, *define*

$$\Omega_k(t_1, \ldots, t_n) = \int_{\prod_{j=1}^{k} a_j}^{t_j} g(s_1, \ldots, s_n) \prod_{j=1}^{k} \Delta s_j, \quad k \in \{1, \ldots, n\} \tag{44}$$

*then for* $p, q > 0$ *and* $p/q > 1$

$$\begin{aligned} &\int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{\Omega_n^{p/q}(t_1, \ldots, t_n)}{\prod_{i=1}^{n} \sigma_i^{\gamma_i}(t_i)} \prod_{i=1}^{n} \Delta t_i \\ &\leq \left(\frac{p}{q}\right)^{\frac{np}{q}} \prod_{i=1}^{n} \left(\frac{1}{1-\gamma_i}\right)^{p/q} \int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{1}{\prod_{i=1}^{n} (\sigma_i(t_i))^{\gamma_i - p/q}} g^{p/q}(t_1, \ldots, t_n) \prod_{i=1}^{n} \Delta t_i \end{aligned} \tag{45}$$

*holds, where $n$ is any positive integer.*

**Proof.** To prove the result, we use the principle of mathematical induction. For $n = 1$ the statement is true by Theorem 1.4. Let the statement be true for $1 \leq n \leq k$.

To prove the result for $n = k + 1$. The left-hand side of (45) can be written as

$$\int_{\prod_{i=1}^{n} a_i}^{\infty} \frac{\Omega_{k+1}^{p/q}(t_1, \ldots, t_{k+1})}{\prod_{i=1}^{k+1} \sigma_i^{\gamma_i}(t_i)} \prod_{i=1}^{k+1} \Delta t_i \tag{46}$$

Denote $\int_{a_{k+1}}^{\infty} \frac{\Omega_{k+1}^{p/q}(t_1,\ldots,t_{k+1})}{\sigma_{k+1}^{\gamma_{k+1}}(t_{k+1})} \Delta t_{k+1} = I_{k+1}$. Apply (5) with $\frac{\partial}{\Delta t_{k+1}} v(t_{k+1}) = \frac{1}{\sigma_{k+1}^{\gamma_{k+1}}(t_{k+1})}$ and $u(t_{k+1}) = \Omega_{k+1}^{p/q}(t_1, \ldots, t_{k+1})$. Thus

$$\begin{aligned} I_{k+1} &= v(t_{k+1}) \Omega_{k+1}^{p/q}(t_1, \ldots, t_{k+1}) \big|_{a_{k+1}}^{\infty} \\ &+ \int_{a_{k+1}}^{\infty} v^{\sigma_{k+1}}(t_{k+1}) \left(-\frac{\partial}{\Delta t_{k+1}} \Omega_{k+1}^{p/q}(t_1, \ldots, t_{k+1})\right) \Delta t_{k+1}, \end{aligned} \tag{47}$$

where $v(t_{k+1}) = \int_{a_{k+1}}^{t_{k+1}} 1/\sigma_{k+1}^{\gamma_{k+1}}(s_{k+1}) \Delta s_{k+1}$. Use chain rule (6) and the fact that $\sigma_{k+1}(s_{k+1}) \geq s_{k+1}$ to get

$$\begin{aligned} \frac{\partial}{\Delta s_{k+1}} (s_{k+1}^{1-\gamma_{k+1}}) &= (1 - \gamma_{k+1}) \int_0^1 [h_{k+1} \sigma_{k+1}(s_{k+1}) + (1 - h_{k+1}) s_{k+1}]^{-\gamma_{k+1}} dh_{k+1} \\ &\geq (1 - \gamma_{k+1}) \frac{1}{\sigma_{k+1}^{\gamma_{k+1}}(s_{k+1})}, \end{aligned}$$

which gives

$$v^{\sigma_{k+1}}(t_{k+1}) = \int_{a_{k+1}}^{\sigma_{k+1}(t_{k+1})} \frac{1}{\sigma_{k+1}^{\gamma_{k+1}}(s_{k+1})} \Delta s_{k+1} \leq \frac{1}{(1 - \gamma_{k+1})} (\sigma_{k+1}(t_{k+1}))^{1-\gamma_{k+1}}. \tag{48}$$

Combine (47), (48) and use the facts $\Omega_{k+1}(t_1, \ldots, t_k, \infty) = 0$, $v(a_{k+1}) = 0$ to get

$$I_{k+1} \leq \frac{1}{(1 - \gamma_{k+1})} \int_{a_{k+1}}^{\infty} \frac{-\frac{\partial}{\Delta t_{k+1}} \Omega_{k+1}^{p/q}(t_1, \ldots, t_{k+1})}{(\sigma_{k+1}(t_{k+1}))^{\gamma_{k+1}-1}} \Delta t_{k+1}. \tag{49}$$

Apply chain rule 2 (7) to find

$$-\frac{\partial}{\Delta t_{k+1}} \Omega_{k+1}^{p/q}(t_1, \ldots, t_{k+1}) = -\left(\frac{p}{q}\right) \Omega_{k+1}^{\frac{p}{q}-1}(t_1, \ldots, t_k, c_{k+1}) \frac{\partial}{\Delta t_{k+1}} \Omega_{k+1}(t_1, \ldots, t_{k+1}),$$

where, $c_{k+1} \in [t_{k+1}, \sigma_{k+1}(t_{k+1})]$. Since

$$\frac{\partial}{\Delta t_{k+1}}\Omega_{k+1}(t_1,\ldots,t_{k+1}) = -\int_{\prod_{i=1}^{k}a_i}^{\infty} g(s_1,\ldots,s_k,t_{k+1})\prod_{i=1}^{k}\Delta s_i$$
$$\doteq \Omega_k(t_1,\ldots,t_{k+1}) \leq 0,$$

and $c_{k+1} \geq t_{k+1}$, one has that

$$-\frac{\partial}{\Delta t_{k+1}}\Omega_{k+1}^{p/q}(t_1,\ldots,t_{k+1}) \leq \frac{p}{q}\Omega_{k+1}^{\frac{p}{q}-1}(t_1,\ldots,t_{k+1})\,\Omega_k\,(t_1,\ldots,t_{k+1}). \tag{50}$$

Substitute (50) into (49)

$$I_{k+1} \leq \frac{p}{q(1-\gamma_{k+1})}\int_{a_{k+1}}^{\infty}\frac{\Omega_{k+1}^{\frac{p}{q}-1}(t_1,\ldots,t_{k+1})}{(\sigma_{k+1}(t_{k+1}))^{\gamma_{k+1}-1}}\Omega_k(t_1,\ldots,t_{k+1})\Delta t_{k+1}. \tag{51}$$

Apply Hölder's inequality on the right-hand side of (51) with indices $p/q$ and $p/(p-q)$ to obtain

$$I_{k+1} \leq \frac{p}{q(1-\gamma_{k+1})}\left[\int_{a_{k+1}}^{\infty}\left[\frac{(\sigma_{k+1}^{\gamma_{k+1}}(t_{k+1}))^{\frac{p-q}{p}}}{(\sigma_{k+1}(t_{k+1}))^{\gamma_{k+1}-1}}\Omega_k(t_1,\ldots,t_{k+1})\right]^{p/q}\Delta t_{k+1}\right]^{q/p}$$
$$\times [I_{k+1}]^{\frac{p-q}{p}}.$$

After simplification, we get

$$I_{k+1} \leq \left(\frac{p}{q(1-\gamma_{k+1})}\right)^{p/q}\int_{a_{k+1}}^{\infty}\frac{\Omega_k^{p/q}(t_1,\ldots,t_{k+1})}{(\sigma_{k+1}(t_{k+1}))^{-\frac{p}{q}+\gamma_{k+1}}}\Delta t_{k+1}. \tag{52}$$

Substitute (52) into (46)

$$\begin{aligned}&\int_{\prod_{i=1}^{k+1}a_i}^{\infty}\frac{\Omega_{k+1}^{p/q}(t_1,\ldots,t_{k+1})}{\prod_{i=1}^{k+1}\sigma_i^{\gamma_i}(t_i)}\prod_{i=1}^{k+1}\Delta t_i\\ &\leq \int_{\prod_{i=1}^{k}a_i}^{\infty}\frac{1}{\prod_{i=1}^{k}\sigma_i^{\gamma_i}(t_i)}\left(\frac{p}{q(1-\gamma_{k+1})}\right)^{p/q}\int_{a_{k+1}}^{\infty}\frac{\Omega_k^{p/q}(t_1,\ldots,t_{k+1})}{(\sigma_{k+1}(t_{k+1}))^{-\frac{p}{q}+\gamma_{k+1}}}\prod_{i=1}^{k+1}\Delta t_i.\end{aligned} \tag{53}$$

Exchange integrals on right-hand side of (53) $k$-times by using (9)

$$\begin{aligned}&= \left(\frac{p}{q(1-\gamma_{k+1})}\right)^{p/q}\int_{a_{k+1}}^{\infty}\frac{1}{(\sigma_{k+1}(t_{k+1}))^{-\frac{p}{q}+\gamma_{k+1}}}\\ &\times\left\{\int_{\prod_{i=1}^{k}a_i}^{\infty}\frac{\Omega_k^{p/q}(t_1,\ldots,t_{k+1})}{\prod_{i=1}^{k}\sigma_i^{\gamma_i}(t_i)}\prod_{i=1}^{k}\Delta t_i\right\}\Delta t_{k+1}.\end{aligned} \tag{54}$$

Use the induction hypothesis for $\Omega_k(t_1,\ldots,t_{k+1})$ in (54) instead for $\Omega_k(t_1,\ldots,t_k)$ for fixed $t_{k+1}\in\mathbb{T}_{k+1}$ and again apply (9) $k$ times to get

$$\begin{aligned}&\int_{\prod_{i=1}^{k+1}a_i}^{\infty}\frac{\Omega_{k+1}^{p/q}(t_1,\ldots,t_{k+1})}{\prod_{i=1}^{k+1}\sigma_i^{\gamma_i}(t_i)}\prod_{i=1}^{k+1}\Delta t_i\\&\le\left(\frac{p}{q}\right)^{\frac{(k+1)p}{q}}\prod_{i=1}^{k+1}\left(\frac{1}{1-\gamma_i}\right)^{p/q}\int_{\prod_{i=1}^{k+1}a_i}^{\infty}\prod_{i=1}^{k+1}\frac{1}{(\sigma_i(t_i))^{\gamma_i-p/q}}g^{p/q}(t_1,\ldots,t_{k+1})\prod_{i=1}^{k+1}\Delta t_i.\end{aligned}$$

Hence by induction principle, the statement is true $\forall\, n\in\mathbb{N}$. □

**Corollary 6.2.** *Under the conditions of Theorem 6.1, we get the following inequality*

$$\begin{aligned}&\int_{\prod_{i=1}^{n}a_i}^{\infty}\frac{1}{\prod_{i=1}^{n}\sigma_i^{\gamma_i}(t_i)}(\Omega_n(\sigma_1(t_1),\ldots,\sigma_n(t_n)))^{p/q}\prod_{i=1}^{n}\Delta t_i\\&\le\left(\frac{p}{q}\right)^{\frac{np}{q}}\prod_{i=1}^{n}\left(\frac{1}{1-\gamma_i}\right)^{p/q}\int_{\prod_{i=1}^{n}a_i}^{\infty}\prod_{i=1}^{n}\frac{1}{(\sigma_i(t_i))^{\gamma_i-p/q}}g^{p/q}(t_1,\ldots,t_n)\prod_{i=1}^{n}\Delta t_i.\end{aligned}\tag{55}$$

***Proof.*** The fact $\frac{\partial^n\Omega_n}{\Delta t_1\cdots\Delta t_n}\le 0$ implies

$$\begin{aligned}&\int_{\prod_{i=1}^{n}a_i}^{\infty}\frac{1}{\prod_{i=1}^{n}\sigma_i^{\gamma_i}(t_i)}(\Omega_n(\sigma_1(t_1),\ldots,\sigma_n(t_n)))^{p/q}\prod_{i=1}^{n}\Delta t_i\\&\le\int_{\prod_{i=1}^{n}a_i}^{\infty}\frac{1}{\prod_{i=1}^{n}\sigma_i^{\gamma_i}(t_i)}(\Omega_n(t_1,\ldots,t_n))^{p/q}\prod_{i=1}^{n}\Delta t_i.\end{aligned}\tag{56}$$

Now use (45) in (56) to get (55). □

**Remark 6.3.** Consider $\mathbb{T}_1=\cdots=\mathbb{T}_n=\mathbb{R}$, $p/q=\lambda>1$ and $\gamma_1,\ldots,\gamma_n<1$, in Theorem 6.1. Denote $G(t_1,\ldots,t_n)=\int_{\prod_{i=1}^{n}t_i}^{\infty}g(s_1,\ldots,s_n)\prod_{i=1}^{n}ds_i$. Thus, (45) takes the form

$$\begin{aligned}&\int_{\prod_{i=1}^{n}a_i}^{\infty}\frac{1}{\prod_{i=1}^{n}t_i^{\gamma_i}}(G^{\lambda}(t_1,\ldots,t_n))\prod_{i=1}^{n}dt_i\\&\le\prod_{i=1}^{n}\left(\frac{\lambda}{1-\gamma_i}\right)^{\lambda}\int_{\prod_{i=1}^{n}a_i}^{\infty}\frac{1}{\prod_{i=1}^{n}(t_i)^{\gamma_i-\lambda}}\frac{\partial^n}{\partial t_1\ldots\partial t_n}G^{\lambda}(t_1,\ldots,t_n)\prod_{i=1}^{n}dt_i,\end{aligned}$$

which can be considered as a generalization of Wirtinger's inequality [1].

**Remark 6.4.** As a special case of Theorem 6.1, assume that $\mathbb{T}_1=\cdots=\mathbb{T}_n=\mathbb{N}$, $p/q=\lambda>1$, $a_1=\cdots=a_n=1$ and $\gamma_1,\ldots,\gamma_n<1$. In this case (55) becomes the following discrete Hardy and Littlewood inequality

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{1}{\prod_{i=1}^{n}(m_i+1)^{\gamma_i}}\left(\sum_{k_1=m_1+1}^{\infty}\cdots\sum_{k_n=m_n+1}^{\infty}g(k_1,\ldots,k_n)\right)^{\lambda}$$
$$\leq\prod_{i=1}^{n}\left(\frac{\lambda}{1-\gamma_i}\right)^{\lambda}\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{1}{\prod_{i=1}^{n}(m_i+1)^{\gamma_i-\lambda}}g^{\lambda}(m_1,\ldots,m_n).$$

## 7. Hardy and Littlewood-type inequalities for $p/q \leq 2$ and $\gamma < 1$

**Theorem 7.1.** *Assume $i\in\{1,\ldots,n\}$, $\mathbb{T}_i$ is a time scale with $a_i\in(0,\infty)_{\mathbb{T}_i}$ and $\gamma_i<1$, further assume $g:[a_1,\infty)_{\mathbb{T}_1}\times\cdots\times[a_n,\infty)_{\mathbb{T}_n}\to\mathbb{R}_+$ is such that the delta integrals $\int_{\prod_{i=1}^{n}a_i}^{\infty}\prod_{i=1}^{n}(\sigma_i(t_i))^{\frac{p}{q}-\gamma_i}g^{p/q}(t_1,\ldots,t_n)\prod_{i=1}^{n}\Delta t_i$ exist, then for $p,q>0$ and $p/q\leq 2$. Then*

$$\int_{\prod_{i=1}^{n}a_i}^{\infty}\frac{1}{\prod_{i=1}^{n}\sigma_i^{\gamma_i}(t_i)}\left(\int_{\prod_{i=1}^{n}t_i}^{\infty}g(s_1,\ldots,s_n)\prod_{i=1}^{n}\Delta s_i\right)^{p/q}\prod_{i=1}^{n}\Delta t_i$$
$$\leq\left(\frac{p}{q}\right)^{\frac{np}{q}}\prod_{i=1}^{n}\left(\frac{2}{1-\gamma_i}\right)^{p/q}\int_{\prod_{i=1}^{n}a_i}^{\infty}\prod_{i=1}^{n}\frac{1}{(\sigma_i(t_i))^{\gamma_i-p/q}}g^{p/q}(t_1,\ldots,t_n)\prod_{i=1}^{n}\Delta t_i. \tag{57}$$

**Proof:** Use (12) and proceed as in the proof of Theorem 6.1 to get (57). ☐

**Remark 7.2.** In Theorem 7.1, when $\mathbb{T}_1=\cdots=\mathbb{T}_n=\mathbb{R}$, $p/q=\lambda>1$ and $\gamma_i<1$, (57) becomes the following Wirtinger type inequality,

$$\int_{\prod_{i=1}^{n}a_i}^{\infty}\frac{1}{\prod_{i=1}^{n}t_i^{\gamma_i}}(G(t_1,\ldots,t_n))^{\lambda}\prod_{i=1}^{n}dt_i$$
$$\leq\prod_{i=1}^{n}\left(\frac{2\lambda}{1-\gamma_i}\right)^{\lambda}\int_{\prod_{i=1}^{n}a_i}^{\infty}\frac{1}{\prod_{i=1}^{n}(t_i)^{\gamma_i-\lambda}}\left(\frac{\partial^n}{\partial t_1\ldots\partial t_n}G(t_1,\ldots,t_n)\right)^{\lambda}\prod_{i=1}^{n}dt_i,$$

where $G(t_1,\ldots,t_n)\doteq\int_{\prod_{i=1}^{n}t_i}^{\infty}g(s_1,\ldots,s_n)\prod_{i=1}^{n}ds_i$.

**Remark 7.3.** In Theorem 7.1, assume that $\mathbb{T}_1=\cdots=\mathbb{T}_n=\mathbb{N}$, $p/q=\lambda>1$, $a_1=\cdots=a_n=1$ and $\gamma_i<1$. (57) becomes the following discrete Hardy and Littlewood inequality

$$\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{1}{\prod_{i=1}^{n}(m_i+1)^{\gamma_i}}\left(\sum_{k_1=m_1+1}^{\infty}\cdots\sum_{k_n=m_n+1}^{\infty}g(k_1,\ldots,k_n)\right)^{\lambda}$$
$$\leq\prod_{i=1}^{n}\left(\frac{2\lambda}{1-\gamma_i}\right)^{\lambda}\sum_{m_1=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\frac{1}{\prod_{i=1}^{n}(m_i+1)^{\gamma_i-\lambda}}g^{\lambda}(m_1,\ldots,m_n).$$

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