Generators and number fields for torsion points of a special elliptic curve

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Abstract
Let $E$ be an elliptic curve with Weierstrass form $y^2 = x^3 - px$, where $p$ is a prime number and let $E[m]$ be its $m$-torsion subgroup. Let $p_1 = (x_1,y_1)$ and $p_2 = (x_2,y_2)$ be a basis for $E[m]$, then we prove that $\mathbb{Q}(E[m]) = \mathbb{Q}(x_1,x_2,\xi_m,y_1)$ in general. We also find all the generators and degrees of the extensions $\mathbb{Q}(E[m])/\mathbb{Q}$ for $m = 3$ and $m = 4$.

Keywords: Elliptic curves, Torsion points, Algebraic extensions

1. Introduction
Let $E$ be an elliptic curve with Weierstrass form $y^2 = x^3 - px$, where $p$ is a prime number. Let $m$ be a positive number, we denote by $E[m]$ the $m$-torsion subgroup of $E$, by $\mathbb{Q}(E[m])$ the number field generated by the coordinates of the $m$-torsion points of $E$, and by $\mathbb{Q}(E_{2^m})$ the number field generated by the abscissas of $m$-torsion points of $E$. Mazur proves the $m$-torsion subgroup is isomorphic to one of 15 finite groups \[5\]. Let $p_1 = (x_1,y_1)$ and $p_2 = (x_2,y_2)$ be two points in $E$ forming a basis of $E[m]$, then $\mathbb{Q}(E[m]) = \mathbb{Q}(x_1,x_2,y_1,y_2)$. By Artin’s primitive element theorem the extension $\mathbb{Q}(x_1,x_2,y_1,y_2)/\mathbb{Q}$ is monogeneous and we can find unique generator for $\mathbb{Q}(x_1,x_2,y_1,y_2)/\mathbb{Q}$ by combining the above coordinates. As usual, we denote by $\mu_m$ the group of $m$th roots of unity and by $\xi_m$ one of its generators. By Weil pairing, we have $\xi_m \in \mathbb{Q}(E[m])$, so $\mathbb{Q}(\xi_m) \subseteq \mathbb{Q}(E[m])$ for all $m$ \[5\]. In \[3\] Paladino gives a family of elliptic curves such that $\mathbb{Q}(E[3]) = \mathbb{Q}(\xi_3)$ and in \[4\] finds the number fields generated by the 4th torsion points, degrees and Galois groups of an elliptic curve $y^2 = (x - \alpha)(x - \beta)(x - \gamma)$ where $\alpha, \beta, \gamma \in \mathbb{Q}$, and $\alpha \neq \beta \neq \gamma$. In \[1\] Bandini and Paladino determine the number fields generated by the 3-torsion points, degrees and Galois groups of an elliptic curve $y^2 = x^3 + c$ where $c \in \mathbb{Q}^*$. In \[2\] the result of Brau and Jones says that the rational points on the modular

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curve of level 6 yield elliptic curve $E$ satisfying the given containment. In the first part of this paper we prove $\bar{\xi}_m \in \bar{Q}(E_3[m])$ and $Q(E[m]) = Q(x_1, x_2, \bar{\xi}_m, y_1)$ for all $m$. In the second part of this paper we find the number fields of torsion points $E[m]$ for cases $m = 3, 4$, extensions and degrees. These theorems have applications in local–global divisibility problem [4] and modular curves [2].

2. Generators for $Q(E[m])$

Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ form a basis of $E[m]$. We have $Q(E[m]) = Q(x_1, x_2, y_1, y_2)$. We will denote by $L$ the field $Q(x_1, x_2)$ and by $K$ the field $Q(E[m])$. Suppose $(x_3, y_3)$ be the coordinates of the point $p_3 = p_1 + p_2$ and $(x_4, y_4)$ be the coordinates of the point $p_4 = p_1 - p_2$. In next theorem we will prove $\bar{\xi}_m \in Q(E_3[m])$ for all $m$.

Lemma 2.1. Let $\{P, Q\}$ be a basis for $E[m]$. Then $e_m(P, Q)$ is a primitive $m$th root of unity.

Proof. We know that there are $S, T \in E[m]$ such that $e_m(S, T) = \bar{\xi}_m$, a primitive $m$th root of unity. Write $S = aP + bQ$ and $T = cP + dQ$. Then the antisymmetry properties of the Weil pairing imply that

$$\bar{\xi}_m = e_m(S, T) = e_m(P, Q)^{ad-bc}.$$

Since $e_m(P, Q)$ is an $m$th root of unity and a power of it is a primitive $m$th root of unity, it follows that $e_m(P, Q)$ is a primitive $m$th root of unity.

Theorem 2.2. Let $\{p_1, p_2\}$ be a basis for $E[m]$, let $p_3 = p_1 + p_2$ and $p_4 = p_1 - p_2$, and write $p_i = (x_i, y_i)$. Then

$$Q(\bar{\xi}_m) \subseteq Q(x_1, x_2, x_3, x_4) \subseteq Q(E[m]).$$

Proof. The second inclusion is by the definition of $Q(E_3[m])$. For the first inclusion, let $\sigma$ be an automorphism of $Q(E[m])$ that fixes $Q(x_1, x_2, x_3, x_4)$. Then $\sigma(y_i) = \pm y_i$, since $\sigma(y_i^2) = y_i^2$. The equation

$$y_1y_2 = \frac{(x_4 - x_3)(x_1 - x_2)^2}{4}$$

shows that $\sigma(y_1y_2) = y_1y_2$. This means that either $\sigma(y_i) = y_i$ for $i = 1, 2$, or $\sigma(y_i) = -y_i$ for $i = 1, 2$. These mean that either $\sigma(p_i) = p_i$ for $i = 1, 2$, or $\sigma(p_i) = -p_i$ for $i = 1, 2$. In the first case,

$$e_m(p_1, p_2)^\sigma = e_m(\sigma(p_1), \sigma(p_2)) = e_m(p_1, p_2).$$

In the second case,

$$e_m(p_1, p_2)^\sigma = e_m(\sigma(p_1), \sigma(p_2)) = e_m(-p_1, -p_2) = e_m(p_1, p_2).$$

Since $e_m(p_1, p_2)$ is a primitive $m$th root of unity, we find that $Q(\bar{\xi}_m) \subseteq Q(x_1, x_2, x_3, x_4)$.

We know that $Q(x_1, x_2, y_1, y_2) = Q(x_1, x_2, y_1y_2)$. In next theorem we will prove that $Q(E[m])$ is equal to the field $Q(x_1, x_2)$ by adding $\bar{\xi}_m$ and $y_1$.

Theorem 2.3. $Q(E[m]) = Q(x_1, x_2, \bar{\xi}_m, y_1)$.

Proof. We have $Q(x_1, x_2, \bar{\xi}_m, y_1, y_2) = Q(E[m])$. If we do not have the equality in the theorem, then $y_2 \notin Q(x_1, x_2, \bar{\xi}_m, y_1)$. Since $y_2^2$ is in this field, there is an automorphism $\sigma$ such that $\sigma(y_2^2) = y_2$ and $\sigma$ is the identity on $Q(x_1, x_2, \bar{\xi}_m, y_1)$. Then
\[ e_m(p_1, p_2) = e_m(p_1, p_2)^\sigma = e_m(\sigma(p_1), \sigma(p_2)) = e_m(p_1, -p_2) = e_m(p_1, p_2)^{-1}. \]

This implies that \( e_m(p_1, p_2)^2 = 1 \). Since \( e_m(p_1, p_2) \) is a primitive \( m \)th root of unity, we must have \( m = 2 \). But then \( y_1 = y_2 = 0 \), in which case the theorem is true. \( \square \)

3. Number fields \( \mathbb{Q}(E'[m]) \) for cases \( m = 3, 4 \)
It is well known that the abscissas of the 3-torsion points of an elliptic curve \( y^2 = x^3 - px \) are the roots of the polynomial
\[ \varphi_3 = 3x^4 - 6px^2 - p^2, \]
then the roots \( \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4 \) of \( \varphi_3 \) are:
\[ \hat{x}_1 = \sqrt[p - 2p/3, \hat{x}_2 = -\sqrt[p - 2p/3, \hat{x}_3 = \sqrt[p + 2p/3, \hat{x}_4 = -\sqrt[p + 2p/3, \]
In next theorems we will determine the field generated by 3 and 4 torsion points.

Theorem 3.1. Let \( E \) be an elliptic curve with Weierstrass form \( E : y^2 = x^3 - px \), where \( p \) is a prime number. Then
\[ \mathbb{Q}(E[3]) = \mathbb{Q}\left(\sqrt[p - 2p/3, \xi_3\right) \quad \text{with} \quad \left[\mathbb{Q}(E[3]) : \mathbb{Q}\right] = 8, \]
\[ \mathbb{Q}(E[3]) = \mathbb{Q}\left(\sqrt[2p\sqrt{2p - 3}, \xi_3\right) \quad \text{with} \quad \left[\mathbb{Q}(E[3]) : \mathbb{Q}\right] = 16. \]

Proof. We have \( \mathbb{Q}(\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4) = \mathbb{Q}(\hat{x}_1, \hat{x}_3) \). On the other hand we have
\[ \hat{x}_1\hat{x}_3 = \sqrt{\left(p - \frac{2p}{\sqrt{3}}\right)\left(p + \frac{2p}{\sqrt{3}}\right)} = \sqrt{\frac{p^2}{3} = \frac{-3p}{3}}, \]
so \( \mathbb{Q}(\hat{x}_1, \hat{x}_3) = \mathbb{Q}(\hat{x}_1, \hat{x}_1\hat{x}_3) = \mathbb{Q}(\hat{x}_1, \xi_3) = \mathbb{Q}\left(\sqrt[p - 2p/3, \xi_3\right) . \)
We have
\[ \left[\mathbb{Q}\left(\sqrt[p - 2p/3, \xi_3\right) : \mathbb{Q}\right] = \left[\mathbb{Q}\left(\sqrt[p - 2p/3, \xi_3\right) : \mathbb{Q}(\xi_3\right) \left[\mathbb{Q}(\xi_3) : \mathbb{Q}\right]. \]
Put \( \alpha = \sqrt[p - 2p/3, \), then
\[ f(x) = \min(\alpha, Q(\xi_3)) = 3\alpha^4 + 6\alpha\alpha^2 - p^2 = 0 \]
is irreducible over \( \mathbb{Q}(\xi_3) \), because the roots of \( f(x) \) are \( \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4 \). They are irrational, so either \( f(x) \) is irreducible or it has a quadratic factor that has \( \hat{x}_1 \) and some other \( \hat{x}_i \) as roots. Since \( \hat{x}_1, \hat{x}_2 \not\in \mathbb{Q}(\xi_3) \), the other root is not \( \hat{x}_2 \). Suppose the other root is \( \hat{x}_3 \) or \( \hat{x}_4 \). Then (using \( \hat{x}_3 \))
\[ \frac{2p}{3} \left(3 \pm \sqrt{-3}\right) = (\hat{x}_1 + \hat{x}_3)^2 \]
is a square in $\mathbb{Q}(\xi_3)$. But its norm to $\mathbb{Q}$ is $\frac{16p^6}{3}$, which is not a square, so it cannot be a square.

Therefore, there is no quadratic factor and $f(x)$ is irreducible. So

$$[Q(\sqrt[3]{p} - \sqrt[3]{2p} \frac{\sqrt{3}}{3}, \xi_3) : Q(\xi_3)] = 4.$$ 

It is easy to verify that $[Q(\xi_3) : Q] = 2$. Hence

$$[Q(E_3[3] : Q)] = [Q\left(\sqrt[3]{p} - \sqrt[3]{2p} \frac{\sqrt{3}}{3}, \xi_3\right) : Q] = 4 \cdot 2 = 8.$$ 

By Theorem 2.2 we proved that $Q(E[3]) = Q(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{y}_1) = Q(\tilde{x}_1, \tilde{x}_3, \tilde{y}_1)$, where $\tilde{x}_1 = -\tilde{y}_2$. As $\tilde{y}_1^2 = \tilde{x}_1^3 - p\tilde{x}_1$, then

$$\tilde{y}_1 = \sqrt{\tilde{x}_1^3 - p\tilde{x}_1} = \sqrt{\left(\sqrt[3]{p} - \sqrt[3]{2p} \frac{\sqrt{3}}{3}\right)^3 - p\left(\sqrt[3]{p} - \sqrt[3]{2p} \frac{\sqrt{3}}{3}\right)} = \sqrt{\frac{2p\sqrt{2p^3 - 3p}}{3}},$$

and $[Q(\tilde{x}_1, \tilde{x}_3, \tilde{y}_1) : Q(\tilde{x}_1, \tilde{x}_3)] = 2$. We found in previous case that $[Q(\tilde{x}_1, \tilde{x}_3) : Q] = 8$. Hence

$$[Q(E[3]) : Q] = [Q(\tilde{x}_1, \tilde{x}_3, \tilde{y}_1) : Q] = [Q(\tilde{x}_1, \tilde{x}_3) : Q][Q(\tilde{x}_1, \tilde{x}_3) : Q] = 2 \cdot 8 = 16. \quad \square$$

It is well known that the abscissas of the 4-torsion points of an elliptic curve $y^2 = x^3 - px$ are the roots of the polynomial

$$\varphi_4 = x^6 - 5px^4 - 5p^2x^2 + p^3,$$

then the roots $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6$ of $\varphi_4$ are

$$\tilde{x}_1 = i\sqrt{p}, \quad \tilde{x}_2 = +\sqrt{p} + \sqrt{2p}, \quad \tilde{x}_3 = -i\sqrt{p},$$

$$\tilde{x}_4 = \sqrt{p} - \sqrt{2p}, \quad \tilde{x}_5 = -\sqrt{p} + \sqrt{2p}, \quad \tilde{x}_6 = \sqrt{p} - \sqrt{2p}.$$ 

**Theorem 3.2.** Let $E$ be an elliptic curve with Weierstrass form $y^2 = x^3 - px$, where $p$ is a prime number. Then

$$Q(E[4]) = \begin{cases} Q(i, \sqrt{2}, \sqrt{p}) & \text{with } [Q(E[4]) : Q] = 8 \text{ if } p \neq 2, \\ Q(i, \sqrt{2}) & \text{with } [Q(E[4]) : Q] = 4 \text{ if } p = 2. \end{cases}$$

Proof. The points of exact order 4 of $y^2 = x^3 - px$ are $\pm p_1, \pm p_2, \pm p_3, \pm p_4, \pm p_5, \pm p_6$, where

$$p_1 = \left(\sqrt{p}, -\sqrt{p^3} + i\sqrt{p^3}\right), \quad p_2 = \left(\sqrt{p} + \sqrt{2p}, 2\sqrt{p^3} + \sqrt{2} \sqrt[p]{p^3}\right),$$

$$p_3 = \left(-i\sqrt{p}, -\sqrt{p^3} - i\sqrt{p^3}\right), \quad p_4 = \left(\sqrt{p} - \sqrt{2p}, -2\sqrt{p^3} + \sqrt{2} \sqrt[p]{p^3}\right),$$

$$p_5 = \left(-\sqrt{p} + \sqrt{2p}, \frac{2p}{\sqrt[p]{p^3}} + \frac{2p}{i\sqrt[2]{p^3}}\right), \quad p_6 = \left(-\sqrt{p} - \sqrt{2p}, \frac{2p}{\sqrt[p]{p^3}} - \frac{2p}{i\sqrt[2]{p^3}}\right).$$

We have:
\[
\mathbb{Q}(E_4[4]) = \mathbb{Q}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4, \tilde{x}_5, \tilde{x}_6)
\]
\[
= \mathbb{Q}\left(i\sqrt{p}, \sqrt{p} + \sqrt{2p}, -i\sqrt{p}, \sqrt{p} - \sqrt{2p}, -\sqrt{p} + \sqrt{2p}, -\sqrt{2} - \sqrt{2p}\right)
\]
\[
= \mathbb{Q}\left(i, \sqrt{2}, \sqrt{p}\right)
\]

with \([\mathbb{Q}(E_4[4]) : \mathbb{Q}] = 8\) if \(p \neq 2\) and \([\mathbb{Q}(E_4[4]) : \mathbb{Q}] = 4\) if \(p = 2\) \(\square\)

Let \(\{p_1, p_2\}\) be a basis for \(E[4]\), then

\[
\mathbb{Q}(E[4]) = \mathbb{Q}(\tilde{x}_1, \tilde{x}_2, \tilde{y}_1, \tilde{y}_2)
\]
\[
= \mathbb{Q}\left(i\sqrt{p}, \sqrt{p} + \sqrt{2p}, -\sqrt{p^3} + i\sqrt{p^3}, 2\sqrt{p^3} + \sqrt{2}\right)
\]
\[
= \mathbb{Q}\left(i, \sqrt{2}, \sqrt{p^3}\right)
\]

with \([\mathbb{Q}(E[4]) : \mathbb{Q}] = 16\) if \(p \neq 2\) and \([\mathbb{Q}(E[4]) : \mathbb{Q}] = [\mathbb{Q}(i, \sqrt{8})] = 8\) if \(p = 2\) \(\square\)

References


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