Structure and substructure connectivity of circulant graphs and hypercubes

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Abstract
Let $H$ be a connected subgraph of a connected graph $G$. The $H$-structure connectivity of the graph $G$, denoted by $\kappa(G; H)$, is the minimum cardinality of a minimal set of subgraphs $F = \{H_1', H_2', \ldots, H_m'\}$ in $G$, such that every $H_i' \in F$ is isomorphic to $H$ and removal of $F$ from $G$ will disconnect $G$. The $H$-substructure connectivity of the graph $G$, denoted by $\kappa^*(G; H)$, is the minimum cardinality of a minimal set of subgraphs $F = \{J_1', J_2', \ldots, J_n'\}$ in $G$, such that every $J_i' \in F$ is a connected subgraph of $H$ and removal of $F$ from $G$ will disconnect $G$. In this paper, we provide the $H$-structure and the $H$-substructure connectivity of the circulant graph Cir$(n, \Omega)$ where $\Omega = \{1, \ldots, k, n - k, \ldots, n - 1\}, 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$ and the hypercube $Q_n$ for some connected subgraphs $H$.

Keywords Structure connectivity, Substructure connectivity, Circulant graph, Hypercube

Paper type Original Article

1. Introduction
A simple graph $G = (V, E)$ is a finite nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of $G$ called edges. Two distinct vertices $u, v \in V(G)$ are said to be adjacent in $G$ if $u$ and $v$ are connected by an edge and it is represented by $\{u, v\} \in E(G)$. A graph $G$ is said to be trivial if it contains only one vertex and no edges. The connectivity is an important indicator of the reliability and fault tolerability of a network. The vertex connectivity of a connected graph $G$, denoted by $\kappa(G)$, is the minimum cardinality of a vertex subset $S \subseteq V(G)$, whose removal would disconnect $G$ or

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G \ S is the trivial graph. As a generalization of the vertex connectivity \( \kappa(G) \), Cheng-Kuan Lin et al. [6] introduced two new kinds of connectivity, called structure connectivity and substructure connectivity. A set \( F \) of connected subgraphs of a graph \( G \) is a subgraph-cut of \( G \) if \( G \setminus V(F) \) is a disconnected graph or \( K_1 \). Let \( H \) be a connected subgraph of \( G \). Then \( F \) is an \( H \)-structure cut if \( F \) is a subgraph cut, and every element in \( F \) is isomorphic to \( H \). The \( H \)-structure connectivity of \( G \), denoted by \( \kappa(G;H) \), is defined to be the minimum cardinality of all \( H \)-structure cuts of \( G \). \( F \) is an \( H \)-substructure cut if \( F \) is a subgraph-cut, and every element in \( F \) is isomorphic to a connected subgraph of \( H \). The \( H \)-substructure connectivity of \( G \), denoted by \( \kappa^s(G;H) \), is the minimum cardinality of all \( H \)-substructure cuts of \( G \). Since every \( H \)-structure cut is an \( H \)-substructure cut \( \kappa(L(G);H) \leq \kappa(G;H) \). If \( H = K_1 \) then we have \( \kappa(G;H) = \kappa^s(G;H) \).

The vertex connectivity \( \kappa(G) \geq \kappa^s(G;H) \) for every subgraph \( H \) of \( G \) whereas the relation between vertex connectivity and structure connectivity depends on \( H \). For the graph \( G \), given in Figure 1, \( \kappa(G) = 2 \), the structure connectivity of \( G \) with respect to the cycle of length \( 5 \), \( \kappa(G;C_5) = 1 \) and the structure connectivity of \( G \) with respect to the cycle of length \( 4 \), \( \kappa(G;C_4) = 3 \).

Let \( \Gamma \) be a finite group with \( e \) as the identity. A generating set of \( \Gamma \) is a subset \( \Omega \) such that every element of \( \Gamma \) can be expressed as a product of finitely many elements in \( \Omega \). Assume that \( e \notin \Omega \) and \( a \in \Omega \) implies \( a^{-1} \in \Omega \) and such a subset \( \Omega \) is called as a symmetric generating set of \( \Gamma \). Hereafter, we assume that \( \Omega \) is a symmetric generating set of a finite group \( \Gamma \). A Cayley graph is a graph \( G = (V, E) \), where \( V(G) = \Gamma \) and two vertices \( x \) and \( y \) are adjacent if \( xy^{-1} \in \Omega \) and it is denoted by \( \text{Cay}(\Gamma, \Omega) \). The inclusion of the inverse in \( \Omega \) for every element of \( \Omega \) means that \( \text{Cay}(\Gamma, \Omega) \) is undirected. Since \( \Omega \) is a generating set for \( \Gamma \), \( \text{Cay}(\Gamma, \Omega) \) is connected and \( \text{Cay}(\Gamma, \Omega) \) is a regular graph of degree \( |\Omega| \). Cayley graphs are extensively dealt in the literature and various authors including Dejter [3], Lakshimivarahan [4], Lee [5], Tamizh Chelvam [8], and Wang [10] have worked on Cayley graphs. For example, one can refer the survey by Tamizh Chelvam and Sivagami [9] for domination in Cayley graphs. The Cayley graph constructed out of the finite cyclic group \( \mathbb{Z}_n \), \( n \geq 2 \) along with a symmetric generating set \( \Omega \) is called a circulant graph and the same is denoted by \( \text{Cir}(n, \Omega) \). The hypercube \( Q_n \) is the Cayley graph defined on the group \( \mathbb{Z}_2^n \) with the standard orthonormal basis as the generating set. Cheng-Kuan Lin et al. [6] have obtained \( \kappa(Q_n;H) \) and \( \kappa^s(Q_n;H) \) for \( H \in \{K_1, K_{1,1}, K_{1,2}, K_{1,3}, C_4 \} \). Here, we provide an example in Figure 2, to exhibit a structure cut of the circulant graph \( \text{Cir}(10, \{1, 2, 3, 7, 8, 9\}) \) with respect to \( K_3 \). In Figure 2,

![Figure 1](image-url)

**Figure 1.**
\( \kappa(G) = 2, \kappa(G;C_5) = 1, \kappa(G;C_4) = 3. \)
the structure cut is indicated by the dotted lines and note that $\kappa(Cir(10,\{1,2,3,7,8,9\};K_3)) = 2$.

Throughout this paper, $G - X$ denotes the removal of a set $X$ of subgraphs from the graph $G$ and $G \setminus B$ denotes the removal of the set $B \subseteq V(G)$ from the graph $G$. When $X = \{H\}$, $G - X$ is simply denoted as $G - H$. For a graph $G$, the open neighborhood $N(v)$ of a vertex $v \in V(G)$ is the set of all vertices which are adjacent to $v$. The closed neighborhood of $v$ is $N[v] = N(v) \cup \{v\}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the intersection of $G_1$ and $G_2$, denoted by $G_1 \cap G_2$, is the graph whose vertex set is $V_1 \cap V_2$ and the edge set is $E_1 \cap E_2$. The union of two disjoint vertex sets graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$. For basic definitions and properties related to graph theory, one can refer [2]. For undefined definitions related to algebraic graph theory, one can refer [1].

In Section 2, we obtain the $H$-structure and the $H$-substructure connectivity of the circulant graph $Cir(n, \Omega)$ where $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$, for $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ with respect to some of its connected subgraphs. For integers $n \geq 5$ and $m$ with $2 \leq m \leq n-2$, Mane [7] proved that $\kappa(Q_n; C_k) \leq n - m$, where $k$ is a positive even integer with $2^m < k < 2^{m+1}$ and observed that $\kappa(Q_4; C_6) = 2$. In Section 3, for $n \geq 4$, we obtain the exact value for $\kappa(Q_n; C_6)$.

2. Structure and substructure connectivity of circulant graphs

Throughout this section $n \geq 2$, $m$ and $k$ are integers such that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and $\Omega = \{1, 2, \ldots, k, n-k, \ldots, n-1\}$. We take the elements of $Z_n$ as $Z_n = \{0, 1, \ldots, n-1\}$. The following result due to Wang [10] is useful in the paper.

**Theorem 2.1** ([10, Wang]). Let $n$ and $k$ be positive integers such that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$ and $G = Cir(n, \Omega)$. Then $\kappa(G) = |\Omega|$.
By definition and from Theorem 2.1, we have the following corollary.

**Corollary 2.2.** Let \( n \geq 2 \) and \( k \) be positive integers such that \( 1 \leq k \leq \lfloor \frac{n}{2} \rfloor \), \( \Omega = \{1, \ldots, k, n-k, \ldots, n-1\} \) and \( G = \text{Cir}(n, \Omega) \). Then \( \kappa(G; K_1) = |\Omega| \) and \( \kappa^*(G; K_1) = |\Omega| \).

A star graph \( K_{1,m} \) \((m \geq 1)\) is a complete bipartite graph comprised of two partite sets of vertices of sizes 1 and \( m \) respectively, such that two vertices are adjacent if and only if they are in different partite sets. If \( K_{1,m} \) is a subgraph of \( \text{Cir}(n, \Omega) \), then \( m \leq |\Omega| \). If \( k = \lfloor \frac{n}{2} \rfloor \) and \( \Omega = \{1, \ldots, k, n-k, \ldots, n-1\} \), then the circulant graph \( G = \text{Cir}(n, \Omega) \) is the complete graph \( K_n \). Now, we obtain the structure connectivity of \( K_n \) as a circulant graph with respect to \( K_{1,m} \). If \( m + 1 \) does not divide \( n - 1 \), then removal of \( \lambda K_{1,m} \) does not disconnect \( \text{Cir}(n, \Omega) \) for any \( \lambda \). Hence the structure connectivity of \( K_n \) with respect to \( K_{1,m} \) is meaningful only when \( m + 1 \) divides \( n - 1 \).

**Lemma 2.3.** Let \( n \geq 2 \) and \( k \) be positive integers such that \( k = \lfloor \frac{n}{2} \rfloor \), \( \Omega = \{1, \ldots, k, n-k, \ldots, n-1\} \) and \( G = \text{Cir}(n, \Omega) \). For a positive integer \( m \) with \( m \leq n - 2 \), \( \kappa^*(G; K_{1,m}) = \left\lceil \frac{n-1}{m+1} \right\rceil \). Also, \( \kappa(G; K_{1,m}) = \frac{n-1}{m+1} \) if \( m + 1 \) divides \( n - 1 \).

**Proof.** By the assumption on \( n, k \) and \( \Omega \), \( \text{Cir}(n, \Omega) = K_n \). By Theorem 2.1, \( G \) is \((n-1)\)-connected. Let \( F \) be a \( K_{1,m} \)-substructure cut with minimum cardinality of \( G = \text{Cir}(n, \Omega) \).

Suppose \( \kappa^*(G; K_{1,m}) < \left\lceil \frac{n-1}{m+1} \right\rceil \). Then \(|V(F)| < n-1 \) and \( G \setminus V(F) \) is disconnected, which is a contradiction to \( G \) is \((n-1)\)-connected. Hence \( \kappa^*(G; K_{1,m}) \geq \left\lceil \frac{n-1}{m+1} \right\rceil \).

For \( 1 \leq i \leq \left\lfloor \frac{n}{m+1} \right\rfloor - 1 \), let \( H_i \) be the subgraph of \( G \) with \( i(m+1) \) as the central vertex and \( i(m+1) - 1, \ldots, i(m+1) - m \) as the end vertices and hence isomorphic to \( K_{1,m} \). Consider the subgraph \( H_{\left\lfloor \frac{n}{m+1} \right\rfloor} \) of \( G \) with \((n-1)\) as the central vertex and all remaining vertices of \((G - \{H_1, \ldots, H_{\left\lfloor \frac{n}{m+1} \right\rfloor - 1}\}) \setminus \{0, n-1\} \) as end vertices. Note that \( H_{\left\lfloor \frac{n}{m+1} \right\rfloor} \) is isomorphic to a subgraph of \( K_{1,m} \). Clearly \( G - \{H_1, \ldots, H_{\left\lfloor \frac{n}{m+1} \right\rfloor}\} \) is the trivial graph \( K_1 \). Hence \( \kappa^*(G; K_{1,m}) \leq \left\lfloor \frac{n-1}{m+1} \right\rfloor \). Thus \( \kappa^*(G; K_{1,m}) = \left\lfloor \frac{n-1}{m+1} \right\rfloor \).

Suppose \( m + 1 \) divides \( n - 1 \). As mentioned above in the proof, \( H_{\frac{n}{m+1}} \) is a subgraph of \( G \) isomorphic to \( K_{1,m} \) and so \( \kappa(G; K_{1,m}) \leq \frac{n-1}{m+1} \). Now \( \frac{n-1}{m+1} = \kappa^*(G; K_{1,m}) \leq \kappa(G; K_{1,m}) \leq \frac{n-1}{m+1} \).

Hence \( \kappa(G; K_{1,m}) = \frac{n-1}{m+1} \). \( \Box \)

In Lemma 2.3, we have considered \( k = \lfloor \frac{n}{2} \rfloor \) in which case \( G = \text{Cir}(n, \Omega) \) is complete. Now, we consider \( k < \lfloor \frac{n}{2} \rfloor \), so that \( G = \text{Cir}(n, \Omega) \) can never be complete. By considering \( k < \lfloor \frac{n}{2} \rfloor \), we determine the structure and substructure connectivity of \( \text{Cir}(n, \Omega) \) with respect to \( K_{1,m} \) where \( m \leq 2k \).

**Theorem 2.4.** Let \( n \geq 4 \), \( k \) and \( m \) be positive integers such that \( 1 \leq k < \lfloor \frac{n}{2} \rfloor \) and \( m \leq 2k \). Let \( \Omega = \{1, \ldots, k, n-k, \ldots, n-1\} \) and \( G = \text{Cir}(n, \Omega) \). Then the following are equivalent:

(i) \( \kappa^*(G; K_{1,m}) = 1 \);

(ii) \( m + 1 = 2k + 1 = n - 1 \);

(iii) \( \kappa(G; K_{1,m}) = 1 \).

**Proof.** Since \( k < \lfloor \frac{n}{2} \rfloor \), \( |\Omega| = 2k < n-1 \).

(i) \( \Rightarrow \) (ii). Assume that \( \kappa^*(G; K_{1,m}) = 1 \). So that there exists a subgraph \( K_{1,t} \) of \( K_{1,m} \) for some \( t, t \leq m \leq 2k \) such that \( G - K_{1,t} \) is disconnected or a trivial graph. Since \( G \) is vertex transitive, one can have the central vertex of \( K_{1,t} \) as \( u = 0 \). Consider the subgraph \( H \) of \( G \) induced by \( \{0, \pm 1, \ldots, \pm k\} \). The graph \( G - H \) is connected and the vertices of \( H \) other than \( 0 \) are adjacent to either \( k + 1 \) or \( n - (k + 1) \) in \( G \). Note that \( G - H \) is a subgraph of \( G - K_{1,t} \).
Suppose \( t < 2k \), then \( G - K_{1,t} \) is connected, which is a contradiction. Hence \( t = m = 2k \) and \( G - K_{1,2k} = G - K_{1,t} \). It is easy to observe that the graphs \( G - K_{1,2k} \) and \( G - H \) are equal. It is known that \( m = 2k \leq n - 2 \). Suppose \( 2k < n - 2 \), then \( G - K_{1,t} = G - H \) is connected, which is a contradiction. This implies that \( t = m = 2k = n - 2 \). Hence \( m + 1 = 2k + 1 = n - 1 \).

(ii) \( \Rightarrow \) (iii). Assume that \( m + 1 = 2k + 1 = n - 1 \). For \( u \in V(G) \), \( \deg(u) = 2k = n - 2 \) and hence \( G \setminus N[u] = K_{1} \). Since \( |N(u)| = 2k = m, K_{1,m} \) is a subgraph of \( N[u] \) and hence removal of \( N[u] \) from \( G \) is same as removing \( K_{1,m} \) from \( G \). Thus \( \kappa(G; K_{1,m}) = 1 \).

(iii) \( \Rightarrow \) (i). Since \( \kappa(G; K_{1,m}) \leq \kappa(G; K_{1,2k}) = 1 \), \( \kappa(G; K_{1,m}) = 1 \).

**Remark 2.5.** Let \( n (\geq 6) \) and \( k \) be positive integers such that \( 2 \leq k < \lfloor \frac{n}{2} \rfloor, \Omega = \{1, \ldots, k, n-k, \ldots, n-1\}, G = \text{Cir}(n, \Omega) \) and \( m \leq 2k \). Even without the condition \( n > (m + 1) \lfloor \frac{2k}{m+1} \rfloor \), one can talk about \( \kappa(G; K_{1,m}) \), whereas it is not so in the case of \( \kappa(G; K_{1,2k}) \). For, if \( n \leq (m + 1) \lfloor \frac{2k}{m+1} \rfloor \), then for any integer \( \lambda \) with \( |V(\lambda K_{1,m})| \leq n \), removal of \( \lambda K_{1,m} \) does not disconnect \( G \).

Consider \( n \leq (m + 1) \lfloor \frac{2k}{m+1} \rfloor \). If \( \lambda \leq \lfloor \frac{2k}{m+1} \rfloor \), then \( |V(\lambda K_{1,m})| \geq 2k \) and hence by Theorem 2.1, \( G \) is connected after removal of \( \lambda K_{1,m} \) from \( G \). On the other hand if \( \lambda \geq \lfloor \frac{2k}{m+1} \rfloor \), then \( |V(\lambda K_{1,m})| \geq (m + 1) \lfloor \frac{2k}{m+1} \rfloor \geq n \). This along with \( |V(\lambda K_{1,m})| \leq n \) yields \( |V(\lambda K_{1,m})| = n \). Thus \( G = \lambda K_{1,m} \).

Now, we attempt to obtain \( \kappa(\text{Cir}(n, \Omega); K_{1,m}) \) and \( \kappa(\text{Cir}(n, \Omega); K_{1,2k}) \), for \( 2 \leq m + 1 \leq k \) and \( \Omega = \{1, \ldots, k, n-k, \ldots, n-1\} \).

**Theorem 2.6.** Let \( n (\geq 6) \) and \( k \) be positive integers such that \( 2 \leq k < \lfloor \frac{n}{2} \rfloor, \Omega = \{1, \ldots, k, n-k, \ldots, n-1\} \) and \( G = \text{Cir}(n, \Omega) \). If \( m \) is an integer such that \( 2 \leq m + 1 \leq k \) and \( (m + 1) \lfloor \frac{2k}{m+1} \rfloor < n \), then \( \kappa(G; K_{1,m}) = \lfloor \frac{2k}{m+1} \rfloor \) and \( \kappa(G; K_{1,2k}) = \lfloor \frac{2k}{m+1} \rfloor \).

**Proof.** Let \( a_{i} = n - (k - i + 1) \) for \( 1 \leq i \leq k \), \( a_{i} = i - k \) for \( k + 1 \leq i \leq 2k \) and \( b_{j} = j \) for \( k + 1 \leq j \leq n - (k + 1) \). By division algorithm \( 2k = (m + 1)s + r \) and \( k = (m + 1)h + r' \) for some \( r \) and \( r' \) with \( 0 \leq r \leq m \) and \( 0 \leq r' \leq m \).

For \( 1 \leq i \leq s = \frac{2k}{m+1} \), let \( H_{i} \) be defined as follows.

\[ V(H_{i}) = \{ a_{(m+1)i-m}; a_{(m+1)i-(m-1)}; \ldots, a_{(m+1)i} \} \] and edge set

\[ E(H_{i}) = \{ \{ a_{(m+1)i-m-r'}, a_{(m+1)i-(m-r')} \} : j \in \{1, \ldots, m\} / \{r'\} \}. \]

Further when \( r \neq 0 \), let \( H_{s+1} \) be defined as follows.

\[ V(H_{s+1}) = \{ v_{1}, \ldots, v_{m+1} : v_{i} = \begin{cases} a_{2k-(r-i)} & \text{if} \ 1 \leq i \leq r \\ b_{k+i-r} & \text{if} \ r+1 \leq i \leq m+1 \end{cases} \] and edge set

\[ E(H_{s+1}) = \{ \{ v_{r'+1}, v_{j} \} : j \in \{1, \ldots, m+1\} \setminus \{r'\} \}. \]

In \( G \), two vertices \( u \) and \( v \) are adjacent if and only if \( u, v \in \mathbb{Z}_{n} \) has the property that \( |u - v| \leq k \). Since \( |a_{(m+1)i-(m-r')} - a_{(m+1)i-(m-r')}| \leq k \) for every \( 0 \leq j \leq m \) and \( |v_{r'+1} - v_{j}| \leq k \) for every \( 1 \leq j \leq m + 1 \), \( H_{i} \) is indeed a subgraph of \( G \) for every \( 1 \leq i \leq s + 1 \).

Note that, each \( H_{i} \) is isomorphic to \( K_{1,m} \). Let \( H \) be the union of subgraphs given by

\[ H = \{ \bigcup_{i=1}^{s} H_{i} \cup_{i=1}^{s+1} K_{1,m} : \text{if} \ r = 0 \}; \]

\[ \bigcup_{i=1}^{s+1} H_{i} \cup_{i=1}^{s+1} K_{1,m} : \text{if} \ r \neq 0 \} \]

Note that \( V(H) = \lfloor \frac{2k}{m+1} \rfloor K_{1,m} \), \( N(0) \subseteq V(H) \), \( G - H \) is disconnected with \( \{0\} \) as one component. Thus, \( \kappa(G; K_{1,m}) \leq \lfloor \frac{2k}{m+1} \rfloor \) and so \( \kappa(G; K_{1,m}) \leq \lfloor \frac{2k}{m+1} \rfloor \). By Theorem 2.1, \( G \) is \( 2k \)-connected. Suppose there exists a set \( F' = \{ H'_{1}, \ldots, H'_{t} \} \) of subgraphs of \( G \) such that
every $H' \in F'$ is isomorphic to a subgraph of $K_{1,m}$, $t < \lceil \frac{2k}{m+1} \rceil$ and $G - F'$ is disconnected. Let $X = V(F')$. Clearly $|X| < 2k$ and by the assumption $G \setminus X$ is disconnected, which is a contradiction to $G$ is $2k$-connected.

Thus $\kappa'(G; K_{1,m}) \geq \lceil \frac{2k}{m+1} \rceil$ and so $\kappa(G; K_{1,m}) \geq \lceil \frac{2k}{m+1} \rceil$. Hence $\kappa(G; K_{1,m}) = \kappa'(G; K_{1,m}) = \lceil \frac{2k}{m+1} \rceil$. □

Now we obtain $\kappa(\text{Cir}(n, \Omega); K_{1,m})$ and $\kappa'(\text{Cir}(n, \Omega); K_{1,m})$, for $k < m + 1 \leq 2k + 1$ and $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$.

**Lemma 2.7.** Let $n \geq 6$ and $k$ be positive integers such that $2 \leq k < \lfloor \frac{n}{2} \rfloor$, $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$ and $G = \text{Cir}(n, \Omega)$. If $m$ is an integer with $k < m + 1 \leq 2k + 1$ and $n > (m + 1) \lceil \frac{2k}{m+1} \rceil$, then

$$\kappa(G; K_{1,m}) = \kappa'(G; K_{1,m}) = \begin{cases} 2 & \text{if } m + 1 = 2k + 1 = n - 1; \\ 1 & \text{otherwise}. \end{cases}$$

**Proof.** By Theorem 2.6, $\kappa(G; K_{1,m}) = \kappa'(G; K_{1,m}) = 2$ for $m + 1 = k$. This gives that $\kappa(G; K_{1,m}) \leq 2$ and so $\kappa'(G; K_{1,m}) \leq 2$ when $k < m + 1 \leq 2k + 1$. By Theorem 2.4, $\kappa(G; K_{1,m}) = 1 = \kappa'(G; K_{1,m})$ if and only if $m + 1 = 2k + 1 = n - 1$. Hence for the other cases $\kappa(G; K_{1,m}) \geq 2$ and $\kappa'(G; K_{1,m}) \geq 2$. Thus,

$$\kappa(G; K_{1,m}) = \kappa'(G; K_{1,m}) = \begin{cases} 1 & \text{if } m + 1 = 2k + 1 = n - 1; \\ 2 & \text{otherwise}. \end{cases}$$

Now we provide an example for the $K_{1,4}$-substructure connectivity of the circulant graph Cir(16, \{1, 2, 14, 15\}) in Figure 3. Here $n = 16, k = 2, m = 4$ and $k < m + 1$. The substructure cut is $F = \{H_1 \cong K_{1,3}, H_2 \cong K_{1,2}\}$. In Figure 3, the substructure cut $F$ is indicated by the dotted lines.

**Figure 3.** $K_{1,4}$-substructure cut of Cir(16, \{1, 2, 14, 15\}).
3. Structure and substructure connectivity of hypercubes

The \( n \)-dimensional hypercube \( Q_n \) is the Cayley graph defined on the group \( \mathbb{Z}_2^n \) with generating set as the standard orthonormal basis. Note that \( Q_n \) contains \( 2^n \) vertices and \( n2^{n-1} \) edges. Actually two distinct vertices \( x = (x_1x_2 \ldots x_n) \) and \( y = (y_1y_2 \ldots y_n) \) in \( V(Q_n) \) are adjacent if and only if \( x_i \neq y_i \) for exactly one \( i \) (1 \( \leq i \leq n \)). For any vertex \( x = (x_1x_2 \ldots x_n) \) in \( Q_n \), let \( (x)_i = (x'_1x'_2 \ldots x'_n) \) where \( x'_j = x_j \) for every \( j \neq i \) and \( x'_i = 1 - x_i \). Note that \( \{(x)_i\}_{1 \leq i \leq n} \) is the neighborhood set of \( x \) in \( Q_n \). For each \( t = 0, 1 \), we have two \( (n - 1) \)-dimensional subgraphs \( Q'_n \) of \( Q_n \) where \( V(Q'_n) = \{x | x = (x_1x_2 \ldots x_n) \in V(Q_n) \text{ and } x_n = t\} \) and \( E(Q'_n) = \{\{x,y\} | x,y \in V(Q'_n) \} \). Obviously, \( Q'_n \) is isomorphic to \( Q_{n-1} \) for each \( t = 0, 1 \). The path \( P_m \) of length \( m \) is a walk with \( m + 1 \) distinct vertices and \( m \) distinct edges.

The cycle \( C_m \) of length \( m \) is a closed path that contains \( m \) distinct vertices.

Cheng-Kuan Lin et al. [6] proved the following theorem for the substructure connectivity of hypercube \( Q_n \) with respect to the cycle \( C_4 \).

**Theorem 3.1** ([6, Theorem 10]). For \( n \geq 4 \), \( \kappa^e(Q_n; C_4) = [\frac{n}{2}] \).

For integers \( n \geq 5 \), \( k \) and \( m \) with \( k \) is a positive even integer, \( 2^m < k < 2^{m+1} \) and \( 2 \leq m \leq n - 2 \), Mane [7] considered the substructure connectivity of \( Q_n \) with respect to the cycle \( C_4 \). In fact Mane [7] proved that \( \kappa(Q_n; C_k) \leq n - m \) and \( \kappa(Q_4; C_6) = 2 \). In this section, for \( n \geq 4 \), we obtain the exact value for \( \kappa(Q_n; C_6) \).

First, we obtain the structure connectivity and the substructure connectivity of hypercube \( Q_n \) with respect to \( P_3 \), the path of length 3.

**Corollary 3.2.** For \( n \geq 4 \), \( \kappa(Q_n; P_3) = \kappa^e(Q_n; P_3) = [\frac{n}{2}] \).

**Proof.** By Theorem 3.1, \( \kappa^e(Q_n; C_4) = [\frac{n}{2}] \). Since all subgraphs of \( P_3 \) are also subgraphs of \( C_4 \), we have \( \kappa^e(Q_n; P_3) \geq [\frac{n}{2}] \).

For \( 1 \leq i \leq [\frac{n}{2}] - 1 \), consider the paths of length 3, \( R_i : (a_1 \cdots a_m) - (b_1 \cdots b_m) - (c_1 \cdots c_m) - (d_1 \cdots d_m) \) where

\[
\begin{align*}
a_{ij} &= \begin{cases} 1 & \text{if } j = 2i - 1; \\ 0 & \text{otherwise.} \end{cases} \\
b_{ij} &= \begin{cases} 1 & \text{if } j = 2i - 1, 2i; \\ 0 & \text{otherwise.} \end{cases} \\
c_{ij} &= \begin{cases} 1 & \text{if } j = 2i; \\ 0 & \text{otherwise.} \end{cases} \\
d_{ij} &= \begin{cases} 1 & \text{if } j = 2i, 2i + 1; \\ 0 & \text{otherwise.} \end{cases}
\end{align*}
\]

For odd \( n \), let \( R_{[\frac{n}{2}]} : (0 \ldots 01) - (10 \ldots 01) - (10 \ldots 011) - (10 \ldots 010) \) and for even \( n \), let \( R_{[\frac{n}{2}]} : (0 \ldots 010) - (0 \ldots 011) - (0 \ldots 01) - (10 \ldots 01) \). The removal of these paths \( R_n \) for \( 1 \leq i \leq [\frac{n}{2}] \), of length 3 disconnects \( Q_n \) with \( (0 \ldots 00) \) as an isolated vertex. Hence \( \kappa(Q_n; P_3) \leq [\frac{n}{2}] \).

Thus, we have \( [\frac{n}{2}] \leq \kappa^e(Q_n; P_3) \leq \kappa(Q_n; P_3) \leq [\frac{n}{2}] \) and so \( \kappa(Q_n; P_3) = \kappa^e(Q_n; P_3) = [\frac{n}{2}] \). \( \square \)

In the following lemma, we obtain an upper bound for the structure connectivity of \( Q_n \) with respect to \( C_6 \).

**Lemma 3.3.** For \( n \geq 3 \), \( \kappa(Q_n; C_6) \leq [\frac{n}{3}] \).
Hence, \( \kappa \) and Lemma 3.4 (Lemma 3.5. If connected to a vertex in a subset of collection of 6-cycles of \( Q_n \)).

If connected and every vertex in connectivity of the hypercube with respect to \( \text{length 6} \). Given below:

\[
B_i : (a_1 \cdots a_m) - (b_1 \cdots b_m) - (c_1 \cdots c_m) - (d_1 \cdots d_m) - (e_1 \cdots e_m) - (f_1 \cdots f_m) - (a_1 \cdots a_m) \text{where}
\]

\[
a_i = \begin{cases} 
1 & \text{if } j = 3i - 2; \\
0 & \text{otherwise}; 
\end{cases}
\]

\[
b_i = \begin{cases} 
1 & \text{if } j = 3i - 2 \text{ or } 3i - 1; \\
0 & \text{otherwise}; 
\end{cases}
\]

\[
e_i = \begin{cases} 
1 & \text{if } j = 3i - 1; \\
0 & \text{otherwise}; 
\end{cases}
\]

\[
d_i = \begin{cases} 
1 & \text{if } j = 3i - 1 \text{ or } 3i; \\
0 & \text{otherwise}; 
\end{cases}
\]

\[
e_i = \begin{cases} 
1 & \text{if } j = 3i; \\
0 & \text{otherwise}; 
\end{cases}
\]

\[
f_i = \begin{cases} 
1 & \text{if } j = 3i - 2 \text{ or } 3i; \\
0 & \text{otherwise}. 
\end{cases}
\]

If \( r = 1 \), let \( B_{q+1} : (0 \ldots 001) - (0 \ldots 011) - (0 \ldots 0111) - (0 \ldots 01111) - (0 \ldots 01101) - (0 \ldots 0001) - (0 \ldots 0001) \).

If \( r = 2 \), let \( B_{r+1} : (0 \ldots 010) - (0 \ldots 0110) - (0 \ldots 0111) - (0 \ldots 0101) - (0 \ldots 001) - (0 \ldots 011) - (0 \ldots 000) \).

The removal of cycles \( B_1, B_2, \ldots, B_{6|6|} \) disconnects \( Q_n \) with \( (0 \ldots 00) \) as an isolated vertex. Hence \( \kappa (Q_n; C_6) \leq \left[ \frac{3}{4} \right] \).

For each \( n \geq 3 \), \( Z_6^n \) is a collection of 6-cliques of \( Q_n \) and the same is taken as \( \{ \{ u, v, w, x, y, z \} | \{ u, v \}, \{ v, w \}, \{ w, x \}, \{ x, y \}, \{ y, z \}, \{ z, u \} \in E(Q_n) \} \). Let \( Z_6^n = \{ X_1, \ldots, X_m \} \) be a subset of collection of 6-cliques of \( Q_n \). For \( i = 0, 1, (Z_6^n)_{i} \subseteq Z_6^n \) is the subgraph \( \bigcup_{j=1}^{m} X_j \cap Q_n \) of \( Q_n \). Cheng-Kuan Lin et al. [6] obtained the substructure connectivity of hypercube \( Q_n \) with respect to \( K_{1,2} \) and the same is taken as an isolated vertex. Hence \( \kappa (Q_n; C_6) \leq \left[ \frac{3}{4} \right] \).

**Lemma 3.4** ([6, Theorem 6]). For \( n \geq 3 \), \( \kappa(Q_n; K_{1,2}) = \left[ \frac{7}{8} \right] \).

**Lemma 3.5.** If \( |Z_6^1| < 2 \), then \( Q_4 - Z_6^{1} \) is connected.

**Proof.** If \( |Z_6^1| = 0 \), then \( Q_4 - Z_6^{1} = Q_4 \), hence is connected. Assume that \( |Z_6^1| = 1 \) and \( Z_6^{1} = \{ C : u_1 - u_2 - u_3 - u_4 - u_5 - u_6 - u_1 \} \).

Suppose \( u_i \in Q_4^1 \) for all \( 1 \leq i \leq 6 \). Since \( Q_4^1 \) is connected and every vertex of \( Q_4^1 - Z_6^{1} \) is connected to a vertex in \( Q_4^1 \), we get that \( Q_4 - Z_6^{1} \) is connected.

If \( u_i \in Q_4^1 \) for all \( 1 \leq i \leq 6 \), then by similar arguments as above, \( Q_4 - Z_6^{1} \) is connected. Assume that \( V(C) \cap V(Q_4^1) \neq \emptyset \) and \( V(C) \cap V(Q_4^1) \neq \emptyset \). Without loss of generality one can assume that \( |V(C) \cap V(Q_4^1)| \leq |V(C) \cap V(Q_4^1)| \). Then we have two cases.

**Case 1.** Let \( |V(C) \cap V(Q_4^1)| = 2 \) and \( |V(C) \cap V(Q_4^1)| = 4 \). Clearly \( Q_4^1 - (Z_6^{1})_0 \) is connected and every vertex in \( Q_4^1 - (Z_6^{1})_1 \) is adjacent to a vertex in \( Q_4^1 - (Z_6^{1})_0 \). Hence \( Q_4 - Z_6^{1} \) is connected.
Theorem 3.6. For an integer \( n \geq 4 \), \( \kappa(Q_n; C_6) \geq \lceil \frac{n}{3} \rceil \).

Proof. By induction on \( n \). By Lemma 3.5, the result is true for \( n = 4 \). Assume as induction hypothesis that the statement holds for \( Q_k \), \( 4 \leq i \leq n-1 \). To complete the proof, one has to prove that if \( |Z_0^k| \leq \lceil \frac{n}{3} \rceil - 1 \), then \( Q_n - Z_0^n \) is connected.

Case 1. Assume that either \( V(Z_0^n) \subseteq V(C_k^1) \) or \( V(Z_0^n) \subseteq V(C_k^4) \). Without loss of generality, let us assume that \( V(Z_0^n) \subseteq V(C_k^1) \). Note that \( C_k^1 \) is connected and every vertex of \( Q_n - Z_0^n \) is connected to a vertex in \( C_k^1 \) and hence \( Q_n - Z_0^n \) is connected.

Case 2. Suppose \( V(Z_0^n) \cap V(C_k^1) \neq \emptyset \) and \( V(Z_0^n) \cap V(C_k^4) \neq \emptyset \).

Case 2.1. Assume that, for every 6-cycle \( X \) of \( Z_0^n \), \( V(X) \subseteq V(C_k^1) \) or \( V(X) \subseteq V(C_k^4) \). In this case, we have the number of 6-cycles of \( Z_0^n \) and \( C_k^1 \) is at most \( \lceil \frac{n}{3} \rceil - 2 \) and \( |V(Z_0^n) \cap V(C_k^1)| \leq \lceil \frac{n}{3} \rceil - 2 \). Note that \( \lceil \frac{n}{3} \rceil - 1 \leq \lceil \frac{n}{3} \rceil - 2 \) and so \( \lceil \frac{n}{3} \rceil - 2 < \lceil \frac{n}{3} \rceil - 1 \). By the induction hypothesis, \( \kappa(Q_k; C_6) \geq \lceil \frac{n}{3} \rceil - 1 \) and thus \( Q_n - (Z_0^n)_{\emptyset} \) is connected for \( i \in \{0, 1\} \).

Since \( 6(\lceil \frac{n}{3} \rceil - 2) < 6(\frac{n+3}{3} - 2) = 2(n-3) < 2(n-2) < 2n-2 \), for \( i = 0, 1 \), \( Q_n - (Z_0^n)_{\emptyset} \) contains more than \( \frac{n}{2} - 1 \) vertices. Hence there exists a vertex \( u \in Q_n - (Z_0^n)_{\emptyset} \) which is adjacent to \( (u)^n-1 \in (Q_n - (Z_0^n)_{\emptyset}) \). Hence \( Q_n - Z_0^n \) is connected.

Case 2.2. Assume that \( V(Z_0^n) \cap V(C_k^1) \neq \emptyset \) and \( V(Z_0^n) \cap V(C_k^4) \neq \emptyset \) and there is a 6-cycle \( X \) of \( Z_0^n \) such that \( V(X) \cap V(C_k^1) \neq \emptyset \) for each \( i = 0, 1 \). Let \( Z_0^n = \{X_1, X_2, \ldots, X_m\} \), \( m \leq \lceil \frac{n}{3} \rceil - 1 \) where each \( X_i \) is a 6-cycle. For any \( X_i \in Z_0^n \), the elements of \( V(X_i) \) differs from one another in at most 3 coordinates. Let us name those coordinates as \( k_1, k_2, k_3 \), i.e., if 0 (or 1) is the \( j \)th coordinate of an element in \( V(X_i) \) for \( j \neq k_1, k_2, k_3 \), then every element of \( V(X_i) \) has 0 (or 1) as the jth coordinate.

In total, we have 3m such coordinates \( k_1, k_2, k_3 \), for \( 1 \leq i \leq m \) (not necessarily distinct) corresponding to all elements in \( Z_0^n \). Further, \( 3m \leq 3(\lceil \frac{n}{3} \rceil - 1) - 3 < (\frac{n+3}{3}) - 3 = n \). Thus there exists \( k \in \{1, 2, \ldots, n\} \) such that \( k \notin \{k_1, k_2, k_3\} \) for each \( 1 \leq i \leq m \). This means that \( k \)th coordinate of \( V(X_i) \) is same in all the 6 elements of \( X_i \), i.e., the \( k \)th coordinate of all the elements of \( V(X_i) \) are equal.

Let us partition the vertices of \( Q_n \) into two subsets \( V_i = \{x = (x_1 \ldots x_n) \in V(Q_n) : x_k = j, k \text{is the index identified above}, j \in \{0, 1\} \} \). By the above arguments, for every \( i \), \( 1 \leq i \leq m \), either \( V(X_i) \subseteq V_0 \) or \( V(X_i) \subseteq V_1 \). Note that both the induced subgraphs \( V_0 \) and \( V_1 \) of \( Q_n \) are isomorphic to \( Q_{n-1} \). Now, if \( Z_0^n \subseteq V_j \) for some \( j \in \{0, 1\} \), proceeding as in Case 1, \( Q_n - Z_0^n \) is connected. Otherwise, proceeding as in Case 2.1, \( Q_n - Z_0^n \) is connected. Thus, \( \kappa(Q_n; C_6) \geq \lceil \frac{n}{3} \rceil \).

Figure 4 illustrates the \( C_6 \)-structure connectivity of \( Q_n \). In Figure 4, the structure cut is indicated with the dotted lines.

Since \( Q_3 \) is connected and by Lemma 3.3, \( \kappa(Q_3; C_6) = \lceil \frac{n}{3} \rceil = 1 \), \( \kappa(Q_3; C_6) = 1 = \lceil \frac{n}{3} \rceil \). Also, by Lemma 3.3 and 3.6, we have the following result.

Theorem 3.7. For \( n \geq 3 \), \( \kappa^e(Q_n; C_6) \leq \kappa(Q_n; C_6) = \lceil \frac{n}{3} \rceil \).
References


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