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Structure and substructure connectivity of circulant graphs and hypercubes

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Abstract

Let H be a connected subgraph of a connected graph G. The H-structure connectivity of the graph G, denoted by $\kappa(G;H)$, is the minimum cardinality of a minimal set of subgraphs $F = \{H'_1, H'_2, \ldots, H'_m\}$ in G, such that every $H'_i \in F$ is isomorphic to H and removal of F from G will disconnect G. The H-substructure connectivity of the graph G, denoted by $\kappa^s(G;H)$, is the minimum cardinality of a minimal set of subgraphs $F = \{J'_1, J'_2, \ldots, J'_m\}$ in G, such that every $J'_i \in F$ is a connected subgraph of H and removal of F from G will disconnect G. In this paper, we provide the H-structure and the H-substructure connectivity of the circulant graph $\mathrm{Cir}(n, \Omega)$ where $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}, 1 \leq k \leq \lfloor \frac{n}{2} \rfloor$ and the hypercube Q_n for some connected subgraphs H.

Keywords Structure connectivity, Substructure connectivity, Circulant graph, Hypercube

Paper type Original Article

1. Introduction

A simple graph $G = (V, E)$ is a finite nonempty set $V(G)$ of objects called vertices together with a (possibly empty) set $E(G)$ of unordered pairs of distinct vertices of G called edges. Two distinct vertices $u, v \in V(G)$ are said to be adjacent in G if u and v are connected by an edge and it is represented by $\{u, v\} \in E(G)$. A graph G is said to be trivial if it contains only one vertex and no edges. The connectivity is an important indicator of the reliability and fault tolerability of a network. The vertex connectivity of a connected graph G, denoted by $\kappa(G)$, is the minimum cardinality of a vertex subset $S \subseteq V(G)$, whose removal would disconnect G or

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$G \setminus S$ is the trivial graph. As a generalization of the vertex connectivity $\kappa(G)$, Cheng-Kuan Lin et al. [6] introduced two new kinds of connectivity, called structure connectivity and substructure connectivity. A set F of connected subgraphs of a graph G is a subgraph-cut of G if $G \setminus V(F)$ is a disconnected graph or K_1. Let H be a connected subgraph of G. Then F is an H-structure cut if F is a subgraph cut, and every element in F is isomorphic to H. The H-structure connectivity of G, denoted by $\kappa(G;H)$, is defined to be the minimum cardinality of all H-structure cuts of G. F is an H-substructure cut if F is a subgraph-cut, and every element in F is isomorphic to a connected subgraph of H. The H-substructure connectivity of G, denoted by $\kappa^s(G;H)$, is the minimum cardinality of all H-substructure cuts of G. Since every H-structure cut is an H-substructure cut $\kappa^s(G;H) \leq \kappa(G;H)$. If $H = K_1$ then we have $\kappa(G;H) = \kappa^s(G;H)$.

The vertex connectivity $\kappa(G) \geq \kappa^s(G;H)$ for every subgraph H of G whereas the relation between vertex connectivity and structure connectivity depends on H. For the graph G, given in Figure 1, $\kappa(G) = 2$, the structure connectivity of G with respect to the cycle of length $5, \kappa(G;C_5) = 1$ and the structure connectivity of G with respect to the cycle of length $4, \kappa(G;C_4) = 3$.

Let Γ be a finite group with e as the identity. A generating set of Γ is a subset Ω such that every element of Γ can be expressed as a product of finitely many elements in Ω. Assume that $e \notin \Omega$ and $a \in \Omega$ implies $a^{-1} \in \Omega$ and such a subset Ω is called as a symmetric generating set of Γ. Hereafter, we assume that Ω is a symmetric generating set of a finite group Γ. A Cayley graph is a graph $G = (V, E)$, where $V(G) = \Gamma$ and two vertices x and y are adjacent if $xy^{-1} \in \Omega$ and it is denoted by $\mathrm{Cay}(\Gamma, \Omega)$. The inclusion of the inverse in Ω for every element of Ω means that $\mathrm{Cay}(\Gamma, \Omega)$ is undirected. Since Ω is a generating set for Γ, $\mathrm{Cay}(\Gamma, \Omega)$ is connected and $\mathrm{Cay}(\Gamma, \Omega)$ is a regular graph of degree $|\Omega|$. Cayley graphs are extensively dealt in the literature and various authors including Dejter [3], Lakshmivarahan [4], Lee [5], Tamizh Chelvam [8], and Wang [10] have worked on Cayley graphs. For example, one can refer the survey by Tamizh Chelvam and Sivagami [9] for domination in Cayley graphs. The Cayley graph constructed out of the finite cyclic group $\mathbb{Z}_n$, $n \geq 2$ along with a symmetric generating set Ω is called a circulant graph and the same is denoted by $\mathrm{Cir}(n, \Omega)$. The hypercube Q_n is the Cayley graph defined on the group $\mathbb{Z}_2^n$ with the standard orthonormal basis as the generating set. Cheng-Kuan Lin et al. [6] have obtained $\kappa(Q_n;H)$ and $\kappa^s(Q_n;H)$ for $H \in \{K_1, K_{1,1}, K_{1,2}, K_{1,3}, C_4\}$. Here, we provide an example in Figure 2, to exhibit a structure cut of the circulant graph $\mathrm{Cir}(10, \{1,2,3,7,8,9\})$ with respect to K_3. In Figure 2,

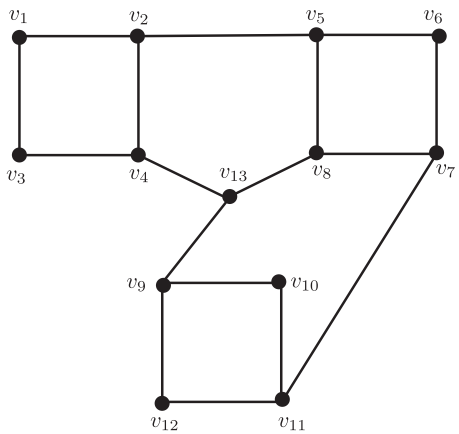

Figure 1.
$\kappa(G) = 2, \kappa(G;C_5) = 1,$
$\kappa(G;C_4) = 3.$

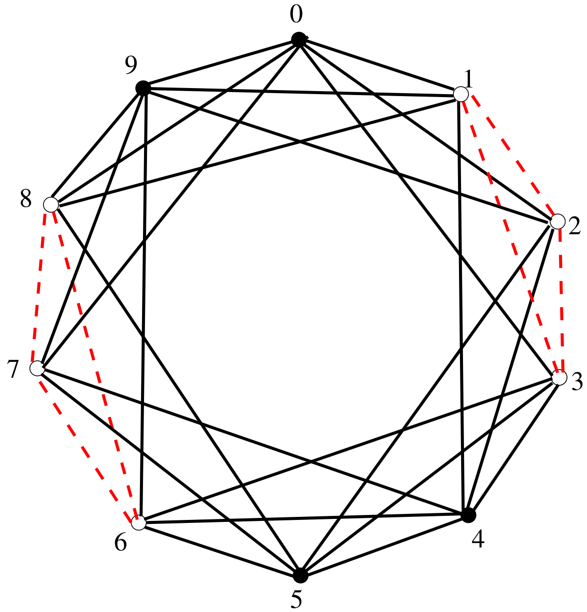

Figure 2.
K_3-structure cut of Cir (10, $\{1,2,3,7,8,9\}$).

the structure cut is indicated by the dotted lines and note that $\kappa^s(Cir(10, \{1,2,3,7,8,9\};K_3)) = 2$.

Throughout this paper, $G - X$ denotes the removal of a set X of subgraphs from the graph G and $G \setminus B$ denotes the removal of the set $B \subseteq V(G)$ from the graph G. When $X = \{H\}$, $G - X$ is simply denoted as $G - H$. For a graph G, the open neighborhood $N(v)$ of a vertex $v \in V(G)$ is the set of all vertices which are adjacent to v. The closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the intersection of G_1 and G_2, denoted by $G_1 \cap G_2$, is the graph whose vertex set is $V_1 \cap V_2$ and the edge set is $E_1 \cap E_2$. The union of two disjoint vertex sets graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted by $G_1 \cup G_2$, is the graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$. For basic definitions and properties related to graph theory, one can refer [2]. For undefined definitions related to algebraic graph theory, one can refer [1].

In Section 2, we obtain the H-structure and the H-substructure connectivity of the circulant graph $\mathrm{Cir}(n, \Omega)$ where $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$, for $1 \le k \le \lfloor \frac{n}{2} \rfloor$ with respect to some of its connected subgraphs. For integers n (≥ 5) and m with $2 \le m \le n-2$, Mane [7] proved that $\kappa(Q_n; C_k) \le n - m$, where k is a positive even integer with $2^m < k < 2^{m+1}$ and observed that $\kappa(Q_4; C_6) = 2$. In Section 3, for $n \ge 4$, we obtain the exact value for $\kappa(Q_n; C_6)$.

2. Structure and substructure connectivity of circulant graphs

Throughout this section $n\,(\ge 2)$, m and k are integers such that $1 \le k \le \lfloor \frac{n}{2} \rfloor$ and $\Omega = \{1, 2, \ldots, k, n-k, \ldots, n-1\}$. We take the elements of $\mathbb{Z}_n$ as $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. The following result due to Wang [10] is useful in the paper.

Theorem 2.1 *([10, Wang]). Let n and k be positive integers such that $1 \le k \le \lfloor \frac{n}{2} \rfloor$, $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$ and $G = \mathrm{Cir}(n, \Omega)$. Then $\kappa(G) = |\Omega|$.*

By definition and from Theorem 2.1, we have the following corollary.

Corollary 2.2. *Let $n\,(\geq 2)$ and k be positive integers such that $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$, $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$ and $G = \mathrm{Cir}(n, \Omega)$. Then $\kappa(G; K_1) = |\Omega|$ and $\kappa^s(G; K_1) = |\Omega|$.*

A star graph $K_{1,m}$ $(m \geq 1)$ is a complete bipartite graph comprised of two partite sets of vertices of sizes 1 and m respectively, such that two vertices are adjacent if and only if they are in different partite sets. If $K_{1,m}$ is a subgraph of $\mathrm{Cir}(n, \Omega)$, then $m \leq |\Omega|$. If $k = \lfloor \frac{n}{2} \rfloor$ and $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$, then the circulant graph $G = \mathrm{Cir}(n, \Omega)$ is the complete graph K_n. Now, we obtain the structure connectivity of K_n, as a circulant graph with respect to $K_{1,m}$. If $m+1$ does not divide $n-1$, then removal of $\lambda K_{1,m}$ does not disconnect $\mathrm{Cir}(n, \Omega)$ for any λ. Hence the structure connectivity of K_n with respect to $K_{1,m}$ is meaningful only when $m+1$ divides $n-1$.

Lemma 2.3. *Let $n\,(\geq 2)$ and k be positive integers such that $k = \lfloor \frac{n}{2} \rfloor$, $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$ and $G = \mathrm{Cir}(n, \Omega)$. For a positive integer m with $m \leq n-2$, $\kappa^s(G;\, K_{1,m}) = \lceil \frac{n-1}{m+1} \rceil$. Also, $\kappa(G;\, K_{1,m}) = \frac{n-1}{m+1}$ if $m+1$ divides $n-1$.*

Proof. By the assumption on n, k and Ω, $\mathrm{Cir}(n, \Omega) = K_n$. By Theorem 2.1, G is $(n-1)$-connected. Let F be a $K_{1,m}$-substructure cut with minimum cardinality of $G = \mathrm{Cir}(n, \Omega)$. Suppose $\kappa^s(G;\, K_{1,m}) < \lceil \frac{n-1}{m+1} \rceil$. Then $|V(F)| < n-1$ and $G \setminus V(F)$ is disconnected, which is a contradiction to G is $(n-1)$-connected. Hence $\kappa^s(G;\, K_{1,m}) \geq \lceil \frac{n-1}{m+1} \rceil$.

For $1 \leq i \leq \lceil \frac{n-1}{m+1} \rceil - 1$, let H_i be the subgraph of G with $i(m+1)$ as the central vertex and $i(m+1)-1, \ldots, i(m+1)-m$ as the end vertices and hence isomorphic to $K_{1,m}$. Consider the subgraph $H_{\lceil \frac{n-1}{m+1} \rceil}$ of G with $(n-1)$ as the central vertex and all remaining vertices of $(G - \{H_1, \ldots, H_{\lceil \frac{n-1}{m+1} \rceil - 1}\}) \setminus \{0, n-1\}$ as end vertices. Note that $H_{\lceil \frac{n-1}{m+1} \rceil}$ is isomorphic to a subgraph of $K_{1,m}$. Clearly $G - \{H_1, \ldots, H_{\lceil \frac{n-1}{m+1} \rceil}\}$ is the trivial graph K_1. Hence $\kappa^s(G;\, K_{1,m}) \leq \lceil \frac{n-1}{m+1} \rceil$. Thus $\kappa^s(G;\, K_{1,m}) = \lceil \frac{n-1}{m+1} \rceil$.

Suppose $m+1$ divides $n-1$. As mentioned above in the proof, $H_{\frac{n-1}{m+1}}$ is a subgraph of G isomorphic to $K_{1,m}$ and so $\kappa(G; K_{1,m}) \leq \frac{n-1}{m+1}$. Now $\frac{n-1}{m+1} = \kappa^s(G; K_{1,m}) \leq \kappa(G; K_{1,m}) \leq \frac{n-1}{m+1}$. Hence $\kappa(G; K_{1,m}) = \frac{n-1}{m+1}$. $\square$

In Lemma 2.3, we have considered $k = \lfloor \frac{n}{2} \rfloor$ in which case $G = \mathrm{Cir}(n, \Omega)$ is complete. Now, we consider $k < \lfloor \frac{n}{2} \rfloor$, so that $G = \mathrm{Cir}(n, \Omega)$ can never be complete. By considering $k < \lfloor \frac{n}{2} \rfloor$, we determine the structure and substructure connectivity of $\mathrm{Cir}(n, \Omega)$ with respect to $K_{1,m}$ where $m \leq 2k$.

Theorem 2.4. *Let $n\,(\geq 4), k$ and m be positive integers such that $1 \leq k < \lfloor \frac{n}{2} \rfloor$ and $m \leq 2k$. Let $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$ and $G = \mathrm{Cir}(n, \Omega)$. Then the following are equivalent:*

(i) $\kappa^s(G; K_{1,m}) = 1$;

(ii) $m+1 = 2k+1 = n-1$;

(iii) $\kappa(G; K_{1,m}) = 1$.

Proof. Since $k < \lfloor \frac{n}{2} \rfloor$, $|\Omega| = 2k < n-1$.

(i) $\Rightarrow$ (ii). Assume that $\kappa^s(G; K_{1,m}) = 1$. So that there exists a subgraph $K_{1,t}$ of $K_{1,m}$ for some $t, t \leq m \leq 2k$ such that $G - K_{1,t}$ is disconnected or a trivial graph. Since G is vertex transitive, one can have the central vertex of $K_{1,t}$ as $u = 0$. Consider the subgraph H of G induced by $\langle \{0, \pm 1, \ldots, \pm k\} \rangle$. The graph $G - H$ is connected and the vertices of H other than 0 are adjacent to either $k+1$ or $n-(k+1)$ in G. Note that $G - H$ is a subgraph of $G - K_{1,t}$.

Suppose $t < 2k$, then $G - K_{1,t}$ is connected, which is a contradiction. Hence $t = m = 2k$ and $G - K_{1,t} = G - K_{1,2k}$. It is easy to observe that the graphs $G - K_{1,2k}$ and $G - H$ are equal. It is known that $m = 2k \le n - 2$. Suppose $2k < n - 2$, then $G - K_{1,t} = G - H$ is connected, which is a contradiction. This implies that $t = m = 2k = n - 2$. Hence $m + 1 = 2k + 1 = n - 1$.

(ii) $\Rightarrow$ (iii). Assume that $m + 1 = 2k + 1 = n - 1$. For $u \in V(G)$, $\deg(u) = 2k = n - 2$ and hence $G \setminus N[u] = K_1$. Since $|N(u)| = 2k = m$, $K_{1,m}$ is a subgraph of $\langle N[u] \rangle$ and hence removal of $\langle N[u] \rangle$ from G is same as removing $K_{1,m}$ from G. Thus $\kappa(G; K_{1,m}) = 1$.

(iii) $\Rightarrow$ (i). Since $\kappa^s(G; K_{1,m}) \le \kappa(G; K_{1,m}) = 1$, $\kappa^s(G; K_{1,m}) = 1$. $\square$

Remark 2.5. Let $n\,(\ge 6)$ and k be positive integers such that $2 \le k < \lfloor \frac{n}{2} \rfloor$, $\Omega = \{1, \ldots, k, n - k, \ldots, n - 1\}$, $G = \mathrm{Cir}(n, \Omega)$ and $m \le 2k$. Even without the condition $n > (m + 1)\lceil \frac{2k}{m+1} \rceil$, one can talk about $\kappa^s(G; K_{1,m})$, whereas it is not so in the case of $\kappa(G; K_{1,m})$. For, if $n \le (m + 1)\lceil \frac{2k}{m+1} \rceil$, then for any integer λ with $|V(\lambda K_{1,m})| \le n$, removal of $\lambda K_{1,m}$ does not disconnect G.

Consider $n \le (m + 1)\lceil \frac{2k}{m+1} \rceil$. If $\lambda < \lceil \frac{2k}{m+1} \rceil$, then $|V(\lambda K_{1,m})| < 2k$ and hence by Theorem 2.1, G is connected after removal of $\lambda K_{1,m}$ from G. On the other hand if $\lambda \ge \lceil \frac{2k}{m+1} \rceil$, then $|V(\lambda K_{1,m})| \ge (m + 1)\lceil \frac{2k}{m+1} \rceil \ge n$. This along with $|V(\lambda K_{1,m})| \le n$ yields $|V(\lambda K_{1,m})| = n$. Thus $G = \lambda K_{1,m}$.

Now, we attempt to obtain $\kappa(\mathrm{Cir}(n, \Omega); K_{1,m})$ and $\kappa^s(\mathrm{Cir}(n, \Omega); K_{1,m})$, for $2 \le m + 1 \le k$ and $\Omega = \{1, \ldots, k, n - k, \ldots, n - 1\}$.

Theorem 2.6. *Let $n\,(\ge 6)$ and k be positive integers such that $2 \le k < \lfloor \frac{n}{2} \rfloor$, $\Omega = \{1, \ldots, k, n - k, \ldots, n - 1\}$ and $G = \mathrm{Cir}(n, \Omega)$. If m is an integer such that $2 \le m + 1 \le k$ and $(m + 1)\lceil \frac{2k}{m+1} \rceil < n$, then $\kappa(G; K_{1,m}) = \lceil \frac{2k}{m+1} \rceil$ and $\kappa^s(G; K_{1,m}) = \lceil \frac{2k}{m+1} \rceil$.*

Proof. Let $a_i = n - (k - i + 1)$ for $1 \le i \le k$, $a_i = i - k$ for $k + 1 \le i \le 2k$ and $b_j = j$ for $k + 1 \le j \le n - (k + 1)$. By division algorithm $2k = (m + 1)s + r$ and $k = (m + 1)h + r'$ for some r and r' with $0 \le r \le m$ and $0 \le r' \le m$.

For $1 \le i \le s = \frac{2k - r}{m+1}$, let H_i be defined as follows.
$V(H_i) = \{a_{(m+1)i-m}, a_{(m+1)i-(m-1)}, \ldots, a_{(m+1)i}\}$ and edge set

$$E(H_i) = \{\{a_{(m+1)i-(m-r')}, a_{(m+1)i-(m-j)}\} : j \in \{0, \ldots, m\} \,/\, \{r'\}\}.$$

Further when $r \ne 0$, let H_{s+1} be defined as follows.

$$V(H_{s+1}) = \left\{ v_1, \ldots, v_{m+1} : v_i = \begin{cases} a_{2k-(r-i)} & \text{if } 1 \le i \le r \\ b_{k+i-r} & \text{if } r + 1 \le i \le m + 1 \end{cases} \right\} \text{ and edge set}$$

$$E(H_{s+1}) = \{\{v_{r'+1}, v_j\} : j \in \{1, \ldots, m + 1\} \setminus \{r' + 1\}\}.$$

In G, two vertices u and v are adjacent if and only if $u, v \in \mathbb{Z}_n$ has the property that $|u - v| \le k$. Since $|a_{(m+1)i-(m-r')} - a_{(m+1)i-(m-j)}| \le k$ for every $0 \le j \le m$ and $|v_{r'+1} - v_j| \le k$ for every $1 \le j \le m + 1$, H_i is indeed a subgraph of G for every $1 \le i \le s + 1$.

Note that, each H_i is isomorphic to $K_{1,m}$. Let H be the union of subgraphs given by $H = \begin{cases} \bigcup_{i=1}^{s} H_i = \bigcup_{i=1}^{s} K_{1,m} & \text{if } r = 0; \\ \bigcup_{i=1}^{s+1} H_i = \bigcup_{i=1}^{s+1} K_{1,m} & \text{if } r \ne 0. \end{cases}$

Note that $V(H) = \lceil \frac{2k}{m+1} \rceil K_{1,m}$, $N(0) \subseteq V(H)$, $G - H$ is disconnected with $\{0\}$ as one component. Thus, $\kappa(G; K_{1,m}) \le \lceil \frac{2k}{m+1} \rceil$ and so $\kappa^s(G; K_{1,m}) \le \lceil \frac{2k}{m+1} \rceil$. By Theorem 2.1, G is $2k$-connected. Suppose there exists a set $F' = \{H'_1, \ldots, H'_t\}$ of subgraphs of G such that every $H'_i \in F'$ is isomorphic to a subgraph of $K_{1,m}$, $t < \lceil \frac{2k}{m+1} \rceil$ and $G - F'$ is disconnected. Let $X = V(F')$. Clearly $|X| < 2k$ and by the assumption $G \setminus X$ is disconnected, which is a contradiction to G is $2k$-connected.

Thus $\kappa^s(G; K_{1,m}) \geq \lceil \frac{2k}{m+1} \rceil$ and so $\kappa(G; K_{1,m}) \geq \lceil \frac{2k}{m+1} \rceil$. Hence $\kappa(G; K_{1,m}) = \kappa^s(G; K_{1,m}) = \lceil \frac{2k}{m+1} \rceil$. □

Now we obtain $\kappa(\mathrm{Cir}(n, \Omega); K_{1,m})$ and $\kappa^s(\mathrm{Cir}(n, \Omega); K_{1,m})$, for $k < m+1 \leq 2k+1$ and $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$.

Lemma 2.7. *Let $n\,(\geq 6)$ and k be positive integers such that $2 \leq k < \lfloor \frac{n}{2} \rfloor$, $\Omega = \{1, \ldots, k, n-k, \ldots, n-1\}$ and $G = \mathrm{Cir}(n, \Omega)$. If m is an integer with $k < m+1 \leq 2k+1$ and $n > (m+1)\lceil \frac{2k}{m+1} \rceil$, then*

$$\kappa(G;\, K_{1,m}) = \kappa^s(G;\, K_{1,m}) = \begin{cases} 1 & \text{if } m+1 = 2k+1 = n-1; \\ 2 & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.6, $\kappa(G; K_{1,m}) = \kappa^s(G; K_{1,m}) = 2$ for $m+1 = k$. This gives that $\kappa(G; K_{1,m}) \leq 2$ and so $\kappa^s(G; K_{1,m}) \leq 2$ when $k < m+1 \leq 2k+1$. By Theorem 2.4, $\kappa(G; K_{1,m}) = 1 = \kappa^s(G; K_{1,m})$ if and only if $m+1 = 2k+1 = n-1$. Hence for the other cases $\kappa(G; K_{1,m}) \geq 2$ and $\kappa^s(G; K_{1,m}) \geq 2$. Thus,

$$\kappa(G;\, K_{1,m}) = \kappa^s(G;\, K_{1,m}) = \begin{cases} 1 & \text{if } m+1 = 2k+1 = n-1; \\ 2 & \text{otherwise.} \; \square \end{cases}$$

Now we provide an example for the $K_{1,4}$-substructure connectivity of the circulant graph $\mathrm{Cir}(16, \{1, 2, 14, 15\})$ in Figure 3. Here $n = 16, k = 2, m = 4$ and $k < m+1$. The substructure cut is $F = \{H_1 \cong K_{1,3}, H_2 \cong K_{1,2}\}$. In Figure 3, the substructure cut F is indicated by the dotted lines.

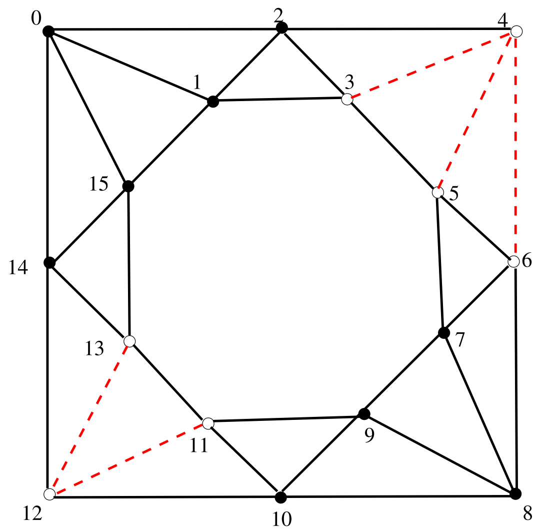

Figure 3.
$K_{1,4}$-substructure cut of $\mathrm{Cir}(16, \{1, 2, 14, 15\})$.

3. Structure and substructure connectivity of hypercubes

The n-dimensional hypercube Q_n is the Cayley graph defined on the group $\mathbb{Z}_2^n$ with generating set as the standard orthonormal basis. Note that Q_n contains 2^n vertices and $n2^{n-1}$ edges. Actually two distinct vertices $x = (x_1x_2\dots x_n)$ and $y = (y_1y_2\dots y_n)$ in $V(Q_n)$ are adjacent if and only if $x_i \neq y_i$ for exactly one i ($1 \leq i \leq n$). For any vertex $x = (x_1x_2\dots x_n)$ in Q_n, let $(x)^i = (x_1^ix_2^i\dots x_n^i)$ where $x_j^i = x_j$ for every $j \neq i$ and $x_i^i = 1-x_i$. Note that $\{(x)^i\}_{1\leq i\leq n}$ is the neighborhood set of x in Q_n. For each $t = 0, 1$, we have two $(n-1)$-dimensional subgraphs Q_n^t of Q_n where $V(Q_n^t) = \{x | x = (x_1x_2\dots x_n) \in V(Q_n)\, and\, x_n = t\}$ and $E(Q_n^t) = \{\{x, y\} | \{x, y\} \in E(Q_n)\, and\, x, y \in V(Q_n^t)\}$. Obviously, Q_n^t is isomorphic to Q_{n-1} for each $t = 0, 1$. The path P_m of length m is a walk with $m+1$ distinct vertices and m distinct edges. The cycle C_m of length m is a closed path that contains m distinct vertices.

Cheng-Kuan Lin et al. [6] proved the following theorem for the substructure connectivity of hypercube Q_n with respect to the cycle C_4.

Theorem 3.1 *([6, Theorem 10]). For $n \geq 4$, $\kappa^s(Q_n; C_4) = \lceil \frac{n}{2} \rceil$.*

For integers $n\,(\geq 5), k$ and m with k is a positive even integer, $2^m < k < 2^{m+1}$ and $2 \leq m \leq n-2$, Mane [7] considered the substructure connectivity of Q_n with respect to the cycle C_6. In fact Mane [7] proved that $\kappa(Q_n; C_k) \leq n-m$, and $\kappa(Q_4; C_6) = 2$. In this section, for $n \geq 4$, we obtain the exact value for $\kappa(Q_n; C_6)$.

First, we obtain the structure connectivity and the substructure connectivity of hypercube Q_n with respect to P_3, the path of length 3.

Corollary 3.2. *For $n \geq 4$, $\kappa(Q_n; P_3) = \kappa^s(Q_n; P_3) = \lceil \frac{n}{2} \rceil$.*

Proof. By Theorem 3.1, $\kappa^s(Q_n; C_4) = \lceil \frac{n}{2} \rceil$. Since all subgraphs of P_3 are also subgraphs of C_4, we have $\kappa^s(Q_n; P_3) \geq \lceil \frac{n}{2} \rceil$.

For $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$, consider the paths of length 3, $R_i : (a_{i1}\cdots a_{in}) - (b_{i1}\cdots b_{in}) - (c_{i1}\cdots c_{in}) - (d_{i1}\cdots d_{in})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } j = 2i-1; \\ 0 & \text{otherwise.} \end{cases}$$

$$b_{ij} = \begin{cases} 1 & \text{if } j = 2i-1, 2i; \\ 0 & \text{otherwise.} \end{cases}$$

$$c_{ij} = \begin{cases} 1 & \text{if } j = 2i; \\ 0 & \text{otherwise.} \end{cases}$$

$$d_{ij} = \begin{cases} 1 & \text{if } j = 2i, 2i+1; \\ 0 & \text{otherwise.} \end{cases}$$

For odd n, let $R_{\lceil \frac{n}{2} \rceil} : (0\dots01) - (10\dots01) - (10\dots011) - (10\dots010)$ and for even n, let $R_{\lceil \frac{n}{2} \rceil} : (0\dots010) - (0\dots011) - (0\dots01) - (10\dots01)$. The removal of these paths R_i, for $1 \leq i \leq \lceil \frac{n}{2} \rceil$, of length 3 disconnects Q_n with $(0\dots00)$ as an isolated vertex. Hence $\kappa(Q_n; P_3) \leq \lceil \frac{n}{2} \rceil$.

Thus, we have $\lceil \frac{n}{2} \rceil \leq \kappa^s(Q_n; P_3) \leq \kappa(Q_n; P_3) \leq \lceil \frac{n}{2} \rceil$ and so $\kappa(Q_n; P_3) = \kappa^s(Q_n; P_3) = \lceil \frac{n}{2} \rceil$. □

In the following lemma, we obtain an upper bound for the structure connectivity of Q_n with respect to C_6.

Lemma 3.3. *For $n \geq 3$, $\kappa(Q_n; C_6) \leq \lceil \frac{n}{3} \rceil$.*

Proof. By division algorithm, $n = 3q + r, 0 \leq r \leq 2$. For $1 \leq i \leq q$, consider the cycles B_i of length 6 given below:

$B_i : (a_{i1} \cdots a_{in}) - (b_{i1} \cdots b_{in}) - (c_{i1} \cdots c_{in}) - (d_{i1} \cdots d_{in}) - (e_{i1} \cdots e_{in}) - (f_{i1} \cdots f_{in}) - (a_{i1} \cdots a_{in})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } j = 3i - 2; \\ 0 & \text{otherwise;} \end{cases}$$

$$b_{ij} = \begin{cases} 1 & \text{if } j = 3i - 2 \text{ or } 3i - 1; \\ 0 & \text{otherwise;} \end{cases}$$

$$c_{ij} = \begin{cases} 1 & \text{if } j = 3i - 1; \\ 0 & \text{otherwise;} \end{cases}$$

$$d_{ij} = \begin{cases} 1 & \text{if } j = 3i - 1 \text{ or } 3i; \\ 0 & \text{otherwise;} \end{cases}$$

$$e_{ij} = \begin{cases} 1 & \text{if } j = 3i; \\ 0 & \text{otherwise;} \end{cases}$$

$$f_{ij} = \begin{cases} 1 & \text{if } j = 3i - 2 \text{ or } 3i; \\ 0 & \text{otherwise.} \end{cases}$$

If $r = 1$, let $B_{q+1} : (0 \ldots 001) - (0 \ldots 011) - (0 \ldots 0111) - (0 \ldots 01111) - (00 \ldots 01101) - (0 \ldots 01001) - (00 \ldots 001)$.

If $r = 2$, let $B_{q+1} : (0 \ldots 010) - (0 \ldots 0110) - (0 \ldots 0111) - (0 \ldots 0101) - (0 \ldots 001) - (0 \ldots 011) - (00 \ldots 010)$.

The removal of cycles $B_1, B_2, \ldots, B_{\lceil \frac{n}{3} \rceil}$ disconnects Q_n with $(0 \ldots 00)$ as an isolated vertex. Hence $\kappa(Q_n; C_6) \leq \lceil \frac{n}{3} \rceil$. □

For each $n \geq 3$, Z_6^n is a collection of 6-cles of Q_n and the same is taken as $\{\{u, v, w, x, y, z\} | \{u, v\}, \{v, w\}, \{w, x\}, \{x, y\}, \{y, z\}, \{z, u\} \in E(Q_n)\}$. Let $Z_6^n = \{X_1, \ldots, X_m\}$ be a subset of collection of 6-cycles of Q_n. For $i = 0, 1$, $(Z_6^n)_i \subseteq Z_6^n$ is the subgraph $\bigcup_{j=1}^{m} X_j \cap Q_n^i$ of Q_n. Cheng-Kuan Lin et al. [6] obtained the substructure connectivity of hypercube Q_n with respect to $K_{1,2}$ and the same the stated below to obtain a lower bound for the substructure connectivity of the hypercube with respect to C_6.

Lemma 3.4 *([6, Theorem 6])*. For $n \geq 3$, $\kappa(Q_n; K_{1,2}) = \lceil \frac{n}{2} \rceil$.

Lemma 3.5. *If $|Z_6^4| < 2$, then $Q_4 - Z_6^4$ is connected.*

Proof. If $|Z_6^4| = 0$, then $Q_4 - Z_6^4 = Q_4$, hence is connected. Assume that $|Z_6^4| = 1$ and $Z_6^4 = \{C : u_1 - u_2 - u_3 - u_4 - u_5 - u_6 - u_1\}$.

Suppose $u_i \in Q_4^0$ for all $1 \leq i \leq 6$. Since Q_4^1 is connected and every vertex of $Q_4^0 - Z_6^4$ is connected to a vertex in Q_4^1, we get that $Q_4 - Z_6^4$ is connected.

If $u_i \in Q_4^1$ for all $1 \leq i \leq 6$, then by similar arguments as above, $Q_4 - Z_6^4$ is connected.

Assume that $V(C) \cap V(Q_4^0) \neq \phi$ and $V(C) \cap V(Q_4^1) \neq \phi$. Without loss of generality one can assume that $|V(C) \cap V(Q_4^0)| \leq |V(C) \cap V(Q_4^1)|$. Then we have two cases.

Case 1. Let $|V(C) \cap V(Q_4^0)| = 2$ and $|V(C) \cap V(Q_4^1)| = 4$. Clearly $Q_4^0 - (Z_6^4)_0$ is connected and every vertex in $Q_4^1 - (Z_6^4)_1$ is adjacent to a vertex in $Q_4^0 - (Z_6^4)_0$. Hence $Q_4 - Z_6^4$ is connected.

Case 2. Let $|V(C) \cap V(Q_4^0)| = 3$ and $|V(C) \cap V(Q_4^1)| = 3$. Note that subgraphs induced by $|V(C) \cap V(Q_4^0)$ and $|V(C) \cap V(Q_4^1)$ are subgraph isomorphic to $K_{1,2}$ in Q_4^0 and Q_4^1 respectively. Further both Q_4^0 and Q_4^1 are isomorphic to Q_3. By Lemma 3.4, removal of a $K_{1,2}$ does not disconnect Q_4^0 and Q_4^1. Thus both $Q_4^0 - (Z_6^4)_0$ and $Q_4^0 - (Z_6^4)_0$ are connected. Also there exists a vertex $x \in Q_4^0 - (Z_6^4)_0$ adjacent to $(x)^4 \in Q_4^1 - (Z_6^4)_1$. Thus $Q_4 - Z_6^4$ is connected. □

In the following lemma, we obtain a lower bound for the structure connectivity of Q_n with respect to C_6.

Lemma 3.6. *For an integer $n \geq 4$, $\kappa(Q_n; C_6) \geq \lceil \frac{n}{3} \rceil$.*

Proof. By induction on n. By Lemma 3.5, the result is true for $n = 4$. Assume as induction hypothesis that the statement holds for Q_i, $4 \leq i \leq n-1$. To complete the proof, one has to prove that if $|Z_6^n| \leq \lceil \frac{n}{3} \rceil - 1$, then $Q_n - Z_6^n$ is connected.

Case 1. Assume that either $V(Z_6^n) \subseteq V(Q_n^0)$ or $V(Z_6^n) \subseteq V(Q_n^1)$. Without loss of generality, let us assume that $V(Z_6^n) \subseteq V(Q_n^0)$. Note that Q_n^1 is connected and every vertex of $Q_n^0 - Z_6^n$ is connected to a vertex in Q_n^1 and hence $Q_n - Z_6^n$ is connected.

Case 2. Suppose $V(Z_6^n) \cap V(Q_n^1) \neq \phi$ and $V(Z_6^n) \cap V(Q_n^0) \neq \phi$.

Case 2.1. Assume that, for every 6-cycle X of Z_6^n, $V(X) \subset V(Q_n^1)$ or $V(X) \subset V(Q_n^0)$. In this case, we have the number of 6-cycles of Z_6^n and Q_n^1 is at most $\lceil \frac{n}{3} \rceil - 2$ and $|V(Z_6^n) \cap V(Q_n^0)| \leq \lceil \frac{n}{3} \rceil - 2$. Note that $\lceil \frac{n}{3} \rceil - 1 \leq \lceil \frac{n-1}{3} \rceil$ and so $\lceil \frac{n}{3} \rceil - 2 < \lceil \frac{n-1}{3} \rceil$. By the induction hypothesis, $\kappa(Q_n^i; C_6) \geq \lceil \frac{n-1}{3} \rceil$ and thus $Q_n^i - (Z_6^n)_i$ is connected for $i \in \{0, 1\}$. Since $6(\lceil \frac{n}{3} \rceil - 2) < 6(\frac{n+3}{3} - 2) = 2(n-3) < 2(n-2) \leq 2^{n-2}$, for $i = 0, 1$, $Q_n^i - (Z_6^n)_i$ contains more than $\frac{2^{n-1}}{2}$ vertices. Hence there exists a vertex $u \in Q_n^1 - (Z_6^n)_1$ which is adjacent to $(u)^{n-1} \in Q_n^0 - (Z_6^n)_0$. Hence $Q_n - Z_6^n$ is connected.

Case 2.2. Assume that $V(Z_6^n) \cap V(Q_n^1) \neq \phi$ and $V(Z_6^n) \cap V(Q_n^0) \neq \phi$ and there is a 6-cycle $X \in Z_6^n$ such that $V(X) \cap V(Q_n^i) \neq \phi$ for each $i = 0, 1$.

Let $Z_6^n = \{X_1, X_2, \ldots, X_m\}$, $m \leq \lceil \frac{n}{3} \rceil - 1$ where each X_i is a 6-cycle. For any $X_i \in Z_6^n$, the elements of $V(X_i)$ differ from one another in at most 3 coordinates. Let us name those coordinates as k_{i1}, k_{i2}, k_{i3}. i.e., if 0 (or 1) is the pth coordinate of an element in $V(X_i)$ for $p \neq k_{i1}, k_{i2}, k_{i3}$, then every element of $V(X_i)$ has 0 (or 1) as the pth coordinate.

Hence in total we have $3m$ such coordinates k_{i1}, k_{i2}, k_{i3}, for $1 \leq i \leq m$ (not necessarily distinct) corresponding to all elements in Z_6^n. Further, $3m \leq 3\lceil \frac{n}{3} \rceil - 3 < 3(\frac{n+3}{3}) - 3 = n$. Thus there exists $k \in \{1, 2 \ldots, n\}$ such that $k \notin \{k_{i1}, k_{i2}, k_{i3}\}$ for each $1 \leq i \leq m$. This means that kth coordinate of $V(X_i)$ is same in all the 6 elements of X_i. i.e., the kth coordinate of all the elements of $V(X_i)$ are equal.

Let us partition the vertices of Q_n into two subsets $V_j = \{x = (x_1 \ldots x_n) \in V(Q_n) : x_k = j, k$ is the index identified above$\}, j \in \{0, 1\}$. By the above arguments, for every i, $1 \leq i \leq m$, either $V(X_i) \subseteq V_0$ or $V(X_i) \subseteq V_1$. Note that both the induced subgraphs $\langle V_0 \rangle$ and $\langle V_1 \rangle$ of Q_n are isomorphic to Q_{n-1}. Now, if $Z_6^n \subseteq V_j$ for some $j \in \{0, 1\}$, proceeding as in Case 1, $Q_n - Z_6^n$ is connected. Otherwise, proceeding as in Case 2.1, $Q_n - Z_6^n$ is connected. Thus, $\kappa(Q_n; C_6) \geq \lceil \frac{n}{3} \rceil$. □

Figure 4 illustrates the C_6-structure connectivity of Q_4. In Figure 4, the structure cut is indicated with the dotted lines.

Since Q_3 is connected and by Lemma 3.3, $\kappa(Q_3; C_6) \leq \lceil \frac{3}{3} \rceil = 1$, $\kappa(Q_3; C_6) = 1 = \lceil \frac{3}{3} \rceil$. Also, by Lemma 3.3 and 3.6, we have the following result.

Theorem 3.7. *For $n \geq 3$, $\kappa^s(Q_n; C_6) \leq \kappa(Q_n; C_6) = \lceil \frac{n}{3} \rceil$.*

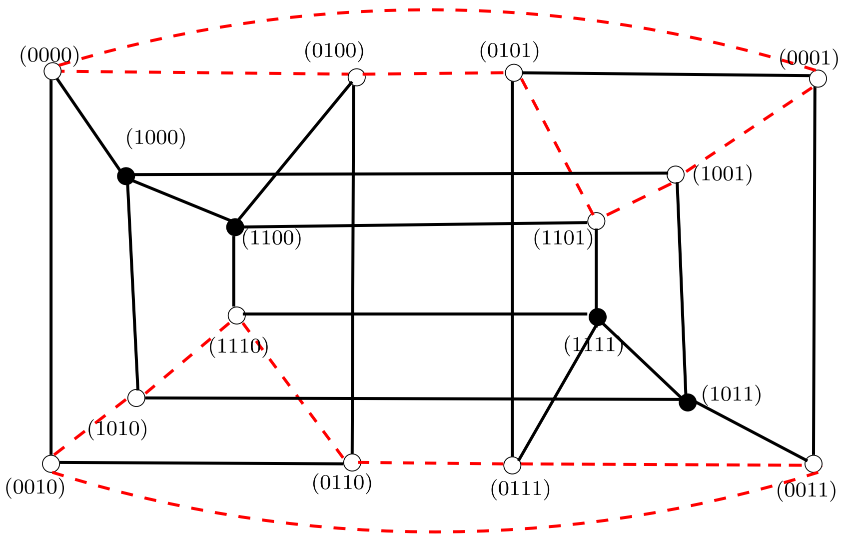

Figure 4.
C_6-structure cut of Q_4.

References

[1] N. Biggs, Algebraic Graph Theory, Cambridge University Press, 1993.

[2] G. Chatr, P. Zhang, Introduction To Graph Theory, Tata McGraw-Hill, 2006.

[3] Italo J. Dejter, Perfect domination in regular grid graphs, Australas. J. Combin. 42 (2008) 99–114.

[4] S. Lakshmivarahan, S.K. Dhall, Ring, torus, hypercube architectures algorithms for parallel computing, Parallel Comput. 25 (1999) 1877–1906.

[5] J. Lee, Independent perfect domination sets in Cayley graphs, J. Graph Theory 37 (4) (2001) 213–219.

[6] Cheng-Kuan Lin, Lili Zhang, Jianxi Fan, Dajin Wang, Structure connectivity and substructure connectivity of hypercubes, Theoret. Comput. Sci. 634 (2016) 97–107, http://dx.doi.org/10.1016/j.tcs.2016.04.014.

[7] S.A. Mane, Structure connectivity of hypercubes, AKCE Int. J. Graphs Comb. 15 (1) (2018) 49–52.

[8] T. Tamizh Chelvam, I. Rani, Independent domination number of Cayley graphs on $\mathbb{Z}_n$., J. Combin. Math. Combin. Comput. 69 (2009) 251–255.

[9] T. Tamizh Chelvam, M. Sivagami, Domination in Cayley graphs: A survey, AKCE Int. J. Graphs Comb. 16 (2019) 27–40, http://dx.doi.org/10.1016/j.akcej.2017.11.005.

[10] J.F. Wang, An investigation of the network reliability properties of circulant graphs (Doctoral dissertation), Stevens Institute of Technology, 1983.

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