Reconstruction of a homogeneous polynomial from its additive decompositions when identifiability fails

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Abstract
Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate complex variety. For any $q \in \mathbb{P}^r$ let $r_X(q)$ be its X-rank and $\mathcal{S}(X,q)$ the set of all finite subsets of X such that $|S| = r_X(q)$ and $q \in \langle S \rangle$, where $\langle\ \rangle$ denotes the linear span. We consider the case $|\mathcal{S}(X,q)| > 1$ (i.e. when q is not X-identifiable) and study the set $W(X)_q := \cap_{S \in \mathcal{S}(X,q)} \langle S \rangle$, which we call the non-uniqueness set of q. We study the case $\dim X = 1$ and the case X a Veronese embedding of $\mathbb{P}^n$. We conclude the paper with a few remarks concerning this problem over the reals.

Keywords X-rank, Veronese embedding, Symmetric tensor rank, Additive decomposition, Real X-rank
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1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field $\mathbb{K}$ with characteristic 0. For any set $A \subset \mathbb{P}^r$ let $\langle A \rangle$ denote its linear span. Fix any $q \in \mathbb{P}^r$. The *X-rank* $r_X(q)$ of X is the minimal cardinality of a finite set $S \subset X$ such that $q \in \langle S \rangle$. The notion of X-rank includes the notion of tensor rank of a tensor (take X a multi projective space and $X \subset \mathbb{P}^r$ its Segre embedding) and the notion of additive decomposition of a homogeneous polynomial or its symmetric tensor rank (take as X a projective space and as $X \subset \mathbb{P}^r$ one of its Veronese embeddings). See [3,13,18,19] for a long list of applications of these notions.

Notation 1. Let $\mathcal{S}(X,q)$ denote the set of all $S \subset X$ such that $|S| = r_X(q)$ and $q \in \langle S \rangle$. Set $W(X)_q := \cap_{S \in \mathcal{S}(X,q)} \langle S \rangle$.

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The set $W(X)_q$ is the main actor of this paper. We often write W_q if X is clear from the context.

Remark 1. Note that W_q is a linear subspace of $\mathbb{P}^r$ containing q and that if $W_q = \{q\}$, and $\mathcal{S}(X,q) = \mathcal{S}(X,q')$ for some $q' \in \mathbb{P}^r$, then $q' = q$. We will call W_q the *non-uniqueness set of q*. We have $\dim W_q = r_X(q) - 1$ if and only if $\langle S \rangle = \langle S' \rangle$ for all $S, S' \in \mathcal{S}(X,q)$. In particular $Wq = \{q\}$ and $q \notin X$ imply $|\mathcal{S}(X,q)| > 1$.

In this paper we prove one result on the Veronese variety (i.e. on the additive decomposition of homogeneous polynomials) (Theorem 3) and three results for the case $\dim X = 1$ (Theorems 1 and 2 and Proposition 1). The proof of the result on the Veronese variety uses one of the results for curves.

We first prove the following two cases (with X a curve) in which $W_q = \{q\}$.

Theorem 1. *Fix an even integer $r \geq 2$. Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate curve. There is a non-empty Zariski open subset $\mathcal{U} \subset \mathbb{P}^r$ such that $r_X(q) = r/2 + 1$ for all $q \in \mathcal{U}$ and the following properties hold:*

(a) We have $\{q\} = \cap_{S \in \mathcal{S}(X,q)} \langle S \rangle$ for all $q \in \mathcal{U}$.

(b) For all $(q,q') \in \mathcal{U} \times \mathbb{P}^r$ if $\mathcal{S}(X,q') = \mathcal{S}(X,q)$, then $q' = q$.

Theorem 2. *Fix an integer $d \geq 2$ and let $X \subset \mathbb{P}^d$ be the rational normal curve. Take any $q \in \mathbb{P}^d$ such that $\mathcal{S}(X,q)$ is not a singleton. Then $W_q = \{q\}$. Moreover, if $\mathcal{S}(X,q) = \mathcal{S}(X,q')$ for some $q' \in \mathbb{P}^d$, then $q' = q$.*

Take a non-degenerate $X \subset \mathbb{P}^r$ and $q \in \mathbb{P}^r$. For any integer $t > 0$ the t-secant variety $\sigma_t(X)$ of X is the closure in $\mathbb{P}^r$ of the union of all linear spaces $\langle S \rangle$ with $S \subset X$ and $|S| = t$. The *border rank* or *border X-rank* $b_X(q)$ of $q \in \mathbb{P}^r$ is the minimal integer $b \geq 1$ such that $q \in \sigma_b(X)$. We say that a finite set $A \subset \mathbb{P}^r$ *irredundantly spans q* if $q \in \langle A \rangle$ and $q \notin \langle A' \rangle$ for any $A' \subsetneq A$. We use Theorem 2 to prove the following result for the order d Veronese embedding of $\mathbb{P}^n$.

Theorem 3. *Fix integers n, d, b, k, such that $n \geq 2$, $d \geq 8$, $4 \leq 2b \leq d$ and $d + 2 - b \leq k \leq 2d - 2$. Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^r$, $r = \binom{n+d}{n} - 1$, be the orderd Veronese embedding. Let $L \subset \mathbb{P}^n$ be a line. Set $Y := \nu_d(L)$. Fix $q' \in \langle Y \rangle$ such that $b_Y(q') = b$ and $r_Y(q') = d + 2 - b$. Fix a general $U \subset \mathbb{P}^n$ such that $|U| = k - d - 2 + b$. Let $q \in \mathbb{P}^r$ be any point irredundantly spanned by $\{q'\} \cup \nu_d(U)$. Then:*

(1) *$r_X(q) = k$ and $\mathcal{S}(X,q) \supseteq \{E \cup U\}_{E \in \mathcal{S}(Y,q')}$.*

(2) *If $k \leq 2d - 3$, then $\mathcal{S}(X,q) = \{E \cup U\}_{E \in \mathcal{S}(Y,q')}$ and $W_q = \langle U \cup \{q'\} \rangle$.*

In Section 4 we consider the following problem. For any positive integer t let $\mathcal{S}(X,q,t)$ be the set of all $S \subset X$ such that $|S| = t$ and S irredundantly spans q. We have $\mathcal{S}(X,q,t) = \emptyset$ for all $t < r_X(q)$ and $\mathcal{S}(X,q,r_X) = \mathcal{S}(X,q) \neq \emptyset$. By the definition of irredundantly spanning set we have $\mathcal{S}(X,q,t) = \emptyset$ for all $t \geq r + 2$. Since X is integral and non-degenerate, for all (X,q) we have $\mathcal{S}(X,q,r+1) \neq \emptyset$ and $\mathcal{S}(X,q,r+1)$ contains a general subset of X with cardinality $r + 1$. There are easy examples of triples (X,q,t) such that $r > t > r_X(q)$ and $\mathcal{S}(X,q,t) = \emptyset$ (Remark 3). It easy to check that $\mathcal{S}(X,q,t) \neq \emptyset$ for all t such that $r + 1 - \dim X \leq t \leq r$ (Lemma 2). Set $W(X)_{q,t} := \cap_{S \in \mathcal{S}(X,q,t)} \langle S \rangle$, with the convention $W(X)_{q,t} := \mathbb{P}^r$ if $\mathcal{S}(X,q,t) \neq \emptyset$. We often write $W_{q,t}$ instead of $W(X)_{q,t}$.

In Section 4 we prove the following result.

Proposition 1. *Let $X \subset \mathbb{P}^r$, $r \geq 4$, be an integral and non-degenerate curve. Then there exists a non-empty Zariski open subset $\mathcal{U}$ of $\mathbb{P}^r$ such that $W_{q,t} = \{q\}$ for all $q \in \mathcal{U}$ and all $\lfloor (r+2)/2 \rfloor \leq t \leq r$.*

In Section 5 we briefly discuss the case of real algebraic subvarieties of $\mathbb{P}^r(\mathbb{R})$. In particular we show that a statement similar to Theorem 1 over $\mathbb{R}$ is true if we take as $\mathcal{U}$ a non-zero open subset of $\mathbb{P}^r(\mathbb{R})$, for the euclidean topology (Theorem 4), but it fails if we ask for a non-empty open subset of $\mathbb{P}^r(\mathbb{R})$ for the Zariski topology (Remark 5).

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2. Preliminary observations

The *cactus rank* or *cactus X-rank* $c_X(q)$ of $q \in \mathbb{P}^r$ is the minimal degree of a zero-dimensional scheme $Z \subset X$ such that $q \in \langle Z \rangle$. Let $\mathcal{Z}(X, q)$ denote the set of all zero-dimensional schemes $Z \subset X$ such that $\deg(Z) = c_X(q)$ and $q \in \langle Z \rangle$.

Remark 2. Let $X \subset \mathbb{P}^d$, $d \geq 2$, be a degree d rational normal curve. We use [18, §1.3] and [14] for the following observations. Fix $q \in \mathbb{P}^r$.

(i) We have $b_X(q) = c_X(q)$ ([18, Lemma 1.38]) and $|\mathcal{Z}(X, q)| = 1$ ([18, Part (i) of Theorem 1.43]).

(ii) If $c_X(q) < r_X(q)$, then $c_X(q) + r_X(q) = d + 2$ and $\mathcal{S}(X, q)$ is infinite. Let Z be the only element of $\mathcal{Z}(X, q)$, $d + 2 - c_X(q)$ is the minimal degree of a scheme $A \subset X$ such that $q \in \langle A \rangle$ and $A \not\supseteq Z$.

(iii) If $r_X(q) > c_X(q)$, then $\{q\} = \langle Z \rangle \cap \langle S \rangle$, where $\{Z\} = \mathcal{Z}(X, q)$ and S is any element of $\mathcal{S}(X, q)$ (this also follows from the fact that $h^1(\mathbb{P}^1, L) = 0$ for any line bundle L on $\mathbb{P}^1$ with $\deg(L) \geq -1$, as in the proof of Claim 1).

(iv) then If $r_X(q) > b_X(q)$, then $\dim \mathcal{S}(X, q) = d + 3 - 2b$ ([14, eq. (9)]).

(v) If d is odd and $r_X(q) = (d+1)/2$ (i.e. $r_X(q) = b_X(q)$ is the generic rank), then $\mathcal{S}(X, q) = \mathcal{Z}(X, q)$ and $|\mathcal{S}(X, q)| = 1$ ([18, Theorem 1.43]).

(vi) Assume d even and $r_X(q) = d/2 + 1$ and so q has the generic rank and $b_X(q) = r_X(q)$, but we do not assume that q is general in $\mathbb{P}^d$. Fix $S, S' \in \mathcal{S}(X, q)$ such that $S \neq S'$.

Claim 1. $\langle S \rangle \cap \langle S' \rangle = \{q\}$.

Proof of Claim 1. Since $S \neq S'$ and $S \in \mathcal{S}(X, q)$, we have $q \notin \langle S \cap S' \rangle$. The Grassmann's formula gives $h^1(\mathbb{P}^1, \mathcal{I}_{S \cup S'}(d)) > 0$. Since $h^1(\mathbb{P}^1, L) = 0$ for any line bundle L on $\mathbb{P}^1$ with $\deg(L) \geq -1$ and $q \notin X$, we have $S \cap S' = \emptyset$ and $h^1(\mathbb{P}^1, \mathcal{I}_{S \cup S'}(d)) = 1$. Thus the Grassmann's formula implies $\dim(\langle S \rangle \cap \langle S' \rangle) = 0$, proving Claim 1.

Obviously Claim 1 implies $W_q = \{q\}$ in this case, which by [18, Part (i) of Theorem 1.43] is the only case in which $r_X(q) = b_X(q)$ and $\mathcal{Z}(X, q)$ is not a singleton.

Note that (iii) implies that each $q \in \mathbb{P}^r$ with $c_X(q) \neq r_X(q)$ is uniquely determined by the zero-dimensional scheme evincing its cactus rank and by one single set evincing its rank (any $S \in \mathcal{S}(X, q)$ would do the job). Obviously part (i) implies that most $q \in \mathbb{P}^r$ (the ones with $r_X(q) = b_X(q)$) are not uniquely determined by $\mathcal{S}(X, q)$. By parts (i) and (ii) for each $q \in \mathbb{P}^r$ such that $r_X(q) = b_X(q)$ there are exactly ∞^t, $t := r_X(q) - 1$, points $o \in \mathbb{P}^r$ with $\mathcal{S}(X, o) = \mathcal{S}(X, q)$.

In the proof of Theorem 3 we use the following result ([5, Theorem 1], [4, Theorem 2]); we use the assumption $d \geq 6$ to have $4d - 5 \geq 3d + 1$ and hence to apply a small part of [5, Theorem 1].

Lemma 1 *([5, Theorem 1], [4, Theorem 2])***.** *Fix an integer $d \geq 6$. Let $S \subset \mathbb{P}^n$, $n \geq 2$, be a finite set such that $|S| \leq 4d - 5$. We have $h^1(\mathcal{I}_S(d)) > 0$ if and only if there is $F \subseteq S$ in one of the following cases:*

(1) $|F| = d+1$ *and* F *is contained in a line;*

(2) $|F| = 2d+2$ *and* F *is contained in a reduced conic* D*; if* $D = L_1 \cup L_2$ *with each* L_i *a line we have* $L_1 \cap L_2 \notin F$ *and* $|F \cap L_1| = |F \cap L_2| = d+1$*;*

(3) $|F| = 3d$*,* F *is contained in the smooth part of a reduced plane cubic* C *and* F *is the complete intersection of* C *and a degree* d *hypersurface;*

(4) $|F| = 3d+1$ *and* F *is contained in a plane cubic.*

3. Proofs of Theorems 1–3

Proof of Theorem 1. To prove part (b) it is sufficient to prove part (a), because $\mathcal{S}(X,q') = \mathcal{S}(X,q)$ implies $\{q'\} \subseteq Wq' = W_q$ and $W_q = \{q\}$ for $q \in \mathcal{U}$.

Since part (a) is trivial in the case $r = 2$, we assume $r \geq 4$. Since no non-degenerate curve is defective ([23, Corollary 1.5 and Remark 1.6]), there is a non-empty Zariski open subset $\mathcal{V} \subset \mathbb{P}^r$ such that $r_X(q) = r/2+1$ and $\dim \mathcal{S}(X,q) = 1$ for all $q \in \mathcal{V}$.

For each set $S \subset X$ such that $|S| = r/2+1$ and $\dim\langle S\rangle = r/2$ let $\ell_S : \mathbb{P}^r \setminus \langle S\rangle \to \mathbb{P}^{r/2-1}$ denote the linear projection from $\langle S\rangle$. For a general S we have $\langle S\rangle \cap X = S$ (scheme-theoretically) by Bertini's theorem and the trisecant lemma ([24, Corollary 2.2]) and $\ell_{S|X\setminus S}$ is birational onto its image, again by the trisecant lemma and the assumption $r \geq 4$.

Fix a general $S \subset X$ such that $|S| = r/2+1$. Let $X_S \subset \mathbb{P}^{r/2-1}$ be the closure of $\ell_S(X\setminus S)$ in $\mathbb{P}^{r/2-1}$. There is a finite set $E \subset X_S$ containing $X_S \setminus \ell_S(X \setminus S)$ and such that for each $p \in X_S \setminus E$ there is a unique $o \in X \setminus S$ such that $\ell_S(o) = p$. For any set $A \subset X_S \setminus E$ let $A_S \subset X \setminus S$ denote the only set such that $\ell_S(A_S) = A$. Any general $A \subset X_S \setminus E$ such that $|A| = r/2+1$ is linearly dependent, but each proper subset of A is linearly independent. Thus $\langle S\rangle \cap \langle A_S\rangle$ is a single point, $q_{S,A}$, and $q_{S,A} \notin \langle B\rangle$ for any $B \subsetneq A_S$. For a general A we get as A_S a general subset of X with cardinality $r/2+1$. Thus for a general A we have $q_{S,A} \notin S'$ for any $S' \subsetneq S$. We start with $S \in \mathcal{S}(X,o)$ for a general $o \in \mathbb{P}^r$. Thus $r_X(q) = r/2+1$ for a general $q \in \langle S\rangle$. Thus for a general A we get $S \in \mathcal{S}(q_{S,A})$ and $A_S \in \mathcal{S}(q_{S,A})$. By construction we have $\{q_{S,A}\} = \langle S\rangle \cap \langle A_S\rangle$. For a general A the point $q_{S,A}$ is general in $\langle S\rangle$. By the generality of S we get that the points $q_{S,A}$'s (with (S,A) varying, but general), cover a non-empty Zariski open subset of $\mathbb{P}^r$. □

Proof of Theorem 2. Set $b := b_X(q)$. Since $\mathcal{S}(X,q)$ is not a singleton, we have $q \notin X$ and hence $b \geq 2$. Part (vi) of Remark 2 covers the case $r_X(q) = b$ and hence we may assume $r_X(q) > b$. Thus $r_X(q) = d+2-b$. By part (v) of Remark 2 we have $\mathcal{S}(X,q) \geq 2$. We will prove the stronger assumption that $\{q\} = \cap_{A\in\Gamma}\langle A\rangle$, where Γ is any irreducible family contained in $\mathcal{S}(X,q)$ and with $\dim \Gamma = d+3-2b$; we do not assume that Γ is closed in $\mathcal{S}(X,q)$.

(a) First assume $b = 2$. We use the proof of [20, Proposition 5.1]. Fix $a \in \mathbb{P}^d\setminus\{q\}$. Let $H \subset \mathbb{P}^d$ be a general hyperplane containing q. Since $q \notin X$, Bertini's and Bezout's theorems give that $X \cap H$ is formed by d distinct points. Since X is connected, the exact sequence

$$0 \to \mathcal{I}_X \to \mathcal{I}_X(1) \to \mathcal{I}_{X\cap H,H}(1) \to 0$$

gives that $X \cap H$ spans H. Thus $q \in \langle X \cap H\rangle$. Since $r_X(q) = d$, we get $X \cap H \in \mathcal{S}(X,q)$. The generality of H gives $a \notin H$, concluding the proof that $W_q = \{q\}$.

(b) Step (a) and part (i) (resp. part (vi)) of Remark 2 for the case d odd (resp. d even) and q with generic rank cover all cases with $d \leq 4$. Thus we may assume $d \geq 5$ and use induction on d. Fix a general $o \in X$. Let $\ell_o : \mathbb{P}^d \setminus \{o\} \to \mathbb{P}^{d-1}$ denote the linear projection from o. Let

$Y \subset \mathbb{P}^{d-1}$ denote the closure of $\ell_o(X \setminus \{o\})$ in $\mathbb{P}^{d-1}$. Y is a rational normal curve of $\mathbb{P}^{d-1}$. Set $q' := \ell_o(q)$ and $Z' := \ell_o(Z)$ (by the generality of o we have $o \notin \langle Z \rangle$ and hence Z' is well-defined, $\deg(Z') = b$ and $\dim\langle Z' \rangle = b-1$). The generality of o also implies that $q \notin \langle Z'' \cup \{o\} \rangle$ for any $Z'' \subsetneq Z$ (here we use that X is a smooth curve and hence Z has only finitely many subschemes). Thus $q' \in \langle Z' \rangle$ and $q' \notin \langle Z'' \rangle$ for all $Z'' \subsetneq Z'$. Since Y is a degree $d-1$ rational normal curve and $b \le d/2$, parts (i) and (ii) of Remark 2 imply $b_Y(q') = b$ and $\mathcal{Z}(Y, q') = \{Z'\}$. Fix an irreducible family $\Gamma \subseteq \mathcal{S}(X, q)$ such that $\dim \Gamma = d+3-2b$ (it exists by part (v) of Remark 2). Let $\mathcal{B}$ denote the set of all $A \in \Gamma$ such that $o \in A$. By part (v) of Remark 2 and the generality of o we have $\mathcal{B} \neq 0$ and $\dim \mathcal{B} = d+2-2b$. Set $\mathcal{A} := \{\ell_o(B \setminus \{o\})\}_{B \in \mathcal{B}}$. Since Y is a rational normal curve, parts (i) and (ii) of Remark 2 imply $\mathcal{A} \subseteq \mathcal{S}(Y, q')$. We have $\dim \mathcal{A} = (d-1)+3-2b$. The inductive assumption gives $\{q'\} = \cap_{A \in \mathcal{A}} \langle A \rangle$. Thus $\langle \{o, q\} \rangle = \cap_{B \in \mathcal{B}} \langle B \rangle$. Since $\dim \Gamma = \dim \mathcal{S}(X, q) = d+3-2b$ (part (v) of Remark 2) and o is general in X, there is $S \in \Gamma$ such that $o \notin S$. Thus $\cap_{A \in \Gamma} \langle A \rangle = \{q\}$. □

Proof of Theorem 3. By Autarky ([19, Exercise 3.2.2.2]) we may assume $U \neq \emptyset$. Since U is general in $\mathbb{P}^n$, we have $\dim \langle \nu_d(U) \cup Y \rangle = \min\{r, \dim\langle Y \rangle + |U|\}$. Since $\dim \langle Y \rangle = d$ and $d + |U| < r$, we have $\langle \nu_d(U) \rangle \cap \langle Y \rangle = \emptyset$. By Theorem 2 we have $W(Y)_{q'} = \{q'\}$. Take $E \in \mathcal{S}(Y, q')$ and set $A := U \cup E$. The set $\{q'\} \cup U$ irredundantly spans q and $\langle Y \rangle \cap \langle \nu_d(U) \rangle = \emptyset$. Hence we have $E \cap U = \emptyset$ and $|A| = k$. Since $|A| = k$ and $q \in \langle \nu_d(A) \rangle$, we have $r_X(q) \le k$. Since U is general in $\mathbb{P}^n$, we have $h^0(\mathcal{I}_A(t)) = \max\{0, h^0(\mathcal{I}_E(t)) - |U|\}$ for all $t \in \mathbb{N}$; to use this equality we need to fix one element, E, of $\mathcal{S}(Y, q')$, before choosing a general U.

Note that we have $W_q = \langle \nu_d(U) \cup \{q'\} \rangle$ for any q such that $\mathcal{S}(X, q) = \{E \cup U\}_{E \in \mathcal{S}(Y, q')}$ by Theorem 2. Assume either $r_X(q) < k$ or $k \le 2d-3$ and the existence of $B \in \mathcal{S}(X, q) \setminus \{E \cup U\}_{E \in \mathcal{S}(Y, q')}$. In the former case take $B \in \mathcal{S}(X, q)$. Set $S := A \cup B$. In both cases we have $|B| \le |A|$ and $|A| + |B| \le 4d-5$. Since $h^1(\mathcal{I}_S(d)) > 0$ ([6, Lemma 1]) there is $F \subseteq S$ in one of the cases listed in Lemma 1.

(a) Assume the existence of a plane cubic $T \subset \mathbb{P}^n$ such that $|T \cap S| \ge 3d$.

(a1) Assume $n = 2$. Thus T is an effective divisor of $\mathbb{P}^2$. Consider the residual exact sequence of T in $\mathbb{P}^2$:

$$0 \to \mathcal{I}_{S \setminus S \cap T}(d-3) \to \mathcal{I}_S(d) \to \mathcal{I}_{S \cap T, T}(d) \to 0 \tag{1}$$

Since $|S \setminus S \cap T| \le 4d-5-3d = d-5$, we have $h^1(\mathcal{I}_{S \setminus S \cap T}(d-3)) = 0$. Thus either [7, Lemma 5.1] or [8, Lemmas 2.4 and 2.5] give $A \setminus A \cap T = B \setminus B \cap T$. Assume for the moment $L \nsubseteq T$. Bezout gives $|L \cap T| \le 3$. Since U is general and $h^0(\mathcal{O}_{\mathbb{P}^2}(3)) = 10$, we have $|U \cap T| \le 9$. Thus $|B \cap T| \ge 3d-12 > 12 \ge |T \cap A|$ and hence $|B| > |A|$, a contradiction. Now assume $L \subset T$. Since $h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 6$ and $U \cap L = \emptyset$, we get $|A \cap T| \le d+8-b$. Thus $|B \cap T| \ge 2d - 5 + b$ and again $|B| > |A|$, a contradiction.

(a2) Assume $n > 2$. Let $M \subset \mathbb{P}^n$ be a general hyperplane containing the plane $\langle T \rangle$ (so $M = \langle T \rangle$ if $n = 3$). Since S is a finite set and M is a general hyperplane containing $\langle T \rangle$, we have $S \cap M = S \cap \langle T \rangle$. Consider the residual exact sequence of M in $\mathbb{P}^n$:

$$0 \to \mathcal{I}_{S \setminus S \cap M}(d-1) \to \mathcal{I}_S(d) \to \mathcal{I}_{S \cap M, M}(d) \to 0 \tag{2}$$

Since $|S \setminus A \cap M| \le 4d-5-3d = d-5$, we have $h^1(\mathcal{I}_{S \setminus S \cap M}(d-1)) = 0$. Thus either [7, Lemma 5.1] or [8, Lemmas 2.4 and 2.5] give $A \setminus A \cap M = B \setminus B \cap M$. Since no 4 points of U are coplanar, we have $|A \cap M| \le d+5-b < 3d-d-2+b$. Thus $|B| > |A|$, a contradiction.

(b) Assume the existence of a plane conic D such that $|S \cap D| \geq 2d+2$.

(b1) Assume $n=2$. Consider the residual exact sequence of D in $\mathbb{P}^2$:

$$0 \to \mathcal{I}_{S\setminus S\cap D}(d-2) \to \mathcal{I}_S(d) \to \mathcal{I}_{S\cap D,D}(d) \to 0 \tag{3}$$

First assume $h^1(\mathcal{I}_{S\setminus S\cap D}(d-2)) > 0$. Since $|S \setminus S \cap D| \leq 4d-5-2d-2 = 2(d-3)-1$, there is a line $R \subset \mathbb{P}^2$ such that $|R \cap (S \setminus S \cap D)| \geq d-1$ ([9, Lemma 34]). Thus $|S \cap (D \cup R)| \geq 3d+1$. Step (a1) gives a contradiction. Now assume $h^1(\mathcal{I}_{S\setminus S\cap D}(d-2)) = 0$. Either [7, Lemma 5.1] or [8, Lemmas 2.4 and 2.5] give $A\setminus A\cap D = B\setminus B\cap D$. Assume for the moment $L \nsubseteq D$. Thus $|L \cap D| \leq 2$. Since U is general and $h^0(\mathcal{O}_{\mathbb{P}^2}(2)) = 6$, we have $|U \cap D| \leq 5$. Thus $|B \cap D| > |A \cap D|$ and so $|B| > |A|$, a contradiction. Now assume $L \subset D$. Write $D = L \cup R$ with R a line. Set $\{o\} := L \cap R$. Since $U \cap L = \emptyset$, $|L \cap R| = 1$ and $|U \cap R|' \leq 2$ for each line R', we have $|A \cap D| \leq d+4-b$. Let $Z' \subset L$ be the only degree b zero-dimensional scheme evincing the cactus rank of q' with respect to the rational normal curve $\nu_d(L)$ (part (i) of Remark 2). Set $Z'' := Z' \cup U$ and $Z := Z'' \cup B$. Since $A\setminus A\cap D = B\setminus B\cap D$, we have $Z = Z' \cup (U \cap R) \cup (B \cap D) \cup (U \setminus U \cap D)$. Since $q' \in \langle \nu_d(Z')\rangle$, we have $q \in \langle \nu_d(Z'')\rangle \cap \langle \nu_d(B)\rangle$. Thus $h^1(\mathcal{I}_Z(d)) > 0$. Since $h^1(\mathcal{I}_{U\setminus S\cap D}(d-2)) = h^1(\mathcal{I}_{S\setminus S\cap D}(d-2)) = 0$, the residual sequence (3) of D in $\mathbb{P}^2$ with Z instead of S gives $h^1(D, \mathcal{I}_{Z\cap D,D}(d)) > 0$. Since $D = R \cup L$, using either [16, Corollaire 2] or the residual exact sequences of R and L in $\mathbb{P}^2$ we get that we are in one of the following cases:

(1) $\deg(Z \cap L) \geq d+2$;

(2) $\deg(Z \cap R) \geq d+2$;

(3) $\deg(Z \cap R) = \deg(Z \cap L) = d+1$ and $o \notin Z_{\mathrm{red}}$.

Recall that $A \setminus A \cap B = B \setminus B \cap D$ (and hence) $|B \cap D| \leq |A \cap D|$ and that $|A \cap D| \leq d+4-b$.

(b1.1) Assume $\deg(Z \cap L) \geq d+2$. Since $Z \cap L = Z' \cup (B \cap L)$, we get $|B \cap L| \geq d+b-2$. Consider the residual exact sequence of L in $\mathbb{P}^2$:

$$0 \to \mathcal{I}_{S\setminus S\cap L}(d-1) \to I_S(d) \to \mathcal{I}_{S\cap L,L}(d) \to 0 \tag{4}$$

First assume $h^1(\mathcal{I}_{S\setminus S\cap L}(d-1)) > 0$. Since $h^1(\mathcal{I}_{S\setminus S\cap (R\cup L)}(d-2)) = 0$, the residual exact sequence of R gives $h^1(R, \mathcal{I}_{S\cap R\setminus S\cap R\cap L}(d-1)) > 0$. Thus $|S \cap R \setminus S \cap \{o\}| \geq d+1$. Since $|U \cap R| \leq 2$, we get $|B \cap R \setminus B \cap \{o\}| \geq d-1$ and hence $|B| > |A|$ (because $d \geq 5$), a contradiction.

Now assume $h^1(\mathcal{I}_{S\setminus S\cap L}(d-1)) = 0$. By [7, Lemma 5.1] or [8, Lemmas 2.4 and 2.5] we get $A \setminus A \cap L = B \setminus B \cap L$. Since $|B \cap L| \geq d+2-b = |A \cap L|$, we get $|B \cap L| = d+2-b$. In this case part (1) of Theorem 3 is proved. To prove part (2) we need to prove that $B \cap L \in \mathcal{S}(Y, q')$. Since $|B \cap L| = d+2-b$ and $\deg(Z \cap L) \geq d+2$, we get $Z_1 \cap B = \emptyset$ and $\deg(Z \cap B) = d+2$. Thus $\langle \nu_d(B \cap L)\rangle \cap \langle \nu_d(Z')\rangle$ is a single point, q'', and $B \cap L \in \mathcal{S}(Y, q'')$. Since U is general, and $|U| \leq \binom{d+2}{2} - d$, we have $\langle \nu_d(U)\rangle \cap \langle \nu_d(L)\rangle = \emptyset$. Since $B \setminus B \cap L = A \setminus A \cap L = U$, we get $q'' = q'$, proving part (2) in this case.

(b1.2) Assume $\deg(Z \cap R) = \deg(Z \cap L) = d+1$ and $o \notin Z_{\mathrm{red}}$ (i.e. $o \notin Z'_{\mathrm{red}}$). Since $\deg(Z') = b$ and $\deg(Z'' \cap D) = b + |U \cap R| \leq b+2$, we get $|L \cap B| \geq d+1-b$, $|R \cap B| \geq d-1$ and $o \notin B$. Thus $|B \cap D| \geq 2d+2-b$. Since $|B \cap D| \leq |A \cap D| \leq d+4-b$, we obtain a contradiction.

(b1.3) Assume $\deg(Z \cap R) \geq d+2$. Since $|U \cap R| \leq 2$ with strict inequality if $o \in Z'_{\mathrm{red}}$ and every point of $\langle \nu_d(R)\rangle$ has rank $\leq d$ by Sylvester's theorem, we get $|U \cap R| +$

$\deg(Z' \cap R) = 2$, $|B \cap R| = d$ and $U \cap B \cap R = \emptyset$. If $h^1(\mathcal{I}_{S\setminus S\cap R}(d-1)) = 0$, we have $A \setminus A\cap R = B \setminus B \cap R$ by [7, Lemma 5.1] or [8, Lemmas 2.4 and 2.5] and hence $|B| > |A|$, a contradiction. Now assume $h^1(\mathcal{I}_{S\setminus S\cap R}(d-1)) > 0$. Since $h^1(\mathcal{I}_{S\setminus S\cap D}(d-2)) = 0$ in this part of the proof, the residual exact sequence of L gives $h^1(L, \mathcal{I}_{L\cap S\setminus L\cap S\cap R}(d-1)) > 0$ and hence $|S\cap (L\setminus L\cap R)| \geq d+1$. Thus $|B\cap (L \setminus L\cap R)| \geq b-1$. We get $|B\cap D| \geq d+b-1$. Since $A\setminus A\cap D = B \setminus B\cap D = U \setminus U\cap D$, we get $d+b-1 \leq d+4-b$ and hence $b=2$ and $|B\cap L| \leq 2$. Since $|U\cap R| + \deg(Z'\cap R) \geq 2$ and q' uniquely determines Z', R is uniquely determined by Z' and the set $|R\cap U|$. If $o\in Z'_{\mathrm{reg}}$ R is uniquely determined by q' and one point of $R \setminus \{o\}$. Since we took a general U after fixing q', we have $|U\cap R| \leq 1$ if $R\cap L \in Z'_{\mathrm{reg}}$. Hence (varying the points of $U \setminus U\cap R$ (if $U \nsubseteq R$) we may (after fixing q') assume that $U \setminus U\cap R$ is general in $\mathbb{P}^n \setminus D$. Since $A \setminus A\cap D = B \setminus B\cap D = U \setminus U\cap D$, $|U \setminus U\cap D| \leq \binom{d+2}{2} - 2d - 1$ and $U \setminus U\cap D$ is general, we have $\langle \nu_d(U \setminus U\cap D)\rangle \cap \langle \nu_d(D)\rangle$. Thus there is a unique $q'' \in \langle \nu_d(D)\rangle$ such that $q \in \langle\{\nu_d(U \setminus U\cap R), q''\}\rangle$. We have $q'' \in \langle \nu_d(A\cap D)\rangle \cap \langle \nu_d(B\cap D)\rangle \cap \langle\{q', \nu_d(U\cap R)\}\rangle$. Thus is it sufficient to prove parts (1) and (2) of the theorem for q'', $A\cap D$ and $B\cap D$ instead of q, A and B, i.e. in the rest of this step we assume $U = U\cap R$. Since $|B| \leq |A|$, we have $|B\cap L| \leq 2$. Thus $h^1(\mathcal{I}_{Z'\cup(L\cap B)}(d-1)) = 0$. if $o \notin Z'_{\mathrm{red}} \cap B$, [7, Lemma 5.1] gives a contradiction, because $Z' \nsubseteq B$. Now assume $o\in Z'_{\mathrm{red}}\cap B$. Since $o\in Z'_{\mathrm{red}}$ and Z' is uniquely determined by q', we observed that $|U\cap R| \leq 1$. Thus (under the assumption $U\subset D$), we have $|U| \leq 1$. Since $|B\cap R| = d$, $o = L \cap R \in B$ by assumption and $Z' \nsubseteq R$, we get $\deg(Z\cap R) \leq d+1$, a contradiction.

(b2) Assume $n > 2$. Let $M\subset \mathbb{P}^n$ be a general hyperplane containing the plane $\langle D\rangle$. Thus $S \cap M = S \cap \langle D\rangle$. Since U is general, no 4 points of U are coplanar. Thus $|U\cap M| = |U\cap \langle D\rangle| \leq 3$.

(b2.1) Assume $h^1(I_{S\setminus S\cap M}(d-1)) > 0$. Since $|S \setminus S\cap M| \leq |A| + |B| - 2d - 2 \leq 2(d-1)+1$, there is a line $R'\subset \mathbb{P}^n$ such that $|R'\cap (S \setminus S\cap M)| \geq d+1$. If $R'\subset \langle D\rangle$, then $R'\cup D$ is a plane cubic and we may apply step (a1). Thus we may assume $R' \nsubseteq \langle D\rangle$. Let $N\subset \mathbb{P}^n$ be a general hyperplane containing N. Since S is a finite set, the generality of M and N gives $S \cap (M\cup N) = S \cap (D\cup R')$. Consider the residual exact sequence

$$0 \to \mathcal{I}_{S\setminus S\cap(M\cup N)}(d-2) \to \mathcal{I}_S(d) \to \mathcal{I}_{S\cap(M\cup N), M\cup N}(d) \to 0 \tag{5}$$

of $M\cup N$ in $\mathbb{P}^n$. Since $|S \setminus S \cap (M\cup N)| \leq |A| + |B| - 3d - 3 \leq d-1$, we have $h^1(I_{S\setminus S\cap(M\cup N)}(d-2)) = 0$. Thus either [7, Lemma 5.1] or [8, Lemmas 2.4 and 2.5] give $A \setminus A \cap (M\cup N) = B \setminus B\cap (M\cup N)$. We have $A\cap (M\cup N) \subseteq E \cup (U\cap (M\cup N))$ and hence $|A \setminus A \cap (M\cup N)| \geq k-d-2+b-5$. Since $|A\cap (M\cup N)| \leq d+7-b$, we get $|B \cap (M\cup N)| \geq 3d+3-d-7+b = 2d-4+b$. Since $|B \setminus B\cap (M\cup N)| \geq k-d-2+b-5$, we get a contradiction.

(b2.2) Assume $h^1(\mathcal{I}_{S\setminus S\cap M}(d-1)) = 0$. Either [7, Lemma 5.1] or [8, Lemmas 2.4 and 2.5] give $A \setminus A\cap M = B \setminus B\cap M$. Since $|U\cap M| = |U\cap \langle D\rangle| \leq 3$, we have $|U \setminus U\cap M| \geq k-d-5+b$. Assume for the moment $L \nsubseteq \langle D\rangle$. We get $|E\cap M| \leq 1$ and hence $|A \setminus A\cap M| \geq k-4$. Since $A \setminus A\cap M = B \setminus B\cap M$, we get $|S\cap M| \leq |A| + |B| - 2k - 8$ and hence $2d+2 \leq 8$, a contradiction. Now assume $L\subset \langle D\rangle$. If $L \nsubseteq D$ we get (since $|L\cap D| \leq 2$) $|S\cap M| \geq 3d+b$. Since $A \setminus A \cap M = B \setminus B\cap M$ and $|U \setminus U\cap M| \geq k-d-1+b$, we get $|S| \geq 2d+2b+k-1$, a contradiction.

(c) Assume the existence of a line $R\subset \mathbb{P}^n$ such that $|R\cap S| \geq d+2$. Let $M\subset \mathbb{P}^n$ be a general hyperplane containing R (so $M = R$ if $n = 2$). Since S is a finite set and M is a general hyperplane containing R, we have $M\cap S = R\cap S$. Since U is a general subset of $\mathbb{P}^n$ with cardinality $k-d-2+b$, no 3 of its points are collinear (and hence $|U\cap R| \leq 2$) and $U\cap L = \emptyset$. Let $M\subset \mathbb{P}^n$ be a general hyperplane containing R (so $M = R$ if $n = 2$). Since S is a finite set and M is a general hyperplane containing R, we have $M\cap S = R\cap S$.

(c1) Assume $h^1(\mathcal{I}_{S\setminus S\cap M}(d-1)) > 0$. Since $|S \setminus S \cap M| \le |A| + |B| - d - 2 \le 3(d-1) - 1$, either there is a line R_1 such that $|R_1 \cap (S \setminus S \cap M)| \ge d+1$ or there is a conic D_1 such that $|D_1 \cap (S \setminus S \cap M)| \ge 2d$. If R and R_1 (resp. R and D_1) are contained in a plane, and in particular if $n = 2$, step (b) (resp. step (a)) gives a contradiction, because $|S \cap (R \cup R_1)| \ge 2d+3$ (resp. $|S \cap (R \cup D_1)| \ge 3d+2$). Thus we may assume that this is not the case and in particular we may assume $n > 2$. Let N be a general hyperplane containing R_1 (resp. D_1). We use the residual exact sequence (5). Note that $S \cap (M \cup N) = S \cap (R \cup R_1)$ (resp. $S \cap (M \cup N) = S \cap (R \cup \langle D_1 \rangle)$).

(c1.1) Assume $h^1(\mathcal{I}_{S\setminus S\cap(M\cup N)}(d-2)) > 0$. We exclude the existence of D_1, because $|S \cap (R \cup D_1)| \ge 3d+2$ and hence $|S \setminus S \cap (M \cup N)| \le d-1$. Thus in this case we may assume the existence of R_1. Since $|S \cap (R \cup R_1)| \ge 2d+3$, we have $|S \setminus S \cap (M \cup N)| \le |A| + |B| - 2d - 3 \le 2(d-2)+1$. By [9, Lemma 34] there is a line R_2 such that $|R_2 \cap S \setminus S \cap (M \cup N)| \ge d$. Let M' be a general hyperplane containing R_2. Consider the residual exact sequence of $M' \cup M \cup N$. We have $h^1(\mathcal{I}_{S\setminus S\cap(M\cup N\cup M')}(d-3)) = 0$, because $|S \setminus S \cap (M \cup N \cup M')| \le 2k - d - 2 - d - 1 - d \le d-4$. Either [7, Lemma 5.1] or [8, Lemmas 2.4 and 2.5] give $A \setminus A \cap (M \cup N \cup M') = B \setminus B \cap (M \cup N \cup M')$. Since M, N and M' are general, we have $S \cap (M \cup N \cup M') = S \cap (R \cup R_1 \cup R_2)$. Since U is general, no 3 of the points of U are collinear. Thus $|U \cap (R \cup R_1 \cup R_2)| \le 6$. Hence $|A \setminus A \cap (M \cup N \cup M')| \ge k - d - 8 + b$. Since $A \setminus A \cap (M \cup N \cup M') = B \setminus B \cap (M \cup N \cup M')$, we get $|S \cap (M \cup N \cup M')| \le 2k - 2k + 2d + 16 - 2b$. Hence $2d + 16 - 2b \ge 3d+3$, a contradiction.

(c1.2) Assume $h^1(\mathcal{I}_{S\setminus S\cap(M\cup N)}(d-2)) = 0$. Either [7, Lemma 5.1] or [8, Lemmas 2.4 and 2.5] give $A \setminus A \cap (M \cup N) = B \setminus B \cap (M \cup N)$. Since $U \cap (M \cup N) = U \cap (R \cup R_1)$, we have $|U \setminus U \cap (M \cup N)| \ge k - d - 6 + b$. Assume for the moment $L \notin \{R, R_1\}$. We get $|L \cap (M \cup N)| \le 2$. Thus $|A \setminus A \cap (M \cup N)| \ge k + b - 8$. Since $A \setminus A \cap (M \cup N) = B \setminus B \cap (M \cup N)$, we get $|S \cap (M \cup N)| \le 16 - b < 2d+3$ (even when instead of $|S|$ we take $2k$). Thus we may assume that either $L = R$ or $L = R'$. In both cases, writing $D := R \cup R'$ we are in the case solved in step (b1).

(c2) Assume $h^1(\mathcal{I}_{S\setminus S\cap M}(d-1)) = 0$. Either [7, Lemma 5.1] or [8, Lemmas 2.4 and 2.5] give $A \setminus A \cap M = B \setminus B \cap M$.

(c2.1) Assume $R = L$. We get $U = A \setminus A \cap L = B \setminus B \cap L$. Thus $B = U \cup (B \cap L)$. Since $\langle \nu_d(U) \rangle \cap \langle Y \rangle = \emptyset$, $q \in \langle \nu_d(U) \cup Y \rangle$, $q \notin \langle \nu_d(U) \rangle$, $q \notin \langle \nu_d(Y) \rangle$ (because $U \ne \emptyset$) and $\langle \nu_d(U) \rangle \cap \langle Y \rangle = \emptyset$, there are uniquely determined $q_1 \in \langle \nu_d(U) \rangle$ and $q_2 \in \langle Y \rangle$ such that $q \in \langle \{q_1, q_2\} \rangle$. The uniqueness of q_2 gives $q_2 = q'$. Since $\langle \nu_d(U) \rangle \cap \langle Y \rangle = \emptyset$ and $q \in \langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$, we get $q' \in \langle \nu_d(B \cap L) \rangle$. Thus $|B \cap L| \ge r_Y(q') = |A \cap L|$. Since $|B| \le |A|$ and $A \setminus A \cap L = B \setminus B \cap L$, we get $|B| = |A|$ and $B = U \cup F$ with $F \cap U = \emptyset$ and $F \in \mathcal{S}(Y, q')$. Thus the theorem is true in this case.

(c2.2) Assume $R \ne L$. Since $|L \cap R| \le 1$, we get $|E \cap R| \le 1$. Since $|U \cap R| \le 2$, we get $|A \cap R| \le 3$ and hence $|B \cap R| \ge d - 1 > |A \cap R|$. Since $A \setminus A \cap R = B \setminus B \cap R$, we get $|B| > |A|$, a contradiction. □

4. Irredundantly spanning sets

Lemma 2. *If $r + 1 - \dim X \le t \le r$, then $\mathcal{S}(X, q, t) \ne \emptyset$.*

Proof. The case $t = r + 1 - \dim X$ is an obvious consequence of the proof of [20, Proposition 5.1]. Assume $r + 2 - \dim X \le t \le r$. Let $Y \subset \mathbb{P}^r$ be the intersection of X and $(t + \dim X - r - 1)$ general quadric hypersurfaces. By Bertini's theorem Y is an integral and non-degenerate subvariety of $\mathbb{P}^r$. Thus for any q we have $\mathcal{S}(X, q, t) \supseteq \mathcal{S}(Y, q, t)$. Since $t = r + 1 - \dim Y$, we get $\mathcal{S}(Y, q, t) \ne \emptyset$. □

Remark 3. Let $X \subset \mathbb{P}^d$, $d \geq 4$, be a rational normal curve. Fix $q \in \mathbb{P}^d$ such that $r_X(q) = 2$. Since any subset of X with cardinality at most $d+1$ is linearly independent, the definition of irredundantly spanning set gives $\mathcal{S}(X, q, t) = \emptyset$ for all t such that $3 \leq t \leq d-1$.

Proof of Proposition 1. Since a finite intersection of non-empty Zariski open subsets of $\mathbb{P}^r$ is open and non-empty and the interval $\lfloor (r+2)/2 \rfloor \leq t \leq r$ contains only finitely many integers, it is sufficient to prove the statement for a fixed t. The case $t = r$ is true by Remark 3. The case r even and $t = r/2 + 1$ is true by Theorem 2. Thus when r is even we may assume $r/2 + 2 \leq t \leq r$. Since we saw that the case $r = t$ is always true, we proved the proposition for $r = 4$. Thus we may assume $r \geq 5$ and that the proposition is true for all curves in a lower dimensional projective space. Fix a general $p \in X$ and call $\ell : \mathbb{P}^r \setminus \{p\} \to \mathbb{P}^{r-1}$ the linear projection from p. Let $Y \subset \mathbb{P}^{r-1}$ be the closure of $\ell(X \setminus \{p\})$ in $\mathbb{P}^{r-1}$. Y is an integral and non-degenerate curve. Since p is general in X, it is a smooth point of X and hence $\ell_{|X \setminus \{p\}}$ extends to a surjective morphism $\mu : X \to Y$ with $\mu(p)$ associated to the tangent line of X at p. Thus $Y = \mu(X)$. By the trisecant lemma ([24, Corollary 2.2]) and the generality of p we have $\deg(L \cap X) \leq 2$ for every line $L \subset \mathbb{P}^r$ such that $p \in L$. Hence $\ell_{|X \setminus \{p\}}$ is birational onto its image and there is a finite set $F \subset X$ containing p such that $\mu_{|X \setminus F}$ induces an isomorphism between $X \setminus F$ and $Y \setminus \mu(F)$. Fix the integer t such that $\lfloor (r+2)/2 \rfloor \leq t \leq r$ and write $z := t-1$. By the inductive assumption and, if r is odd and $t = \lfloor (r+2)/2 \rfloor$, Theorem 2 applied to the projective space $\mathbb{P}^{r-1}$ there is a non-empty Zariski open subset $\mathcal{V}$ of $\mathbb{P}^{r-1}$ such that $W(Y)_{q,z} = \{q\}$ for all $q \in \mathcal{V}$. Fix $a \in \mathcal{V}$ and finitely many $S_i \in \mathcal{S}(Y, a, z)$, $1 \leq i \leq e$, such that $\{a\} = \cap_{i=1}^{e} \langle S_i \rangle$. Restricting if necessary $\mathcal{V}$ we may assume that (for a choice of sufficiently general $S_1(a), \ldots, S_e(a)$) we have $S_i(a) \cap \mu(F) = \emptyset$ for all i and all a. Hence there is a unique $A_i(a) \subset X \setminus F$ such that $\mu(A_i(a)) = S_i(a)$. Since $p \in F$, $B_i(a) := A_i(a) \cup \{p\}$ has cardinality t, $1 \leq i \leq e$. Set $\mathcal{U}_p := \ell^{-1}(\mathcal{V}) \subset \mathbb{P}^r \setminus \{p\}$. For each $a \in \mathcal{V}$, set $L_a := \{p\} \cup \ell^{-1}(a)$. Each L_a is a line containing p, $\mathcal{U}_p$ is the union of all $L_a \setminus \{p\}$, $a \in \mathcal{V}$, and $L_a = \cap_{i=1}^{e} \langle B_i(a) \rangle$. Fix $a \in \mathcal{V}$ and $b \in L_a \setminus \{p\}$. Note that each $B_i(a)$ irredundantly spans b. Fix another general $o \in X$, $o \neq p$. We get in the similar way a set $\mathcal{U}_o$. It is easy to check that $W_{q,t} = \{q\}$ for all $q \in \mathcal{U}_0 \cap \mathcal{U}_p$. Thus we may take $\mathcal{U} = \mathcal{U}_p \cap \mathcal{U}_0$. $\square$

5. Real varieties and real ranks

Up to now we worked over an algebraically closed field $\mathbb{K}$ with characteristic zero. In this section we take $\mathbb{K} = \mathbb{C}$, but we consider varieties $X \subset \mathbb{P}^r$ defined over $\mathbb{R}$. Not only we fix the real structure of X but we assume that the embedding $X \hookrightarrow \mathbb{P}^r$ is defined over $\mathbb{R}$. We call $X(\mathbb{C})$ and $\mathbb{P}^r(\mathbb{R})$ the set of all complex points of X and $\mathbb{P}^r$. For any $q \in \mathbb{P}^r(\mathbb{C})$ we have defined the X-rank $r_X(q)$ and the set $\mathcal{S}(X, q)$. In this section we write $r_{X(\mathbb{C})}(q)$ instead of $r_X(q)$ and $\mathcal{S}(X(\mathbb{C}), q)$ instead of $\mathcal{S}(X, q)$. Since X is defined over $\mathbb{R}$, the set $X(\mathbb{R})$ of its real points is well-defined. Since the embedding $X \hookrightarrow \mathbb{P}^r$ is defined over $\mathbb{R}$, we have $X(\mathbb{R}) = X(\mathbb{C}) \cap \mathbb{P}^r(\mathbb{R})$. Easy examples show that a nice X defined over $\mathbb{R}$ may have $X(\mathbb{R}) = \emptyset$. For instance take the smooth plane conic $C := \{x_0^2 + x_1^2 + x_3^2 = 0\}$ (we have $C(\mathbb{C}) \cong \mathbb{P}^1(\mathbb{C})$). Felix Klein proved that for every integer $g \geq 0$ there is a smooth curve $X(\mathbb{C})$ of genus g defined over $\mathbb{R}$ and with $X(\mathbb{R}) = \emptyset$ ([17, Proposition 3.1]). Thus the assumption that $X(\mathbb{R})$ is large is necessary. We assume that X has a smooth point defined over $\mathbb{R}$ (in symbols, we assume $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$). Set $n := \dim X = \dim_{\mathbb{C}} X(\mathbb{C})$. The sets $\mathbb{P}^r(\mathbb{C})$ and $X(\mathbb{C})$ also have a euclidean topology. With the euclidean topology $X_{\mathrm{reg}}(\mathbb{R})$ is a topological (and C^∞) manifold with pure dimension n and the assumption $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$ says that this manifold is non-empty. The assumption $X_{\mathrm{reg}}(\mathbb{R}) \neq \emptyset$ is equivalent to assuming that $X(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})$, because $\mathrm{Sing}(X(\mathbb{C}))$ is a union of complex varieties of dimension $< n$. For any set $S \subset \mathbb{P}^r(\mathbb{C})$ let $\langle S \rangle_{\mathbb{C}}$ be the complex linear projective subspace of $\mathbb{P}^r(\mathbb{C})$ spanned by S, i.e. the linear space that in the previous sections we called $\langle S\rangle$. For any $S\subset\mathbb{P}^r(\mathbb{R})$ we write $\langle S\rangle_{\mathbb{R}}$ for the minimal real projective subspace of $\mathbb{P}^r(\mathbb{R})$ containing S. Since $S\subset\mathbb{P}^r(\mathbb{R})$ we have $\langle S\rangle_{\mathbb{R}}=\langle S\rangle_{\mathbb{C}}\cap\mathbb{P}^r(\mathbb{R})$. Since $X(\mathbb{R})=X(\mathbb{C})\cap\mathbb{P}^r(\mathbb{R})$, $X(\mathbb{R})$ is Zariski dense in $X(\mathbb{C})$ and $X(\mathbb{C})$ spans $\mathbb{P}^r(\mathbb{R})$. Thus for each $q\in\mathbb{P}^r(\mathbb{R})$ the $X(\mathbb{R})$-rank (i.e. the minimal cardinality of a set $S\subset X(\mathbb{R})$ such that $q\in\langle S\rangle_{\mathbb{R}}$) is a well-defined integer. For any $q\in\mathbb{P}^r(\mathbb{R})$ let $\mathcal{S}(X(\mathbb{R}),q)$ denote the set of all $S\subset X(\mathbb{R})$ such that $q\in\langle S\rangle_{\mathbb{R}}$ and $|S|=r_{X(\mathbb{R})}(q)$. The interested reader may find the definition of a real semialgebraic set in [12, §2.1]. The set $\mathcal{S}(X(\mathbb{R}),q)$ is semialgebraic ([12, Proposition 2.2.7]). Set

$$W_q(X(\mathbb{R})) := \cap_{S\in\mathcal{S}(X(\mathbb{R}),q)}\langle S\rangle_{\mathbb{R}}.$$

We always have $r_{X(\mathbb{R})}(q)\ge r_{X(\mathbb{C})}(q)$ and in many cases the inequality is strict. For instance, when $X\subset\mathbb{P}^d$, $d\ge 3$, is a degree d rational normal curve for each integer t such that $\lfloor(d+2)/2\rfloor<t\le d$ there is $q\in\mathbb{P}^r(\mathbb{R})$ such that $r_{X(\mathbb{C})}(q)=\lfloor(d+2)/2\rfloor$ and $r_{X(\mathbb{R})}(q)=t$ [10,15]. See [11] for definitions and many examples when $X(\mathbb{C})$ is a smooth curve and [1,2,21,22] for tensors and symmetric tensors. When $r_{X(\mathbb{R})}(q)=r_{X(\mathbb{C})}(q)$ we have $\mathcal{S}(X(\mathbb{R}),q)\subseteq\mathcal{S}(X(\mathbb{C}),q)$ and hence $W_q(X(\mathbb{R}))\supseteq W_q(X(\mathbb{C}))\cap\mathbb{P}^r(\mathbb{R})$. We give below an example with $r_{X(\mathbb{C})}(q)=r_{X(\mathbb{R})}(q)=2$, $W_q(X(\mathbb{R}))$ a real line and $W_q(X(\mathbb{C}))=\{q\}$ (see Example 1).

Theorem 4. *Fix an even integer $r\ge 2$. Let $X\subset\mathbb{P}^r$ be an integral and non-degenerate curve defined over $\mathbb{R}$ and with $X_{\mathrm{reg}}(\mathbb{R})\neq\emptyset$. There is a non-empty euclidean open subset $\mathcal{U}\subset\mathbb{P}^r(\mathbb{R})$ such that $r_{X(\mathbb{R})}(q)=r/2+1$ for all $q\in\mathcal{U}$ and $\{q\}=\cap_{S\in\mathcal{S}(X(\mathbb{R}),q)}\langle S\rangle_{\mathbb{R}}$ for all $q\in U$.*

Note that we also get $\{q\}=\cap_{S\in\mathcal{S}(X(\mathbb{R}),q)}\langle S\rangle_{\mathbb{C}}$, because $\cap_{S\in\mathcal{S}(X(\mathbb{R}),q)}\langle S\rangle_{\mathbb{C}}$ is defined over $\mathbb{R}$ and hence its dimension as a complex projective space is the dimension of the real projective space $(\cap_{S\in\mathcal{S}(X(\mathbb{R}),q)}\langle S\rangle_{\mathbb{C}})\cap\mathbb{P}^r(\mathbb{R})=\cap_{S\in\mathcal{S}(X(\mathbb{R}),q)}\langle S\rangle_{\mathbb{R}}$.

Remark 4. We recall that the Zariski topology of $\mathbb{P}^r(\mathbb{R})$ (i.e. the topology in which the closed sets are the intersection with $\mathbb{P}^r(\mathbb{R})$ of a Zariski closed subset of $\mathbb{P}^r(\mathbb{C})$) may be defined by taking as closed subsets the zero-loci of real homogeneous polynomials. Non-empty euclidean open subsets of $\mathbb{P}^r(\mathbb{R})$ are Zariski dense. To show that in Theorem 4 we cannot take as $\mathcal{U}$ a Zariski open subset of $\mathbb{P}^r(\mathbb{R})$ it is sufficient to find a curve $X\subset\mathbb{P}^r$ with $X_{\mathrm{reg}}(\mathbb{R})\neq\emptyset$ and with two different typical ranks. By [10] one can take the rational normal curve of $\mathbb{P}^r$, $r\ge 4$.

Before proving Theorem 4 we describe in the next remark the topology of the real part $X(\mathbb{R})$ of an integral projective curve defined over $\mathbb{R}$.

Remark 5. Let $X(\mathbb{C})$ be an integral projective curve defined over $\mathbb{R}$. Let $\eta: Y(\mathbb{C})\to X(\mathbb{C})$ denote the normalization map. Both $Y(\mathbb{C})$ and η are defined over $\mathbb{R}$ and hence $Y(\mathbb{R})$ is well-defined and $\eta(Y(\mathbb{R}))\subseteq X(\mathbb{R})$. Since η is an isomorphism over $X_{\mathrm{reg}}(\mathbb{C})$, $X_{\mathrm{reg}}(\mathbb{R})$ is essentially $Y(\mathbb{R})$ minus a finite set. Call g the genus of $Y(\mathbb{C})$. F. Klein described the possible real parts $Y(\mathbb{R})$ of genus g smooth curve defined over $\mathbb{R}$ ([17, Proposition 3.1]). Topologically $Y(\mathbb{R})$ is the union of k pairwise disjoint circles, with k an integer between 0.and $g+1$. Thus the topological space $X(\mathbb{R})$ is obtained from $Y(\mathbb{R})$ by an equivalence relation which only identifies finitely many finite subsets of $Y(\mathbb{R})$ and then, sometimes, one adds to $\eta(Y(\mathbb{R}))$ finitely many isolated real points of $\mathrm{Sing}(X(\mathbb{C}))$, each of them the image of two complex conjugate points of $Y(\mathbb{C})\setminus Y(\mathbb{R})$. Thus $X(\mathbb{R})$ is finite (and hence not Zariski dense in $X(\mathbb{C})$) if and only if $Y(\mathbb{R})=\emptyset$, i.e. if and only if $X_{\mathrm{reg}}(\mathbb{R})=\emptyset$.

Proof of Theorem 4. Since $X_{\mathrm{reg}}(\mathbb{R})\neq\emptyset$, there is a set $J\subset X(\mathbb{R})$ homeomorphic to a non-empty open interval of $\mathbb{R}$ for the euclidean topology (Remark 5). Since J is infinite, it is Zariski dense in $X(\mathbb{C})$. As in the proof of Theorem 1 let $\mathcal{V}\subset\mathbb{P}^r(\mathbb{C})$ be a non-empty Zariski open subset such that $r_{X(\mathbb{C})}(q) = r/2+1$ for all $q \in \mathcal{V}$. The set $\sigma(\mathcal{V})$ is Zariski open in $\mathbb{P}^r(\mathbb{C})$. Since $X(\mathbb{C})$ is defined over $\mathbb{R}$, we have $r_{X(\mathbb{C})}(q) = r/2+1$ for all $q \in \mathcal{V}$. Set $\mathcal{V}' := (\mathcal{V} \cup \sigma(\mathcal{V})) \cap \mathbb{P}^r(\mathbb{R})$. The set $\mathcal{V}'$ is a non-empty Zariski open subset of $\mathbb{P}^r(\mathbb{C})$. Call $J_{r/2+1}$ the set of all subset $S \subset J$ such that $|S| = r/2+1$ and $\langle S\rangle_{\mathbb{C}} \cap (\mathcal{V} \cup \sigma(\mathcal{V}))$. Since $r_{X(\mathbb{C})}(q) = r/2+1$ for each $q \in \mathcal{V} \cup \sigma(\mathcal{V})$, each $S \in J_{r/2+1}$ is linearly independent. Since $\mathcal{V} \cup \sigma(\mathcal{V})$ is open, we have $S \in J_{r/2+1}$ if and only $\langle S\rangle_{\mathbb{R}} \cap \mathcal{V}' \neq \emptyset$. We get a euclidean open subset $\mathcal{U}_1$ of $\mathcal{V}'$ taking the interior of the union of all sets $\langle S\rangle_{\mathbb{R}} \cap \mathcal{V}'$ for some $S \in J_{r/2+1}$. To get $\{q\} = \cap_{S \in \mathcal{S}(X(\mathbb{R}),q)} \langle S\rangle_{\mathbb{R}}$ for all $q \in \mathcal{U}$ we need to restrict the euclidean open set $\mathcal{U}_1$ in the following way. Fix $q \in \mathcal{U}_1$ and take $S \in \mathcal{S}(X(\mathbb{R}),q)$. We run the proof of Theorem 1 with this set S and get a curve X_S defined over $\mathbb{R}$ and, using it, a set A_S defined over $\mathbb{R}$. We only need to restrict $\mathcal{U}_1$ so that for $q \in \mathcal{U}$ the set A_S is defined and $\langle S\rangle_{\mathbb{C}} \cap \langle A_S\rangle_{\mathbb{C}} = \{q\}$. □

Example 1. Fix an integer $r \geq 3$. Let $Y(\mathbb{R}) \subset \mathbb{P}^{r+1}(\mathbb{R})$ be the degree $r+1$ rational normal curve. Let σ denote the complex conjugation of $\mathbb{P}^{r+1}(\mathbb{C})$ and $\mathbb{P}^r(\mathbb{C})$. Fix $p_1, p_2 \in Y(\mathbb{R})$ such that $p_1 \neq p_2$ and $p_3 \in Y(\mathbb{C}) \setminus Y(\mathbb{R})$. Set $p_4 := \sigma(p_3)$. We may take homogeneous coordinates $z_0, \dots, z_{r+1}$ of $\mathbb{P}^{r+1}(\mathbb{R})$ and $\mathbb{P}^{r+1}(\mathbb{C})$ such that $p_3 = [1 : a_1 : \cdots : a_{n+1}]$ with $a_i \in \mathbb{C}$ for all i and $a_i \notin \mathbb{R}$ for at least one i. Set $o_1 := [1 : \mathrm{Re}(a_1) : \cdots : \mathrm{Re}(a_{r+1})]$ and $o_2 := [1 : \mathrm{Im}(a_1) : \cdots : \mathrm{Im}(a_{r+1})]$. We have $o_i \in \mathbb{P}^{r+1}(\mathbb{R})$ and $o_1 \neq o_2$, because $p_3 \notin \mathbb{P}^{r+1}(\mathbb{R})$. Since $r \geq 2$, $o_i \notin X(\mathbb{C})$. We have $|\{p_1, p_2, p_3, p_4\}| = 4$ and hence $\langle\{p_1, p_2, p_3, p_4\}\rangle_{\mathbb{C}}$ is a 3. -dimensional complex linear subspace. Since $\sigma(\{p_1, p_2, p_3, p_4\}) = \{p_1, p_2, p_3, p_4\}$, the linear space $\langle\{p_1, p_2, p_3, p_4\}\rangle_{\mathbb{C}}$ is defined over $\mathbb{R}$, i.e.$\langle\{p_1, p_2, p_3, p_4\}\rangle_{\mathbb{C}} \cap \mathbb{P}^{r+1}(\mathbb{R})$ is a 3-dimensional real linear space (it is the real linear space $\langle\{p_1, p_2, o_1, o_2\}\rangle_{\mathbb{R}}$). Fix $o \in \langle\{p_1, p_2, o_1, o_2\}\rangle_{\mathbb{R}}$ such that o is not in the linear span of any proper subset of $\{p_1, p_2, o_1, o_2\}$. Let $\ell_o : \mathbb{P}^{r+1}(\mathbb{C}) \setminus \{o\} \to \mathbb{P}^r(\mathbb{C})$ denote the linear projection from o. Since $o \in \mathbb{P}^{r+1}(\mathbb{R})$, ℓ_o is defined over $\mathbb{R}$ and $\ell_o^{-1}(\mathbb{P}^r(\mathbb{R})) = \mathbb{P}^{r+1}(\mathbb{R}) \setminus \{o\}$. By Sylvester's theorem we have $o \notin \sigma_2(Y(\mathbb{C}))$. Thus $X(\mathbb{C}) := \ell_o(Y(\mathbb{C}))$ is a smooth and non-degenerate rational curve defined over $\mathbb{R}$. Since $Y(\mathbb{R}) \neq \emptyset$, we have $X(\mathbb{R}) \neq \emptyset$. The complex linear space $V_{\mathbb{C}} := \ell_o(\langle\{p_1, p_2, o_1, o_2\}\rangle_{\mathbb{C}})$ is a plane containing exactly 4 points of $X(\mathbb{C})$ (the points $\ell_o(p_1)$, $\ell_o(p_2)$, $\ell_o(p_3)$ and $\ell_o(p_4)$), because any $r+2$ points of $Y(\mathbb{C})$ are linearly independent. Set $L := \langle\{\ell_o(p_1), \ell_o(p_2)\}\rangle_{\mathbb{C}}$ and $R := \langle\{\ell_o(p_1), \ell_o(p_2)\}\rangle_{\mathbb{C}}$. Since $L \neq R$ and $\dim_{\mathbb{C}} V_{\mathbb{C}}$, the set $L \cap R$ is a unique point, q. Since $\sigma(L) = L$ and $\sigma(R) = R$, we have $\sigma(q) = q$, i.e. $q \in \mathbb{P}^r(\mathbb{R})$. Since $q \notin X(\mathbb{C})$ and $\ell_o(p_1), \ell_o(p_2) \in X(\mathbb{R})$, we have $r_{X(\mathbb{R})}(q) = 2$ and hence $r_{X(\mathbb{C})}(q) = 2$. Since $\{\ell_o(p_1), \ell_o(p_2)\}$, $\{\ell_o(p_3), \ell_o(p_4)\} \in \mathcal{S}(X(\mathbb{C}), q)$, we have $W_q(X(\mathbb{C})) = \{q\}$. Using that any $r+2$ elements of $Y(\mathbb{C})$ are linearly independents, we get that $\{\ell_o(p_1), \ell_o(p_2)\}$ and $\{\ell_o(p_3), \ell_o(p_4)\}$ are the only elements of $\mathcal{S}(X(\mathbb{C}), q)$. Thus $W_q(X(\mathbb{R})) = \langle\{\ell_o(p_1), \ell_o(p_2)\}\rangle_{\mathbb{R}}$ is a line. Since $\mathcal{S}(X(\mathbb{R}), q) = \{\ell_o(p_1), \ell_o(p_2)\}$, q is $X(\mathbb{R})$-identifiable. This is not the first example of some $q \in \mathbb{P}^r(\mathbb{R})$ which is identifiable over $\mathbb{R}$, but not over $\mathbb{C}$ [1,2].

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