On the number of edges of a graph and its complement

Edges of a graph and its complement

Jamel Dammak

Department of Mathematics, Faculty of Sciences of Sfax, Sfax, Tunisia

Gerard Lopez

Institut de Mathématiques de Luminy, CNRS-UPR 9016, Marseille, France, and

Hamza Si Kaddour

Institut Camille Jordan, CNRS UMR 5208, Université Claude Bernard Lyon 1, Univ Lyon, Villeurbanne, France

15

Received 12 March 2019
 Revised 26 May 2019
 Accepted 26 May 2019

Abstract

Let $G = (V, E)$ be a graph. The complement of G is the graph $\bar{G} := (V, [V]^2 \setminus E)$ where $[V]^2$ is the set of pairs $\{x, y\}$ of distinct elements of V . If K is a subset of V , the restriction of G to K is the graph $G_{|K} := (K, [K]^2 \cap E)$. We prove that if $G = (V, E)$ is a graph and k is an integer, $2 \leq k \leq v - 2$, then there is a k -element subset K of V such that $e(\bar{G}_{|K}) \neq e(G_{|K})$, moreover the condition $k \leq v - 2$ is optimal. We also study the case $e(\bar{G}_{|K}) \not\equiv e(G_{|K}) \pmod{p}$ where p is a prime number. Following a question from M. Pouzet, we show this: Let $G = (V, E)$ be a graph with v vertices. If $e(G) \neq e(\bar{G})$ (resp. $e(G) = e(\bar{G})$) then there is an increasing family $(H_n)_{2 \leq n \leq v}$ (resp. $(H_n)_{2 \leq n \leq v-2}$) of n -element subsets H_n of V such that $e(G_{|H_n}) \neq e(\bar{G}_{|H_n})$ for all n . Similarly if $e(G) \not\equiv e(\bar{G}) \pmod{p}$ where p is a prime number, $p > v - 2$, then there is an increasing family $(H_n)_{2 \leq n \leq v}$ of n -element subsets H_n of V such that $e(G_{|H_n}) \not\equiv e(\bar{G}_{|H_n}) \pmod{p}$ for all integer $n \in \{2, 3, \dots, v\}$.

Keywords Set, Matrix, Graph, Edge, Prime number

Paper type Original Article

1. Introduction

Our notations and terminology follow [2]. A *graph* is an ordered pair $G := (V, E)$ (or $(V(G), E(G))$), where E is a subset of $[V]^2$, the set of pairs $\{x, y\}$ of distinct elements of V . Elements of V are the *vertices* of G and elements of E are its *edges*. An edge $\{x, y\}$ is also noted by xy . The cardinality $|V|$ of V is called the *order* of G . Two distinct vertices x and y are *adjacent* if $xy \in E(G)$, otherwise x and y are *non-adjacent*. We denote by $e(G) := |E(G)|$ the number of edges of G . The *degree* of a vertex x of G , denoted by $d_G(x)$, is the number of edges which contain x . The graph G is δ -*regular* (or *regular*) if $d_G(x) = \delta$ for all $x \in V$;

JEL Classification — 05C50, 05C60

© Jamel Dammak, Gerard Lopez and Hamza Si Kaddour. Published in the *Arab Journal of Mathematical Sciences*. Published by Emerald Publishing Limited. This article is published under the Creative Commons Attribution (CC BY 4.0) license. Anyone may reproduce, distribute, translate and create derivative works of this article (for both commercial and non-commercial purposes), subject to full attribution to the original publication and authors. The full terms of this license may be seen at <http://creativecommons.org/licenses/by/4.0/legalcode>

The publisher wishes to inform readers that the article “On the number of edges of a graph and its complement” was originally published by the previous publisher of the *Arab Journal of Mathematical Sciences* and the pagination of this article has been subsequently changed. There has been no change to the content of the article. This change was necessary for the journal to transition from the previous publisher to the new one. The publisher sincerely apologises for any inconvenience caused. To access and cite this article, please use Dammak, J., Lopez, G., Kaddour, H. S. (2019), “On the number of edges of a graph and its complement”, *Arab Journal of Mathematical Sciences*, Vol. 27 No. 1, pp. 15-19. The original publication date for this paper was 29/05/2019.

Arab Journal of Mathematical Sciences
 Vol. 27 No. 1, 2021
 pp. 15-19
 Emerald Publishing Limited
 e-ISSN: 2588-9214
 p-ISSN: 1319-5166
 DOI 10.1016/j.jams.2019.05.006

δ is called the degree of the regular graph G . The *complement* of G is the graph $\bar{G} := (V, [V]^2 \setminus E)$. If K is a subset of V , the *restriction* of G to K , also called the *induced subgraph* of G on K , is the graph $G_{\upharpoonright K} := (K, [K]^2 \cap E)$. For instance, given a set V , $(V, \emptyset)$ is the *empty* graph on V whereas $(V, \{xy : x \neq y \in V\})$ is the *complete* graph.

Our first result is [Theorem 1.1](#), we prove that: given a graph $G = (V, E)$ and k be an integer, $2 \leq k \leq v-2$, we cannot have $e(\bar{G}_{\upharpoonright K}) = e(G_{\upharpoonright K})$ for all k -element subsets K of V , moreover the condition $k \leq v-2$ is optimal, indeed for $k = v-1$ a counterexample is given by (2) of [Theorem 1.1](#).

Theorem 1.1. *Let $G = (V, E)$ be a graph of order v .*

- (1) *If $v \geq 4$, then for each integer k with $2 \leq k \leq v-2$, there is a k -element subset K of V such that $e(\bar{G}_{\upharpoonright K}) \neq e(G_{\upharpoonright K})$.*
- (2) *If G is regular with degree $\frac{v-1}{2}$ then $e(G) = e(\bar{G})$ and $e(\bar{G}-x) = e(G-x)$ for all vertex x .*
- (3) *Let p be a prime number with $p \geq 3$ such that $2d_G(x) \equiv v-1 \pmod{p}$ for all $x \in V$. Then $e(G) \equiv e(\bar{G}) \pmod{p}$ and $e(\bar{G}-x) \equiv e(G-x) \pmod{p}$ for all vertex x .*

Our second result is [Theorem 1.2](#). Given a graph $G = (V, E)$, p a prime number, and k an integer, $2 \leq k \leq v-2$, under some conditions on k , we cannot have $e(\bar{G}_{\upharpoonright K}) \equiv e(G_{\upharpoonright K}) \pmod{p}$ for all k -element subsets K of V .

Theorem 1.2. *Let $G = (V, E)$ be a graph with $v \geq 4$ vertices and let p be a prime number. Let k be an integer, $2 \leq k \leq v-2$*

- (1) *If $(p = 2 \text{ and } k \equiv 2 \pmod{4})$ or $(p \geq 3 \text{ and } k \not\equiv 0, 1 \pmod{p})$, then there is a k -element subset K of V such that $e(\bar{G}_{\upharpoonright K}) \not\equiv e(G_{\upharpoonright K}) \pmod{p}$.*
- (2) *If $p \geq 3$ and $k \equiv 0 \pmod{p}$ then there is a k -element subset K of V such that $e(\bar{G}_{\upharpoonright K}) \not\equiv e(G_{\upharpoonright K}) \pmod{p}$ if and only if G is neither the complete graph nor the empty graph.*

Our third result is [Theorem 1.3](#). It is related to a question that M. Pouzet asked us about the existence, in a graph $G = (V, E)$, of an increasing family $(H_n)_n$ of n -element subsets H_n of V such that $e(G_{\upharpoonright H_n}) \neq e(\bar{G}_{\upharpoonright H_n})$.

Theorem 1.3. *Let $G = (V, E)$ be a graph of v vertices with $v \geq 3$.*

- (1) *If $e(G) \neq e(\bar{G})$ then there is a vertex x such that $e(G-x) \neq e(\bar{G}-x)$.*
- (2) *If $e(G) \neq e(\bar{G})$ then there is an increasing family $(H_n)_{2 \leq n \leq v}$ of n -element subsets H_n of V such that $e(G_{\upharpoonright H_n}) \neq e(\bar{G}_{\upharpoonright H_n})$ for all integer $n \in \{2, 3, \dots, v\}$.*
- (3) *If $e(G) = e(\bar{G})$ and $v \geq 4$ then there is an increasing family $(H_n)_{2 \leq n \leq v-2}$ of n -element subsets H_n of V such that $e(G_{\upharpoonright H_n}) \neq e(\bar{G}_{\upharpoonright H_n})$ for all integer $n \in \{2, 3, \dots, v-2\}$.*
- (4) *Let p be a prime number, $p > v-2$. If $e(G) \not\equiv e(\bar{G}) \pmod{p}$ then there is an increasing family $(H_n)_{2 \leq n \leq v}$ of n -element subsets H_n of V such that $e(G_{\upharpoonright H_n}) \not\equiv e(\bar{G}_{\upharpoonright H_n}) \pmod{p}$ for all integer $n \in \{2, 3, \dots, v\}$.*

2. Incidence matrices

We consider the matrix W_{tk} defined as follows: Let V be a finite set, with v elements. Given non-negative integers t, k , let W_{tk} be the $\binom{v}{t}$ by $\binom{v}{k}$ matrix of 0's and 1's, the rows of which

are indexed by the t -element subsets T of V , the columns are indexed by the k -element subsets K of V , and where the entry $W_{tk}(T, K)$ is 1. $T \subseteq K$ and is 0 otherwise. The matrix transpose of W_{tk} is denoted ${}^tW_{tk}$.

A fundamental result, due to D.H. Gottlieb [4], and independently W. Kantor [5], is this:

Theorem 2.1 (D.H. Gottlieb [4], W. Kantor [5]). *For $t \leq \min(k, v-k)$, W_{tk} has full row rank over the field $\mathbb{Q}$ of rational numbers.*

It is clear that $t \leq \min(k, v-k)$ implies $\binom{v}{t} \leq \binom{v}{k}$ then, from Theorem 2.1, we have the following result.

Corollary 2.2. *For $t \leq \min(k, v-k)$, the rank of W_{tk} over the field $\mathbb{Q}$ of rational numbers is $\binom{v}{t}$ and thus $\text{Ker}({}^tW_{tk}) = \{0\}$.*

Corollary 2.2 and the following theorem are important tools in the proof of our main results. In fact, Theorem 2.3 has made to establish a version modulo a prime of Kelly's combinatorial lemma [6]; it also allows to obtain a version modulo a prime of the particular version of Pouzet's combinatorial lemma [7].

Let n, p be positive integers, the decomposition of $n = \sum_{i=0}^{n(p)} n_i p^i$ in the basis p is also denoted by $[n_0, n_1, \dots, n_{n(p)}]_p$ where $n_{n(p)} \neq 0$ if and only if $n \neq 0$.

Theorem 2.3 [1]. *Let p be a prime number. Let v, t and k be non-negative integers, $k = [k_0, k_1, \dots, k_{k(p)}]_p$, $t = [t_0, t_1, \dots, t_{t(p)}]_p$, $t \leq \min(k, v-k)$. We have:*

- (1) $k_j = t_j$ for all $j < t(p)$ and $k_{t(p)} \geq t_{t(p)}$ if and only if $\text{Ker}({}^tW_{tk}) = \{0\} \pmod{p}$.
- (2) $t = t_{t(p)} p^{t(p)}$ and $k = \sum_{i=t(p)+1}^{k(p)} k_i p^i$ if and only if $d\text{Ker}({}^tW_{tk}) = 1 \pmod{p}$ and $\{(1, 1, \dots, 1)\}$ is a basis of $\text{Ker}({}^tW_{tk})$.

The notation $a|b$ (resp. $a \nmid b$) means a divide b (resp. a not divide b).

Theorem 2.4 (Lucas' Theorem [3]). *Let p be a prime number, t, k be positive integers, $t \leq k$, $t = [t_0, t_1, \dots, t_{t(p)}]_p$ and $k = [k_0, k_1, \dots, k_{k(p)}]_p$. Then*

$$\binom{k}{t} = \prod_{i=0}^{t(p)} \binom{k_i}{t_i} \pmod{p}, \text{ where } \binom{k_i}{t_i} = 0 \text{ if } t_i > k_i.$$

The following result is a consequence of Lucas' theorem.

Corollary 2.5 ([1]). *Let p be a prime number, t, k be positive integers, $t \leq k$, $t = [t_0, t_1, \dots, t_{t(p)}]_p$ and $k = [k_0, k_1, \dots, k_{k(p)}]_p$. Then:*

$$p \mid \binom{k}{t} \text{ if and only if there is } i \in \{0, 1, \dots, t(p)\} \text{ such that } t_i > k_i.$$

3. Proofs of main results

Let $G = (V, E)$ be a graph of order v . Let $T_1, T_2, \dots, T_{\binom{v}{2}}$ be an enumeration of the 2-element subsets of V , let $K_1, K_2, \dots, K_{\binom{v}{k}}$ be an enumeration of the k -element subsets of V .

Let w_G be the row matrix $(g_1, g_2, \dots, g_{\binom{v}{2}})$ where $g_i = 1$ if T_i is an edge of G , 0 otherwise.

We have $w_G W_{2k} = (e(G_{\uparrow K_1}), e(G_{\uparrow K_2}), \dots, e(G_{\uparrow K_{\binom{v}{k}}}))$.

Note that $w_{\bar{G}} = (\bar{g}_1, \bar{g}_2, \dots, \bar{g}_{\binom{v}{2}})$ with $\bar{g}_i = 0$ if T_i is an edge of G , 1 otherwise. We have $w_{\bar{G}} W_{2k} = (e(\bar{G}_{1K_1}), e(\bar{G}_{1K_2}), \dots, e(\bar{G}_{1K_{\binom{v}{k}}}))$.

3.1 Proof of Theorem 1.1

(1) Assume that $e(\bar{G}_{1K}) = e(G_{1K})$ for all k -element subsets K of V , then $w_G W_{2k} = w_{\bar{G}} W_{2k}$. By Corollary 2.2, $\text{Ker}({}^t W_{1k}) = \{0\}$. Then $w_G = w_{\bar{G}}$, so $G = \bar{G}$, which is impossible. So there is a k -element subset K of V such that $e(\bar{G}_{1K}) \neq e(G_{1K})$.

(2) For $x \in V$, $d_G(x) + d_{\bar{G}}(x) = v - 1$. Since $d_G(x) = \frac{v-1}{2}$ then $d_G(x) = d_{\bar{G}}(x)$. We have $\sum_{x \in V} d_G(x) = 2e(G)$ and $\sum_{x \in V} d_{\bar{G}}(x) = 2e(\bar{G})$, then $e(G) = e(\bar{G})$. Now from $e(\bar{G}) = e(\bar{G} - x) + d_{\bar{G}}(x)$ and $e(G) = e(G - x) + d_G(x)$, we deduce that $e(\bar{G} - x) = e(G - x)$.

(3) For $x \in V$, $d_G(x) + d_{\bar{G}}(x) = v - 1$. Since $2d_G(x) \equiv v - 1 \pmod{p}$ then $d_G(x) \equiv d_{\bar{G}}(x) \pmod{p}$. We conclude using similar arguments to those in item (2). $\square$

3.2 Proof of Theorem 1.2

We set $t := 2$. We recall the notation $k = [k_0, k_1, \dots, k_{k(p)}]_p$.

(1) Case 1. $p = 2$ and $k \equiv 2 \pmod{4}$.

We have $k_0 = 0, k_1 = 1, t = [0, 1]_2$. Since $k_0 = t_0$ and $k_1 \geq t_1 = t_{t(p)}$ then, by Theorem 2.3, $\text{Ker}({}^t W_{tk}) = \{0\} \pmod{p}$.

Case 2. $p \geq 3$ and $k \not\equiv 0, 1 \pmod{p}$.

We have $k_0 \geq 2, t = t_0 = 2$. Since $k_0 \geq 2 = t_{t(p)}$ then, by Theorem 2.3, $\text{Ker}({}^t W_{tk}) = \{0\} \pmod{p}$.

In the two cases, $\text{Ker}({}^t W_{tk}) = \{0\} \pmod{p}$. Assume that $e(\bar{G}_{1K}) \equiv e(G_{1K}) \pmod{p}$ for all k -element subsets K of V . Then $w_G W_{2k} = w_{\bar{G}} W_{2k} \pmod{p}$. As $\text{Ker}({}^t W_{tk}) = \{0\} \pmod{p}$, then $w_G = w_{\bar{G}} \pmod{p}$, so $G = \bar{G}$, which is impossible. Then there is a k -element subset K of V such that $e(\bar{G}_{1K}) \not\equiv e(G_{1K}) \pmod{p}$.

(2) If $p \geq 3$ then $t = t_0 = 2 = t_{t(p)}$. Since $k \equiv 0 \pmod{p}$ then $k_0 = 0$, and thus $k = \sum_{i=t(p)+1}^{k(p)} k_i p^i$. By Theorem 2.3, $\{(1, 1, \dots, 1)\}$ is a basis of $\text{Ker}({}^t W_{tk})$.

If G is the complete graph or the empty graph then $e(G_{1K}) \equiv e(\bar{G}_{1K}) \pmod{p}$. Indeed, if G is the complete graph then $e(G_{1K}) = \binom{k}{2}$, $e(\bar{G}_{1K}) = 0$. Since $t_0 = 2 > k_0 = 0$ then, by

Corollary 2.5, $p \mid \binom{k}{2}$. So $e(G_{1K}) \equiv 0 \equiv e(\bar{G}_{1K}) \pmod{p}$. If G is the empty graph,

then $e(G_{1K}) = 0 \equiv e(\bar{G}_{1K}) = \binom{k}{2} \pmod{p}$.

Conversely if G is neither the complete graph nor the empty graph, assume that $e(\bar{G}_{1K}) \equiv e(G_{1K}) \pmod{p}$ for all k -element subsets K of V . Then $w_G W_{2k} = w_{\bar{G}} W_{2k} \pmod{p}$. So $w_G - w_{\bar{G}} = \lambda(1, \dots, 1) \pmod{p}$ with $\lambda \in \{0, 1, -1\}$. As G is neither the complete graph nor the empty graph, there are i, j such that $g_i = 0$ and $g_j = 1$, so $g_i - \bar{g}_i = -1$ and $g_j - \bar{g}_j = 1$. Then $\lambda \neq 1$ and $\lambda \neq -1$. Thus $\lambda = 0$, and $w_G = w_{\bar{G}} \pmod{p}$, so $G = \bar{G}$, which is impossible. Then there is a k -element subset K of V such that $e(\bar{G}_{1K}) \not\equiv e(G_{1K}) \pmod{p}$. $\square$

3.3 Proof of Theorem 1.3

We need the following lemma.

Lemma 3.1. Let $G = (V, E)$ be a graph of order v and let p be a prime number, $p \geq 3$

- (1) Let k be an integer, $2 \leq k \leq v-2$. If $\binom{v-2}{k-2}e(G) \not\equiv \binom{v-2}{k-2}e(\bar{G}) \pmod{p}$ then there is a k -element subset K of V such that $e(G_{\setminus K}) \not\equiv e(\bar{G}_{\setminus K}) \pmod{p}$.
- (2) If $(v-2)e(G) \not\equiv (v-2)e(\bar{G}) \pmod{p}$ then there is $x \in V$ such that $e(G-x) \not\equiv e(\bar{G}-x) \pmod{p}$.

Proof. (1) It is an immediate consequence of the following formula

$$\binom{v-2}{k-2}e(G) = \sum_{\substack{K \subset V \\ |K|=k}} e(G_{\setminus K})$$

- (2) Follows from (1) by taking $k = v-1$. $\square$

Now we prove [Theorem 1.3](#).

(1) Assume that $e(\bar{G}-x) = e(G-x)$ for all $x \in V$. From $e(G) = e(G-x) + d_G(x)$, we obtain $\sum_{x \in V} e(G) = \sum_{x \in V} e(G-x) + \sum_{x \in V} d_G(x)$. Since $\sum_{x \in V} d_G(x) = 2e(G)$ then $(v-2)e(G) = \sum_{x \in V} e(G-x)$. Similarly, $(v-2)e(\bar{G}) = \sum_{x \in V} e(\bar{G}-x)$. Then $e(G) = e(\bar{G})$.

(2) We make the proof by induction on v . We set $H_v := V$. We assume that H_i is defined for all $k \leq i \leq v$. Let us define H_{k-1} . As $e(G_{\setminus H_k}) \not\equiv e(\bar{G}_{\setminus H_k})$ then by (1), there is $x \in H_k$ such that $e(G_{\setminus H_k} - x) \not\equiv e(\bar{G}_{\setminus H_k} - x)$. We set $H_{k-1} := H_k \setminus \{x\}$. So $H_{k-1} \subset H_k$ and $e(G_{\setminus H_{k-1}}) \not\equiv e(\bar{G}_{\setminus H_{k-1}})$.

(3) By applying (1) of [Theorem 1.1](#) for the graph G and $k = v-2$ we obtain $x \neq y \in V$ such that $e(G - \{x, y\}) \not\equiv e(\bar{G} - \{x, y\})$. If $v = 4$ we are done. If $v \geq 5$, then $v-2 \geq 3$ and here we conclude using (2).

(4) By induction on v . Since $e(G) \not\equiv e(\bar{G}) \pmod{p}$ and $p > v-2$ then $(v-2)e(G) \not\equiv (v-2)e(\bar{G}) \pmod{p}$. By (2) of [Lemma 3.1](#), there is $x \in V$ such that $e(\bar{G}-x) \not\equiv e(G-x) \pmod{p}$. We conclude by using the induction hypothesis. $\square$

References

- [1] A. Ben Amira, J. Dammak, H. Si Kaddour, On a generalization of Kelly's combinatorial lemma, Turk. J. Math. 36 (2014) 949–964.
- [2] J.A. Bondy, Basic graph theory: paths and circuits, in: R.L. Graham, M. Grötschel, L. Lovász (Eds.), Handbook of Combinatorics, Vol. 1, North-Holland, 1995, pp. 3–110.
- [3] N.J. Fine, Binomial coefficients modulo a prime, Amer. Math. Monthly 54 ((10) Part 1) (1947) 589–592.
- [4] D.H. Gottlieb, A certain class of incidence matrices, Proc. Amer. Math. Soc. 17 (1966) 1233–1237.
- [5] W.M. Kantor, On incidence matrices of finite projective and affine spaces, Math. Z. 124 (1972) 315–318.
- [6] P.J. Kelly, A congruence theorem for trees, Pac. J. Math. 7 (1957) 961–968.
- [7] M. Pouzet, Application d'une propriété combinatoire des parties d'un ensemble aux groupes et aux relations, Math. Z. 150 (1976) 117–134.

Corresponding author

Hamza Si Kaddour can be contacted at: sikaddour@univ-lyon1.fr

For instructions on how to order reprints of this article, please visit our website:

www.emeraldgroupublishing.com/licensing/reprints.htm

Or contact us for further details: permissions@emeraldinsight.com