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Coupled fixed points and coupled best proximity points for cyclic Ćirić type operators

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Abstract
The purpose of this paper is to study the coupled fixed point problem and the coupled best proximity problem for single-valued and multi-valued contraction type operators defined on cyclic representations of the space. The approach is based on fixed point results for appropriate operators generated by the initial problems.

Keywords Metric space, Single-valued operator, Multi-valued operator, Fixed point, Coupled fixed point, Best proximity point, Coupled best proximity point, Generalized contraction, Data dependence, Ulam–Hyers stability, Well-posedness

Paper type Original Article

1. Introduction

One of the most important metrical fixed point theorem, Banach contraction principle, has been generalized in several directions, see for example [1]. The concept of coupled fixed point was introduced by Guo and Lakshmikantham (see [2]). A new research direction for the theory of coupled fixed points was developed by many authors (see [3–9]) using contractive type conditions.

Definition 1.1 (*[10]*). Let X be a nonempty set. A pair $(x, y) \in X \times X$ is called coupled fixed point of the operator $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$. If $F(x, x) = x$ then x is called a strong coupled fixed point of F (or, in several papers, a fixed point of F).

Another generalization of the Banach principle was given by Kirk, Srinivasan and Veeramani using the concept of cyclic operators.

JEL Classification — 41A50, 47H09, 47H10, 57H25

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Definition 1.2 ([11]). Let A and B be nonempty subsets of a given set X. An operator $T: A\cup B \to A\cup B$ is called cyclic if $T(A)\subseteq B$ and $T(B)\subseteq A$.

More recently, Choudbury and Maity formulated the following definition.

Definition 1.3 ([12]). Let A and B be nonempty subsets of a given set X. An operator $F: X\times X \to X$ having the property that for any $x\in A$ and $y\in B$, $F(x,y)\in B$ and $F(y,x)\in A$, is called a cyclic operator with respect to A and B.

Definition 1.4 ([13]). Let A and B be nonempty subsets of a metric space (X,d). An operator $F: X\times X\to X$ is called a cyclic Ćirić operator with respect to A and B if F is cyclic with respect to A and B and for some constant $q\in(0,1)$, F satisfies the following condition:

$$d(F(x,y),F(u,v)) \leq q\cdot M(x,v,y,u),$$

where $x,v\in A$, $y,u\in B$, and

$$M(x,v,y,u) = \max\left\{d(x,u), \frac{1}{2}d(u,F(x,y)), \frac{1}{2}d(x,F(u,v)), \frac{1}{2}[d(x,F(x,y)) + d(u,F(u,v))]\right\}.$$

Theorem 1.1 ([13]). *Let A and B be nonempty closed subsets of a complete metric space (X,d), $F: X\times X\to X$ a cyclic Ćirić type operator with respect to A and B, with $A\cap B\neq\emptyset$. Then F has a strong coupled fixed point in $A\cap B$.*

The first aim of this paper is to generalize the above theorem, weakening the contractive condition and excluding the condition $A\cap B\neq\emptyset$. We prove the uniqueness of the strong coupled fixed point and we provide an iterative method for approximating the strong coupled fixed point.

We also present coupled fixed point and coupled best proximity point results for cyclic coupled Ćirić-type multivalued operators.

On the other hand, some qualitative properties of the coupled fixed point set, such as data dependence, generalized Ulam–Hyers stability and well-posedness are studied.

Our approach is based on the following idea: we transform the coupled fixed point/ best proximity point problem into a fixed point/ best proximity point problem for an appropriate operator defined on a cartesian product of the spaces. In this way, many coupled fixed point/ best proximity point results can be obtained using classical fixed point/ best proximity point theorems.

2. Preliminaries

The standard notations and terminologies in nonlinear analysis will be used throughout this paper.

Let (X,d) be a metric space. We denote:

$$P(X) := \{Y\subseteq X \mid Y \text{ is nonempty}\}; P_b(X) := \{Y\in P(X)\mid Y \text{ is bounded}\};$$

$$P_{cl}(X) := \{y\in P(X)\mid Y \text{ is closed}\}; P_{cp}(X) := \{Y\in P(X)\mid Y \text{ is compact}\}.$$

Let us define the following (generalized) functionals used in this paper:

- The gap functional

$$D: P(X)\times P(X)\to \mathbb{R}_{+,} D(A,B) = \inf\{d(a,b)\mid a\in A, b\in B\};$$

- The generalized excess functional
$$\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, \rho(A,B) = \sup\{D(a,B) \mid a \in A\};$$
- The generalized Pompeiu–Hausdorff functional
$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}, H(A,B) = \max\{\rho(A,B), \rho(B,A)\}.$$

There are several conditions upon the comparison function that have been considered in literature. In this paper we shall refer only to:

Definition 2.1 (*[14]*). A function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it satisfies:

(i) φ is increasing;

(ii) $(\varphi^n(t))_{n\in\mathbb{N}}$ converges to 0 as $n \to \infty$, for all $t \in \mathbb{R}_+$.

If the condition (ii) is replaced by the condition:

(iii) $\sum_{k=0}^{\infty} \varphi^k(t) < \infty$, for any $t > 0$, then φ is called a strong comparison function.

Lemma 2.1 (*[1]*). *If $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then $\varphi(t) < t$, for any $t > 0$, $\varphi(0) = 0$ and φ is continuous at* 0.

Lemma 2.2 (*[14]*). *If $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ is a strong comparison function, then the following hold:*

(i) φ is a comparison function;

(ii) the function $s: \mathbb{R}_+ \to \mathbb{R}_+$, defined by

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t),$$

is increasing and continuous at 0.

Example 2.1 (*[15]*). (1) $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = at$, where $a \in [0,1)$, is a strong comparison function;

(2) $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \frac{1}{2}t$, for $t \in [0,1]$ and $\varphi(t) = t - \frac{1}{2}$, for $t > 1$, is a strong comparison function;

(3) $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = at + \frac{1}{2}[t]$, where $a \in (0, \frac{1}{2})$, is a strong comparison function;

(4) $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$, $\varphi(t) = \frac{t}{1+t}$, is a comparison function, but is not a strong comparison function.

For more examples and considerations on comparison functions see [1] and the references therein.

3. Coupled fixed points of cyclic Ćirić type single valued operators

In this section we present some coupled fixed point results for cyclic Ćirić type operators on complete metric spaces.

We introduce now the following new concept.

Definition 3.1 Let (X,d) be a metric space, $A, B \in P_{cl}(X)$, $Y = A \cup B$ and $\varphi: R_+ \to R_+$ a strong comparison function. An operator $F: Y \times Y \to Y$ is called a cyclic coupled φ-contraction of Ćirić type if the following statements hold:

(i) F is cyclic with respect to A and B;

(ii) $$d(F(x,y),F(u,v)) \le \varphi(M(x,v,y,u)), \tag{3.1}$$

for any $x,v \in A$ and $y,u \in B$, where

$$M(x,v,y,u) = \max\Big\{d(x,u),d(v,y),d(x,F(x,y)),d(u,F(u,v)),d(v,F(v,u)), d(y,F(y,x)),\frac{1}{2}[d(x,F(u,v))+d(u,F(x,y))], \frac{1}{2}[d(y,F(v,u))+d(v,F(y,x))]\Big\}.$$

The following theorem (which is a particular case of Theorem 3.2 in [16]) will be used to prove our results presented in this section.

Theorem 3.1 ([16]). *Let (X,d) be a complete metric space, $A,B \in P_{cl}(X)$, $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a strong comparison function and $f : A \cup B \to A \cup B$ be an operator such that $f(A) \subseteq B$ and $f(B) \subseteq A$. If f is a cyclic φ-contraction of Ćirić type, that is*

$$d(f(x),f(y)) \le \varphi\Big(\max\Big\{d(x,y),d(x,f(x)),d(y,f(y)), \frac{1}{2}[d(x,f(y))+d(y,f(x))]\Big\}\Big),$$

for any $x \in A$ and $y \in B$, then the following statements hold:

(1) *f has a unique fixed point $x^* \in A \cap B$ and the Picard iteration $\{x_n\}_{n\ge 0}$ defined by $x_n = f(x_{n-1})$, $n \ge 1$, converges to x^* for any starting point $x_0 \in A \cup B$;*

(2) *the following estimates hold:*

$$d(x_n,x^*) \le s(\varphi^n(d(x_0,x_1))), \quad n \ge 1;$$
$$d(x_n,x^*) \le s(d(x_n,x_{n+1})), \quad n \ge 1;$$

(3) *for any $x \in A \cup B$, $d(x,x^*) \le s(d(x,f(x)))$, where s is given by Lemma 2.2.*

The main result of this section is the following theorem.

Theorem 3.2. Let (X,d) be a complete metric space, $A,B \in P_{cl}(X)$, $Y = A \cup B$ and $F : Y \times Y \to Y$ a cyclic coupled φ-contraction of Ćirić type. Then:

(1) *F has a unique strong coupled fixed point $x^* \in A \cap B$;*

(2) *for any $(x_0,y_0) \in A \times B$, there exists a sequence $\{(x_n,y_n)\}_{n\in\mathbb{N}} \subset X \times X$ defined by*

$$\begin{cases} x_n = F(y_{n-1},x_{n-1}) \\ y_n = F(x_{n-1},y_{n-1}) \end{cases}, n \ge 1,$$

that converges to (x,x);

(3) *the following estimates hold:*

$$\max\{d(x_n,x^*),d(y_n,x^*)\} \le s(\varphi^n(\max\{d(x_0,F(x_0,y_0)),d(y_0,F(y_0,x_0))\})), \quad n \ge 1,$$
$$\max\{d(x_n,x^*),d(y_n,x^*)\} \le s(\max\{d(x_n,x_{n+1}),d(y_n,y_{n+1})\}), \quad n \ge 1;$$

(4) *for any $x,y \in Y$, $d(x,x^*) \le s(\max\{d(x,F(x,y)),d(y,F(y,x))\})$, where s is given by Lemma 2.2.*

Proof. (1)–(2) Changing the roles between x and v and similarly for y and u, the inequality (3.1) becomes:

$$d(F(v,u),F(y,x)) \le \varphi(M(v,x,u,y)), \text{ for } x,v \in A \text{ and } y,u \in B. \tag{3.2}$$

Obviously, $M(x,v,y,u) = M(v,x,u,y)$. From the inequalities (3.1) and (3.2) we obtain

$$\max\{d(F(x,y),F(u,v)), d(F(y,x),F(v,u))\} \le \varphi(M(x,v,y,u)). \tag{3.3}$$

For $z = (x,y) \in A \times B$, $w = (u,v) \in B \times A$, denote

$$d^*(z,w) = \max\{d(x,u), d(y,v)\}. \tag{3.4}$$

Then $(X \times X, d^*)$ is a complete metric space.

Let $T: Y \times Y \to Y \times Y$ be defined by $T(x,y) = (F(x,y),F(y,x))$. We have:

$$\begin{aligned}\frac{1}{2}[d^*(z,T(w)) + d^*(w,T(z))] &= \frac{1}{2}\max\{d(x,F(u,v)), d(y,F(v,u))\}\\ &\quad + \frac{1}{2}\max\{d(u,F(x,y)), d(v,F(y,x))\}\\ &\ge \max\left\{\frac{1}{2}[d(x,F(u,v)) + d(u,F(x,y))],\right.\\ &\qquad \left.\frac{1}{2}[d(y,F(v,u)) + d(v,F(y,x))]\right\}.\end{aligned}$$

Using the above relation, from (3.3) we get

$$\begin{aligned}d^*(T(z),T(w)) \le \varphi\Big(\max\Big\{&d^*(z,w), d^*(z,T(z)), d^*(w,T(w)),\\ &\frac{1}{2}[d^*(z,T(w)) + d^*(w,T(z))]\Big\}\Big),\end{aligned} \tag{3.5}$$

for any $z \in A \times B$, $w \in B \times A$.

Because $F(A \times B) \subseteq B$ and $F(B \times A) \subseteq A$, we have

$$T(A \times B) \subseteq B \times A \text{ and } T(B \times A) \subseteq A \times B. \tag{3.6}$$

(3.5) and (3.6) means that the operator T is a cyclic φ-contraction of Ćirić type. Applying Theorem 3.1, there exists a unique $z^* = (x^*,y^*) \in (A \times B) \cap (B \times A)$ such that $T(z^*) = z^*$ and the Picard iteration $z_n = T(z_{n-1})$ converges to z^* for any starting point $z_0 \in Y$. So

$$\begin{cases} F(x^*,y^*) = x^* \\ F(y^*,x^*) = y^* \end{cases} \tag{3.7}$$

where $x^*, y^* \in A \cap B$.

From unicity of the pair (x^*,y^*) and the symmetry with respect to x^* and y^* of the system (3.7) we conclude $x^* = y^*$.

Then F has a unique strong coupled fixed point $x^* \in A \cap B$ and for any starting point $(x_0,y_0) \in A \times B$ there exists a sequence $\{(x_n,y_n)\}_{n \in \mathbb{N}} \subset Y \times Y$ with

$$\begin{cases} x_n = F(y_{n-1},x_{n-1}) \\ y_n = F(x_{n-1},y_{n-1}) \end{cases}, n \ge 1$$

that converges to (x^*,x^*).

(3) By the second conclusion of Theorem 3.1,

$$d^*(z_n,(x^*,x^*)) \leq s(\varphi^n(d^*(z_0,z_1)))$$

and

$$d^*(z_n,(x^*,x^*)) \leq s(d^*(z_n,z_{n+1})), n \geq 1.$$

Hence

$$\max\{d(x_n,x^*),d(y_n,x^*)\} \leq s(\varphi^n(\max\{d(x_0,F(x_0,y_0)),d(y_0,F(y_0,x_0))\}))$$
$$\max\{d(x_n,x^*),d(y_n,x^*)\} \leq s(\max\{d(x_n,x_{n+1}),d(y_n,y_{n+1}\})), n \geq 1.$$

(4) Using (3) from Theorem 3.1, for any $(x,y) \in Y \times Y$,

$$d^*((x,y),(x^*,x^*)) \leq s(d^*((x,y),T(x,y))).$$

Hence

$$\max\{d(x,x^*),d(y,x^*)\} \leq s(\max\{d(x,F(x,y)),d(y,F(y,x))\}). \quad \square$$

Example 3.1. Let $X = \mathbb{R}, d(x,y) = |x-y|$, for any $x,y \in \mathbb{R}$, $A = [0,2]$, $B = [0,1]$, $Y = A \cup B, F: Y \times Y \to Y, F(x,y) = \frac{x+3y}{9}$.

It is easy to verify that F is cyclic with respect to A and B.

For any $x,\ v \in A$ and $y,\ u \in B$

$$\begin{aligned} d(F(x,y),F(u,v)) &= |\frac{x+3y}{9} - \frac{u+3v}{9}| \\ &= |\frac{x-u}{9} + \frac{y-v}{3}| \\ &\leq |\frac{1}{9}(x-u) + \frac{10}{27}(y-v)| \\ &= \frac{1}{3}|y - \frac{v+3u}{9} + \frac{y+3x}{9} - v| \\ &\leq \frac{1}{3}(|y-F(v,u)| + |v-F(y,x)|) \\ &\leq \frac{2}{3}\cdot\frac{1}{2}[d(y,F(v,u)) + d(v,F(y,x))]. \end{aligned}$$

Then F is a cyclic coupled φ-contraction of Ćirić type, where $\varphi(t) = \frac{2}{3}\cdot t$.
The hypotheses of Theorem 3.2 are satisfied, so by Theorem 3.2, F has a unique strong coupled fixed point $x^* \in A \cap B$. By calculation we get:

$$F(x^*,x^*) = x^* \Leftrightarrow x^* = 0.$$

Our next theorem gives the well-posedness property for the coupled fixed point problem. For the concept of well-posedness for the fixed point problems see [17].

Theorem 3.3. *Let $F: Y \times Y \to Y$ be as in Theorem 3.2. Then the coupled fixed point problem is well posed, that is, if there exists a sequence $\{(a_n,b_n)\}_{n\in\mathbb{N}} \subset Y \times Y$ such that*

$$\begin{cases} d(a_n,F(a_n,b_n)) \to 0 \\ d(b_n,F(b_n,a_n)) \to 0 \end{cases} \text{as } n \to \infty,$$

then $a_n \to x^*$ and $b_n \to x^*$, as $n \to \infty$.

Proof. Using the inequality

$$d(x,x^*) \le s(\max\{d(x,F(x,y)),d(y,F(y,x))\})$$

from Theorem 3.2 for $x := a_n$ and next for $x := b_n$, we have:

$$\begin{cases} d(a_n,x^*) \le s(\max\{d(a_n,F(a_n,b_n)),d(b_n,F(b_n,a_n))\}) \\ d(b_n,x^*) \le s(\max\{d(b_n,F(b_n,a_n)),d(a_n,F(a_n,b_n))\}) \end{cases}, n \in \mathbb{N},$$

and letting $n \to \infty$ we obtain

$$\begin{cases} d(a_n,x^*) \to 0 \\ d(b_n,x^*) \to 0 \end{cases}, n \to \infty.$$

For the data dependence problem we have the following result.

Theorem 3.4. *Let $F: Y \times Y \to Y$ be as in Theorem 3.2. Let G: $Y \times Y \to Y$ be such that:*

(i) G has at least one strong coupled fixed point x_G^*;

(ii) there exists $\eta > 0$ such that

$$d(F(x,x),G(x,x)) \le \eta, \text{ for any } x \in Y.$$

Then $d(x_F^*,x_G^*) \le s(\eta)$, where x_F^* is the unique strong coupled fixed point of F and

$$s(t) = \sum_{k=0}^{\infty} \varphi^k(t), t \in \mathbb{R}_+.$$

Proof. By letting $x := x_G^*$ and $y := x_G^*$ in the inequality

$$d(x,x^*) \le s(\max\{d(x,F(x,y)),d(y,F(y,x))\}),$$

we have

$$d(x_G^*,x_F^*) \le s(d(x_G^*,F(x_G^*,x_G^*))) = s(d(G(x_G^*,x_G^*),F(x_G^*,x_G^*))),$$

and using the monotonicity of s we obtain

$$d(x_F^*,x_G^*) \le s(\eta).$$

Theorem 3.5. *Let F: $Y \times Y \to Y$ be as in* Theorem 3.2 *and F_n: $Y \times Y \to Y$, $n \in N$, be such that:*

(i) for each $n \in \mathbb{N}$ there exists a strong coupled fixed point x_n^* of F_n;

(ii) $\{F_n\}_{n\in\mathbb{N}}$ converges uniformly to F.

Then $x_n^* \to x^*$ as $n \to \infty$, where x^* is the unique strong coupled fixed point of F.

Proof. The sequence $\{F_n\}_{n\in\mathbb{N}}$ converges uniformly to F. Then there exist $\eta_n \in \mathbb{R}_+$, $n \in \mathbb{N}$ such that $\eta_n \to 0$ as $n \to \infty$ and

$$d(F_n(x,y),F(x,y)) \le \eta_n \text{ for any } (x,y) \in Y \times Y.$$

Using Theorem 3.3 for $G := F_n$, $n \in N$, we have

$$d(x_n,x^*) \le s(\eta_n) \text{ as } n \to \infty.$$

We will discuss Ulam–Hyers stability for the coupled fixed point problem corresponding to a cyclic operator.

Definition 3.2. Let (X, d) be a metric space, $Y \in P(X)$ and $F: Y \times Y \to Y$ be an operator. The coupled fixed point problem

$$\begin{cases} F(x,y) = x \\ F(y,x) = y \end{cases}, x, y \in Y \tag{3.8}$$

is called generalized Ulam–Hyers stable if there exists $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for any $\varepsilon_1 > 0, \varepsilon_2 > 0$ and for any solution $(x, y) \in Y \times Y$ of the system

$$\begin{cases} d(x, F(x,y)) \leq \varepsilon_1 \\ d(y, F(y,x)) \leq \varepsilon_2 \end{cases}$$

there exists a solution (x^*, y^*) of the coupled fixed point problem such that

$$\begin{cases} d(x, x^*) \leq \psi(\varepsilon) \\ d(y, y^*) \leq \psi(\varepsilon) \end{cases}, \text{ where } \varepsilon = \max\{\varepsilon_1, \varepsilon_2\}.$$

In particular, if $x^* = y^*$, then we have generalized Ulam–Hyers stability for the strong coupled fixed point problem $F(x, x) = x, x \in Y$.

Theorem 3.6. *Suppose that all the hypotheses of Theorem 3.2 hold. Then the coupled fixed point problem (3.8) is generalized Ulam–Hyers stable.*

Proof. By Theorem 3.2 we have a unique $x^* \in Y$ such that $F(x^*, x^*) = x^*$.

Let $\varepsilon_1 > 0, \varepsilon_2 > 0$ and $(\tilde{x}, \tilde{y}) \in Y \times Y$ such that

$$\begin{cases} d(\tilde{x}, F(\tilde{x}, \tilde{y})) \leq \varepsilon_1 \\ d(\tilde{y}, F(\tilde{y}, \tilde{x})) \leq \varepsilon_2. \end{cases}$$

We know that

$$d(x, x^*) \leq s(\max\{d(x, F(x,y)), d(y, F(y,x))\}), \forall (x,y) \in Y \times Y.$$

Then for

$$\begin{pmatrix} x := \tilde{x} \\ y := \tilde{y} \end{pmatrix}$$

and next for

$$\begin{pmatrix} x := \tilde{y} \\ y := \tilde{x} \end{pmatrix}$$

using the monotonicity of s, we obtain that

$$\max\{d(\tilde{x}, x^*), d(\tilde{y}, x^*)\} \leq s(\max\{d(\tilde{x}, F(\tilde{x}, \tilde{y})), d(\tilde{y}, F(\tilde{y}, \tilde{x}))\}) \leq s(\max\{\varepsilon_1, \varepsilon_2\}).$$

As a conclusion, the coupled fixed point problem (3.8) is generalized Ulam–Hyers stable with $\psi = s$.

4. Coupled fixed points and coupled best proximity points of cyclic Ćirić type multivalued operators

The purpose of this section is to consider the above problems in the multi-valued setting. We present first a new concept of cyclic multi-valued operator.

Definition 4.1. Let (X,d) be a metric space, $A,B\in P(X)$, $Y=A\cup B$ and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ a strong comparison function. A multivalued operator $F:Y\times Y\to P(Y)$ is called a cyclic coupled φ-contraction of Ćirić type multivalued operator if the following statements hold:

(i) F is cyclic with respect to A and B, that is

$$F(A\times B)\subseteq B \text{ and } F(B\times A)\subseteq A;$$

(ii)

$$H(F(x,y),F(u,v))\le\varphi(\tilde{M}(x,v,y,u)),\ \text{ for any } x,v\in A, y,u\in B \tag{4.1}$$

where

$$\tilde{M}(x,v,y,u)=\max\Big\{d(x,u),d(v,y),D(x,F(x,y)),D(u,F(u,v)),D(v,F(v,u)),\\ D(y,F(y,x)),\tfrac{1}{2}[D(x,F(u,v))+D(u,F(x,y))],\tfrac{1}{2}[D(y,F(v,u))+D(v,F(y,x))]\Big\}.$$

Definition 4.2. Let (X,d) be a metric space. Then $Y\in P(X)$ is called proximinal if for any $x\in X$, there exists $y\in Y$ such that

$$d(x,y)=D(x,Y).$$

We denote $P_{prox}=\{y\in P(X)\mid Y \text{ is proximinal}\}$.

Remark 4.1. Let (X,d) be a metric space. Then

$$P_{cp}(X)\subset P_{prox}(X)\subset P_{cl}(X).$$

Remark 4.2. Every closed convex subset of a uniformly Banach space is proximinal, see [18].

For details concerning the above notions see [1,19] and [20].

The following theorem (which is a particular case of Theorem 2.7 in [21]) will be used to prove the first result in this section.

Theorem 4.1. *([21]). Let (X,d) be a complete metric space, $A,B\in P_{cl}(X)$ and $T:A\cup B\to P_{prox}(A\cup B)$ a multivalued cyclic φ-contraction of Ćirić type, that is:*

(i) $T(A)\subseteq B$ and $T(B)\subseteq A$;

(ii) there exists a strong comparison function $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ such that

$$H(T(x),T(y))\le\varphi\Big(\max\Big\{d(x,y),D(x,T(x)),D(y,T(y)),\\ \frac{1}{2}[D(x,T(y))+D(y,T(x))]\Big\}\Big),$$

for any $x\in A$ and $y\in B$.
Then the following statements hold:

(1) *there exists $x^*\in A\cap B$ such that $x^*\in T(x^*)$;*

(2) *for any $x\in A$ and $y\in T(x)$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_0=x$, $x_1=y$ and $x_n\in T(x_{n-1})$, $n\ge 1$, that converges to a fixed point $x^*\in A\cap B$ of T.*

The following lemma presents a well-known result (see for example [22]).

Lemma 4.1. *Let (X, d) be a metric space, d^* the metric defined on $X \times X$ by (3,4) and D^* the gap functional, respectively H^* the generalized Pompeiu–Hausdorff functional generated by d^*. Then for any $a, b \in X$ and any $A, B, C, D \in P_{prox}(X)$, the following statements hold:*

(1) $D^*((a,b), C \times D) = \max(D(a,C), D(b,D))$;

(2) $D^*(A \times B, C \times D) = \max(D(A,C), D(B,D))$;

(3) $H^*(A \times B, C \times D) = \max\{H(A,C), H(B,D)\}$;

(4) $D^*(A \times B, B \times A) = D(A,B)$.

Proof. (1)+(2) Since the sets C and D are proximinal then there exists $c_0 \in C, d_0 \in D$ such that $D(a,C) = d(a,c_0)$ and $D(b,D) = d(b,d_0)$.

Then

$$\begin{aligned} D^*((a,b), C \times D) &= \inf\{d^*((a,b),(c,d)) | c \in C, d \in D\} \\ &= \inf\{\max\{d(a,c), d(b,d)\} | c \in C, d \in D\} \\ &= \max\{d(a,c_0), d(b,d_0)\}. \end{aligned}$$

Similarly, we can prove (2).

(3) $H^*(A \times B, C \times D) =$

$$\max\{\sup_{(a,b)\in A\times B}\{D^*((a,b), C \times D)\}, \sup_{(c,d)\in C\times D}\{D^*((c,d), A \times B)\}\}.$$

Using statement (1), we have

$$\begin{aligned} H^*(A \times B, C \times D) &= \max\{\sup_{(a,b)\in A\times B}\{D(a,C), D(b,D)\}, \sup_{(c,d)\in C\times D}\{D(c,A), D(d,B)\}\} \\ &= \max\{H(A,C), H(B,D)\} \end{aligned}$$

(4) We use statement (2) for $C = A, D = B$.

Lemma 4.2. *Let (X, d) be a metric space, d^* the metric defined on $X \times X$ by* (3.4) *. If a multivalued operator $F : X \times X \to P(X)$ takes proximinal values with respect to d then the multivalued operator $T : X \times X \to P(X \times X)$, $T(x,y) = (F(x,y), F(y,x))$ takes proximinal values with respect to d^*.*

Proof. For any pair $(a,b) \in X \times X, F(a,b)$ is a proximinal set, which means that for any $x \in X$, there exists $c \in F(a,b)$ such that

$$d(x,c) = D(x, F(a,b)).$$

In a similar way, for any $y \in X$, there exists $d \in F(b,a)$ such that

$$d(y,d) = D(y, F(b,a)).$$

Then for any $(x,y) \in X \times X$, there exists $(c,d) \in T(a,b)$ such that

$$\begin{aligned} d^*((x,y),(c,d)) &= \max\{d(x,c), d(y,d)\} \\ &= \max\{D(x, F(a,b)), D(y, F(b,a))\} \\ &= D^*((x,y), T(a,b)). \end{aligned}$$

The first result in this section is the following theorem.

Theorem 4.2. *Let (X, d) be a complete metric space, $A, B \in P_{cl}(X)$, $Y = A \cup B$ and $F : Y \times Y \to P_{prox}(Y)$ a cyclic coupled φ-contraction of Ćirić type multivalued operator.*

Then the following statements hold:

(1) *there exist* $x^*, y^* \in A \cap B$ *such that*

$$x^* \in F(x^*, y^*),\ y^* \in F(y^*, x^*),$$

(that is the pair (x^*, y^*) *is a coupled fixed point of* F*);*

(2) *for each* $(a, b) \in A \times B$ *there exists a sequence* $(a_n, b_n)_{n \in \mathbb{N}^*} \in Y \times Y$ *with* $a_0 = a, b_0 = b$ *and*

$$a_n \in F(b_{n-1}, a_{n-1}), b_n \in F(a_{n-1}, b_{n-1}) \text{ for } n \geq 1$$

that converges to a coupled fixed point $(x^*, y^*) \in A \cap B$ *of* F.

Proof. It is easy to observe that

$$\tilde{M}(x, v, y, u) = \tilde{M}(v, x, u, y), \text{ for any } x, v \in A, y, u \in B.$$

If we change the roles between x and v and similarly for y and u, then the inequality (4.1) becomes

$$H(F(v, u), F(y, x)) \leq \varphi(\tilde{M}(x, v, y, u)). \tag{4.2}$$

From (4.1) and (4.2) we obtain

$$\max\{H(F(x, y), F(u, v)), H(F(y, x), F(v, u))\} \leq \varphi(\tilde{M}(x, v, y, u)).$$

Let $T: Y \times Y \to P(Y \times Y)$, $T(x, y) = (F(x, y), F(y, x))$.

We consider on $Y \times Y$ the metric d^* defined by (3.4), using the same functionals D^* and H^* as in Lemma 4.1.

For $z = (x, y) \in A \times B$, $w = (u, v) \in B \times A$, using Lemma 4.1,

$$\begin{aligned} H^*(T(z), T(w)) &= H^*((F(x, y), F(y, x)), (F(u, v), F(v, u))) \\ &= \max\{H(F(x, y), F(u, v)), H(F(y, x), F(v, u))\} \\ &\leq \varphi(\tilde{M}(x, v, y, u)). \end{aligned} \tag{4.3}$$

By Lemma 4.1,

$$D^*(z, T(z)) = \max\{D(x, F(x, y)), D(y, F(y, x))\},$$
$$D^*(w, T(w)) = \max\{D(u, F(u, v)), D(v, F(v, u))\},$$

$$\begin{aligned} \frac{1}{2}[D^*(w, T(z)) + D^*(z, T(w))] &= \frac{1}{2}[\max\{D(u, F(x, y)), D(v, F(y, x))\} \\ &\quad + \max\{D(x, F(u, v)), D(y, F(v, u))\}] \\ &\geq \max\left\{\frac{1}{2}[D(u, F(x, y)) + D(x, F(u, v))],\right. \\ &\qquad \left.\frac{1}{2}[D(v, F(y, x)) + D(y, F(v, u))]\right\}. \end{aligned}$$

Using the monotonicity of φ, (4.3) becomes

$$H^*(T(z),T(w)) \leq \varphi(\max\left\{d^*(z,w),D^*(z,T(z)),D^*(w,T(w)),\right.$$
$$\left.\frac{1}{2}[D^*(w,T(z))+D^*(z,T(w))]\right\}), \text{ for any } z \in A\times B,$$
$$w \in B\times A,$$

and because T satisfies the cyclic condition

$$T(A\times B)=(F(A\times B),F(B\times A))\subseteq B\times A, T(B\times A)\subseteq A\times B,$$

where $A\times B, B\times A \in P_{cl}(Y\times Y)$, we conclude that T is a multivalued cyclic φ-contraction of Ćirić type.

By Lemma 4.2, the property of the operator F to have proximinal values is transferred to the operator T, so we are in the conditions of Theorem 4.1.

Then there exists $(x^*,y^*)\in(A\times B)\cap(B\times A)$ such that $(x^*,y^*)\in(F(x^*,y^*),F(y^*,x^*))$ and for each $(a,b)\in A\times B$ there exists a sequence $(a_n,b_n)_{n\in\mathbb{N}}\in Y\times Y$ with $a_0=a$, $b_0=b$ and

$$(a_n,b_n)\in(F(b_{n-1},a_{n-1}),F(a_{n-1},b_{n-1})), n\geq 1$$

that converges to (x,y).

Hereinafter we define and study the generalized Ulam–Hyers stability of the following coupled fixed point problem.

Definition 4.3. Let (X,d) be a metric space, $Y\in P(X)$, F: $Y\times Y\to P(Y)$ be a multivalued operator. By definition, the coupled fixed point problem

$$\begin{cases} x\in F(x,y) \\ y\in F(y,x) \end{cases}, x,y\in Y \tag{4.4}$$

is said to be generalized Ulam–Hyers stable if there exists an increasing function $\psi:\mathbb{R}_+\to\mathbb{R}_+$, continuous at 0, with $\psi(0)=0$ such that for each $\varepsilon>0$ and for each solution $(x,y)\in Y\times Y$ of the inequality

$$\max\{D(x,F(x,y)),D(y,F(y,x))\}\leq\varepsilon,$$

there exists a solution $(x^*,y^*)\in Y\times Y$ of the coupled fixed point problem such that

$$\max\{d(x,x^*),d(y,y^*)\}\leq\psi(\varepsilon).$$

Our stability result is a consequence of the following theorem.

Theorem 4.3 *([21]). Let $T:Y\to P_{prox}(Y)$ be as in Theorem 4.2, $\varepsilon>0$ and $x\in Y$ be such that $D(x,T(x))\leq\varepsilon$. Then there exists x^* a fixed point of T such that $d(x,x)\leq s(\varepsilon)$, where s is given by Lemma 2.2.*

Theorem 4.4. *If all the hypotheses of Theorem 4.2 hold, then the coupled fixed point problem (4.4) is generalized Ulam–Hyers stable.*

Proof. Let any $\varepsilon>0$ and let $(\bar{x},\bar{y})\in Y\times Y$ such that

$$\begin{cases} D(\bar{x},F(\bar{x},\bar{y}))\leq\varepsilon \\ D(\bar{y},F(\bar{y},\bar{x}))\leq\varepsilon. \end{cases}$$

As before, we consider $T:Y\times Y\to P(Y\times Y)$,

$$T(x,y)=(F(x,y),F(y,x)).$$

For $z = (\bar{x}, \bar{y})$,

$$D^*(z, T(z)) = \max\{D(\bar{x}, F(\bar{x}, \bar{y})), D(\bar{y}, F(\bar{y}, \bar{x}))\} \leq \varepsilon.$$

Applying Theorem 4.3, there exists a fixed point $z^* = (x^*, y^*)$ of T such that $d^*(z, z^*) \leq s(\varepsilon)$, that is there exists a solution (x^*, y^*) of the coupled fixed point problem (4.4) such that

$$\max\{d(\bar{x}, x^*), d(\bar{y}, y^*)\} \leq s(\varepsilon). \quad \square$$

In the last part of this section we will consider the following best proximity problem for a cyclic coupled multivalued operator:

If (X, d) is a metric space, $A, B \in P(X)$, $Y = A \cup B$, $F : Y \times Y \to P(Y)$ is a coupled multivalued operator satisfying the cyclic condition $F(A \times B) \subseteq B$, $F(B \times A) \subseteq A$, then we are interested in finding $(x^*, y^*) \in A \times B$ such that

$$D(x^*, F(x^*, y^*)) = D(y^*, F(y^*, x^*)) = D(A, B). \tag{4.5}$$

(x^*, y^*) is said to be a coupled best proximity point of F.

Notice that, in particular, if $A \cap B \neq \emptyset$ then (x^*, y^*) is a coupled fixed point of F.

Definition 4.4. Let (X, d) be a metric space, $A, B \in P(X)$, $Y = A \cup B$. A multivalued operator $F : Y \times Y \to P(Y)$ is called a cyclic coupled Ćirić type multivalued operator if:

(i) $F(A \times B) \subseteq B$ and $F(B \times A) \subseteq A$;

(ii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$H(F(x, y), F(u, v)) \leq \varphi(\tilde{M}(x, v, y, u) - D(A, B)) + D(A, B),$$

for any $x, v \in A$, $y, u \in B$.

In 2009, Suzuki, Kikkawa and Vetro introduced the following property.

Definition 4.5. [23] Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the property UC if for $(x_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ sequences in A and $(y_n)_{n\in\mathbb{N}}$ a sequence in B such that $d(x_n, y_n) \to D(A, B)$ and $d(z_n, y_n) \to D(A, B)$ as $n \to \infty$, then $d(x_n, z_n) \to 0$ as $n \to \infty$.

Example 4.1. [24] [23] (1) Any pair of nonempty subsets (A, B) of a metric space (X, d) with $D(A, B) = 0$ satisfies the property UC;

(2) Any pair of nonempty subsets (A, B) of a uniformly convex Banach space with A convex satisfies the property UC.

Lemma 4.3. *Let A and B be nonempty subsets of a metric space (X, d), and d^* be the metric defined on $X \times X$ by (3.4). If (A, B) and (B, A) satisfy the property UC with respect to d then $(A \times B, B \times A)$ satisfy the property UC with respect to d.*

Proof. We denote $D^*(A \times B, B \times A) = D(A, B) = D$. Let $x_n = (a_n, b_n), z_n = (a_n', b_n') \in A \times B, y_n = (\beta_n, \alpha_n) \in B \times A$ such that $d^*(x_n, y_n) \to D$ and $d^*(z_n, y_n) \to D$ as $n \to \infty$. Then

$$\max\{d(a_n, \beta_n), d(b_n, \alpha_n)\} \to D \text{ and}$$

$$\max\{d(a_n', \beta_n), d(b_n', \alpha_n)\} \to D \text{ as } n \to \infty.$$

It is obvious that $d(a_n, \beta_n) \to D, d(a_n', \beta_n) \to D$ and because (A, B) satisfies the property UC we get $d(a_n, a_n') \to 0$.

From $d(b_n, \alpha_n) \to D, d(b_n^{'}, \alpha_n) \to D$ as $n \to \infty$ and using (B, A) satisfies the property UC we get $d(b_n, b_n^{'}) \to 0$.
Finally,

$$d^*(x_n, z_n) = \max\{d(a_n, a_n^{'}), d(b_n, b_n^{'})\} \to 0 \text{ as } n \to \infty.$$

We recall the following result.

Theorem 4.5 ([25]). *Let (X, d) be a complete metric space, $A \in P_{cl}(X), B \in P(X)$ such that (A, B) satisfies the property UC. Let $T: A \cup B \to P_{prox}(X)$ be a multivalued Ćirić type cyclic operator that is:*

(i) *$T(A) \subseteq B$ and $T(B) \subseteq A$;*

(ii) *there exists a comparison function $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$H(T(x), T(y)) \leq \varphi(M(x,y) - D(A,B)) + D(A,B), \text{ where}$$

$$M(x,y) = \max\left\{d(x,y), D(x, T(x)), D(y, T(y)), \frac{1}{2}[D(x, T(y)) + D(y, T(x))]\right\}.$$

Then the following statements hold:

(1) T has a best proximity point $x_A^* \in A$;

(2) there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_0 \in A$, and $x_{n+1} \in T(x_n)$, $n \geq 0$, such that $(x_{2n})_{n\in\mathbb{N}}$ converges to x_A^*.

The next result is a consequence of the above theorem.

Theorem 4.6. *Let (X, d) be a complete metric space, $A, B \in P_{cl}(X)$ such that (A, B) and (B, A) satisfy the property UC, and $Y = A \cup B$. If $F: Y \times Y \to P_{prox}(Y)$ is a cyclic coupled Ćirić type multivalued operator, then the following statements hold:*

(i) F has a coupled best proximity point $(x^*, y^*) \in A \times B$;

(ii) there exist two sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ with

$$(x_0, y_0) \in A \times B, x_{n+1} \in F(x_n, y_n), y_{n+1} \in F(y_n, x_n),$$

such that $((x_{2n}, y_{2n}))_{n\in\mathbb{N}}$ converges to (x^*, y^*).

Proof. Considering again on $Y \times Y$ the metric d^* defined by (3.4), in a similar manner as in Theorem 4.2, we obtain that the operator $T: Y \times Y \to P(Y \times Y)$,

$$T(x,y) = (F(x,y), F(y,x)).$$

is a multivalued Ćirić type cyclic operator which takes proximinal values.

Using Lemma 4.1, the pair $(A \times B, B \times A)$ satisfies the property UC with respect to d^*.

Consequently, we are in the conditions of Theorem 4.5, so T has a best proximity point $(x^*, y^*) \in A \times B$ and there exists a sequence $(x_n, y_n)_{n\in\mathbb{N}}$ with $(x_0, y_0) \in A \times B$ and $(x_{n+1}, y_{n+1}) \in T(x_n, y_n)$ such that $(x_{2n}, y_{2n})_{n\in\mathbb{N}}$ converges to (x^*, y^*) with respect to d^*.

5. An application to a system of integral equations

We apply the results given by Theorem 3.2 to study the existence and the uniqueness of solutions of the following system of integral equations:

$$\begin{cases} x(t) = \int_a^b G(t,s)f(s,x(s),y(s))ds \\ y(t) = \int_a^b G(t,s)f(s,y(s),x(s))ds \end{cases}, t \in [a,b] \tag{5.1}$$

where $a, b \in \mathbb{R}$, $a < b$,

$$G \in C([a,b] \times [a,b], [0,\infty)),$$
$$f \in C([a,b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}).$$

Theorem 5.1. *We suppose that:*

(i) there exist $\alpha, \beta \in C([a,b],\mathbb{R})$ *, with* $\alpha(t) \le \beta(t)$ *, for any* $t \in [a,b]$ *, such that*

$$\begin{cases} \alpha(t) \le \int_a^b G(t,s)f(s,\beta(s),\alpha(s))ds \\ \beta(t) \ge \int_a^b G(t,s)f(s,\alpha(s),\beta(s))ds \end{cases} \quad \text{for any } t \in [a,b]; \tag{5.2}$$

(ii) there exists a strong comparison function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$|f(s,u_1,u_2) - f(s,v_1,v_2)| \le \varphi(\max\{|u_1 - v_1|, |u_2 - v_2|\}),$$

for any $s \in [a,b]$ *and* $u_1, u_2, v_1, v_2 \in \mathbb{R}$*;*

(iii) $\sup_{t\in[a,b]} \int_a^b G(t,s)ds \le 1$ *;*

(iv) $f(s,\cdot,y)$ *is monotone decreasing for any* $s \in [a,b]$ *and any* $y \in \mathbb{R}$*;*

(v) $f(s,x,\cdot)$ *is monotone increasing for any* $s \in [a,b]$ *and any* $x \in \mathbb{R}$*.*
Then the system (5.1) has a unique solution $(x^*, x^*) \in C([a,b],\mathbb{R}^2)$ *, with* $\alpha \le x^* \le \beta$ *.*

Proof. Let us consider

$$X := C([a,b],\mathbb{R}), \text{and the Chebyshev norm } |x|_\infty = \max_{t\in[a,b]}|x(t)|.$$

Then $(X, |\cdot|_\infty)$ is a Banach space. We consider the following closed subsets of X:

$$A = \{x \in X \mid x \le \beta\}, B = \{x \in X | x \ge \alpha\},$$

$Y = A \cup B$ and the operator F: $Y \times Y \to Y$,

$$F(x,y)(t) := \int_a^b G(t,s)f(s,x(s),y(s))ds.$$

The system (5.1) is equivalent to

$$\begin{cases} F(x,y) = x \\ F(y,x) = y \end{cases}, x, y \in Y.$$

We will prove that F is cyclic with respect to A and B, that is

$$F(A \times B) \subseteq B \text{ and } F(B \times A) \subseteq A.$$

Let $x \in A$ and $y \in B \Rightarrow x(s) \le \beta(s), y(s) \ge \alpha(s), \forall s \in [a,b]$.

Using the monotonicity of f we have

$$G(t,s)f(s,x(s),y(s)) \geq G(t,s)f(s,\beta(s),\alpha(s)),$$

and from (i), by integration,

$$\int_a^b G(t,s)f(s,x(s),y(s))ds \geq \alpha(t),$$

which means that

$$F(x,y)(t) \geq \alpha(t), \forall t \in [a,b] \Rightarrow F(x,y) \in B.$$

So $F(A \times B) \subseteq B$. In a similar way we have $F(B \times A) \subseteq A$.

Using the conditions (ii) and (iii), and the monotonicity of φ, for any $x, v \in A$ and $y, u \in B$, we have

$$\begin{aligned} |f(s,x(s),y(s)) - f(s,u(s),v(s))| &\leq \varphi(\max_{s\in[a,b]}\{|x(s) - u(s)|, |y(s) - v(s)|\}) \\ &\leq \varphi(\max\{|x-u|_\infty, |y-v|_\infty\}) \Rightarrow \end{aligned}$$

$$\begin{aligned} |F(x,y)(t) - F(u,v)(t)| &\leq \int_a^b G(t,s)|f(s,x(s),y(s)) - f(s,u(s),v(s))|ds \\ &\leq \varphi(\max\{|x-u|_\infty, |y-v|_\infty\}) \int_a^b G(t,s)ds \\ &\leq \varphi(\max\{|x-u|_\infty, |y-v|_\infty\}), \forall t \in [a,b]. \end{aligned}$$

We have

$$|F(x,y) - F(u,v)|_\infty \leq \varphi(\max\{|x-u|_\infty, |y-v|_\infty\}) \text{ for any } x, v \in A \text{ and } y, u \in B,$$

so the operator F is a cyclic coupled φ-contraction of Ćirić type.

All the conditions of Theorem 3.2 are satisfied, so T has a unique strong coupled fixed point $(x^*, x^*) \in A \cap B$, with $\alpha(t) \leq x^*(t) \leq \beta(t)$, for any $t \in [a,b]$.

Definition 5.1. The system (5.1) is said to be generalized Ulam–Hyers stable if there exists $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0 and $\psi(0) = 0$ such that for any $\varepsilon_1 > 0, \varepsilon_2 > 0$ and for any solution $(x,y) \in C([a,b], \mathbb{R}^2)$, of the system

$$\begin{cases} |x(t) - \int_a^b G(t,s)f(s,x(s),y(s))ds| \leq \varepsilon_1 \\ |y(t) - \int_a^b G(t,s)f(s,y(s),x(s))ds| \leq \varepsilon_2 \end{cases}$$

there exists a solution $(x^*, y^*) \in C([a,b], \mathbb{R}^2)$ of the system (5.1) such that for any $t \in [a,b]$,

$$\begin{cases} |x(t) - x^*(t)| \leq \psi(\varepsilon) \\ |y(t) - y^*(t)| \leq \psi(\varepsilon) \end{cases}, \text{ where } \varepsilon = \max(\varepsilon_1, \varepsilon_2).$$

Theorem 5.2. *Suppose that the hypotheses of Theorem 5.1 hold. Then the system (5.1) is generalized Ulam–Hyers stable.*

Proof. By Theorem 5.1, the system (5.1) has a unique solution $(x^*, x^*) \in C([a,b], \mathbb{R}^2)$, with $\alpha \leq x^* \leq \beta$. Applying Theorem 3.6 to the operator $F: Y \times Y \to Y$,

$$F(x,y)(t) := \int_a^b G(t,s)f(s,x(s),y(s))ds,$$

in the same setting as in the proof of Theorem 5.1, we get the conclusion.

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