On abstract Hilfer fractional integrodifferential equations with boundary conditions

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Abstract
In this paper, we study a Cauchy-type problem for Hilfer fractional integrodifferential equations with boundary conditions. The existence of solutions for the given problem is proved by applying measure of noncompactness technique in an abstract weighted space. Moreover, we use generalized Gronwall inequality with singularity to establish continuous dependence and uniqueness of ϵ-approximate solutions.

Keywords Hilfer fractional integrodifferential equations, Boundary conditions, Mönch fixed point theorem, Measure of noncompactness, Existence, Continuous dependence

Paper type Original Article

1. Introduction
Fractional calculus has emerged as a powerful tool to study complex phenomena in numerous scientific and engineering disciplines such as viscoelasticity, fluid mechanics, physics and heat conduction in materials with memory. For examples and applications, see [2,14,17–21] and references cited therein. Many authors focused on Riemann–Liouville and Caputo type derivatives in investigating fractional differential equations. In [7], Hilfer introduced a new concept of generalized Riemann–Liouville derivative (Hilfer derivative) of order α and type β. This definition facilitated dynamic modeling of non-equilibrium processes based on

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interpolation with respect to parameter of the Riemann–Liouville and Caputo type operators; for instance, see [1,4,6,8,10,11].

Furati et al. [6] established the existence and uniqueness of solutions for the problem:

$$\begin{cases} D_{a^+}^{\alpha,\beta} y(t) = f(t, y(t)), t \in J = (a, b], 0 < \alpha < 1, 0 \le \beta \le 1, \\ I_{a^+}^{1-\gamma} y(a^+) = w, \quad \alpha \le \gamma = \alpha + \beta - \alpha\beta, \end{cases}$$

by applying Banach fixed point theorem in weighted space $C_{1-\gamma}^{\gamma}[J, \mathbb{R}]$. Abbas et al. [1] discussed the above problem by using Kuratowski measure of noncompactness.

Motivated by the works [1,6], we will study a more general problem of Hilfer fractional integrodifferential equations with boundary conditions given by

$$\begin{cases} D_{a^+}^{\alpha,\beta} y(t) = f(t, y(t), (Sy)(t)), t \in J = (a, b], 0 < \alpha < 1, 0 \le \beta \le 1, \\ I_{a^+}^{1-\gamma} [uy(a^+) + vy(b^-)] = w, \quad \alpha \le \gamma = \alpha + \beta - \alpha\beta, \end{cases} \tag{1.1}$$

where $D_{a^+}^{\alpha,\beta}$ is the left-sided Hilfer fractional derivative of order α and type β, $f : J \times X \times X \to X$, X is an abstract Banach space, $u, v, w \in \mathbb{R}, u + v \ne 0$, and S is a linear integral operator defined by $(Sy)(t) = \int_a^t k(t, s)y(s)ds$ with $\zeta = \max\{ \int_a^t k(t, s)ds : (t, s) \in J \times J\}, k \in (J \times J, \mathbb{R})$.

This article is constructed as follows: In Section 2, we recall some preliminaries. Section 3 contains the existence result obtained by using measure of noncompactness and Mönch fixed point theorem. We discuss the ϵ-approximate solution of Hilfer fractional integrodifferential equations in Section 4.

2. Preliminaries

In this section, we present some necessary definitions, notations and preliminaries, which will be used throughout this work.

For $-\infty < a < b < \infty$, let $C[J, X]$ denote the space of all continuous functions on J into X endowed with supremum norm $\|x\|_C := \sup\{\|x(t)\| : t \in J\}$. Define by $C_{1-\gamma}[J, X] = \{f(x) : (a, b] \to X | (x - a)^{1-\gamma} f(x) \in C[J, X]\}$ the weighted space of the abstract continuous functions. Obviously, $C_{1-\gamma}[J, X]$ is a Banach space equipped with the norm $\|f\|_{C_{1-\gamma}} = \left\|(x - a)^{1-\gamma} f(x)\right\|_C$, and $C_{1-\gamma}^n[J, X] = \{f \in C^{n-1}[J, X] : f^{(n)} \in C_{1-\gamma}[J, X]\}$ is the Banach space endowed with the norm

$$\|f\|_{C_{1-\gamma}^n} = \sum_{i=0}^{n-1} \|f^{(k)}\|_C + \|f^{(n)}\|_{C_{1-\gamma}}, n \in \mathbb{N},$$

where, $C_{1-\gamma}^0 := C_{1-\gamma}$

Definition 2.1 (*See [13]*)**.** The left-sided Riemann–Liouville fractional integral of order $\alpha > 0$ of function $f : [a, \infty) \to \mathbb{R}$ is defined by

$$\left(I_{a^+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} f(s)ds, \ t > a,$$

where $a \in \mathbb{R}$ and Γ is the Gamma function.

Definition 2.2 (*See [13]*)**.** The left-sided Riemann–Liouville fractional derivative of order $\alpha \in (n - 1, n]$ of function $f : [a, \infty) \to \mathbb{R}$, is defined by

$$(D^{\alpha}_{a^+}f)(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)ds,\ t > a,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Remark 2.1. If f is an abstract function with values in X, then the integrals appearing in Definitions 2.1 and 2.2 are taken in Bochner's sense.

Definition 2.3 (*See [7]*). The left-sided Hilfer fractional derivative of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$, of function $f(t)$ is defined by

$$(D^{\alpha,\beta}_{a^+}f)(t) = \left(I^{\beta(1-\alpha)}_{a^+} D\left(I^{(1-\beta)(1-\alpha)}_{a^+}\right)\right)(t),$$

where $D := \frac{d}{dt}$.

Remark 2.2 (*See [7]*). From Definition 2.3, we observe that:

(i) the operator $D^{\alpha,\beta}_{a^+}$ can be written as

$$D^{\alpha,\beta}_{a^+} = I^{\beta(1-\alpha)}_{a^+} D I^{(1-\gamma)}_{a^+} = I^{\beta(1-\alpha)}_{a^+} D^{\gamma}, \quad \gamma = \alpha + \beta - \alpha\beta;$$

(ii) The Hilfer fractional derivative can be regarded as an interpolator between the Riemann–Liouville derivative ($\beta = 0$) and Caputo derivative ($\beta = 1$) as

$$D^{\alpha,\beta}_{a^+} = \begin{cases} D I^{(1-\alpha)}_{a^+} = D^{\alpha}_{a^+}, & \text{if } \beta = 0; \\ I^{(1-\alpha)}_{a^+} D = {}^{C}D^{\alpha}_{a^+}, & \text{if } \beta = 1. \end{cases}$$

In the forthcoming analysis, we need the spaces:

$$C^{\alpha,\beta}_{1-\gamma}[J,X] = \left\{f \in C_{1-\gamma}[J,X], D^{\alpha,\beta}_{a^+}f \in C_{1-\gamma}[J,X]\right\},$$

and

$$C^{\gamma}_{1-\gamma}[J,X] = \left\{f \in C_{1-\gamma}[J,X], D^{\gamma}_{a^+}f \in C_{1-\gamma}[J,X]\right\}.$$

Since $D^{\alpha,\beta}_{a^+}f = I^{\beta(1-\alpha)}_{a^+}D^{\gamma}f$, it is obvious that $C^{\gamma}_{1-\gamma}[J,X] \subset C^{\alpha,\beta}_{1-\gamma}[J,X]$.

Now, we state some known results related to our work.

Lemma 2.1 (*See [5]*). *Let $\beta > 0$ and $\alpha > 0$. Then*

$$\left[I^{\alpha}_{a^+}(t-a)^{\beta-1}\right](x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}$$

and

$$\left[D^{\alpha}_{a^+}(t-a)^{\alpha-1}\right](x) = 0, \ \ 0 < \alpha < 1.$$

Lemma 2.2 (*See [5]*). *If $\alpha > 0$ and $\beta > 0$, and $f \in L^1(J)$ for $t \in [a,b]$, then the following properties hold:*

$$\left(I^{\alpha}_{a^+}I^{\beta}_{a^+}f\right)(t) = \left(I^{\alpha+\beta}_{a^+}f\right)(t) \ \text{ and } \ \left(D^{\alpha}_{a^+}I^{\beta}_{a^+}f\right)(t) = f(t).$$

In particular, if $f \in C_{\gamma}[J,X]$ or $f \in C[J,X]$, then the above properties hold for each $t \in (a,b]$ or $t \in [a,b]$ respectively.

Lemma 2.3 *(See [5]). If $0<\alpha<1$, $0\leq\gamma<1$ and that $f\in C_\gamma[J,X]$, $I_{a^+}^{1-\alpha}f\in C_\gamma^1[J,X]$, then*

$$I_{a^+}^{\alpha}D_{a^+}^{\alpha}f(t)=f(t)-\frac{\left(I_{a^+}^{1-\alpha}f\right)(a)}{\Gamma(\alpha)}(t-a)^{\alpha-1},\quad \forall t\in J.$$

Lemma 2.4 *(See [6]). If $0\leq\gamma<1$ and $f\in C_\gamma[J,X]$, then*

$$\left(I_{a^+}^{\alpha}f\right)(a)=\lim_{t\to a^+}I_{a^+}^{\alpha}f(t)=0,\quad 0\leq\gamma<\alpha.$$

Lemma 2.5 *(See [6]). Let $\alpha>0$, $\beta>0$ and $\gamma=\alpha+\beta-\alpha\beta$. If $f\in C_{1-\gamma}^{\gamma}[J,X]$, then*

$$I_{a^+}^{\gamma}D_{a^+}^{\gamma}f=I_{a^+}^{\alpha}D_{a^+}^{\alpha,\beta}f, D_{a^+}^{\gamma}I_{a^+}^{\alpha}f=D_{a^+}^{\beta(1-\alpha)}f.$$

Lemma 2.6 *(See [6]). Let $f\in L^1(J)$ and $D_{a^+}^{\beta(1-\alpha)}f\in L^1(J)$ exists, then*

$$D_{a^+}^{\alpha,\beta}I_{a^+}^{\alpha}f=I_{a^+}^{\beta(1-\alpha)}D_{a^+}^{\beta(1-\alpha)}f.$$

Lemma 2.7 *(Theorem 23, [6]). Let $f:J\times\mathbb{R}\to\mathbb{R}$ be a function such that $f\in C_{1-\gamma}[J,\mathbb{R}]$ for any $y\in C_{1-\gamma}[J,\mathbb{R}]$. Then $y\in C_{1-\gamma}^{\gamma}[J,\mathbb{R}]$ is a solution of the initial value problem:*

$$\begin{cases} D_{a^+}^{\alpha,\beta}y(t)=f(t,y(t)), t\in J=(a,b], 0<\alpha<1, 0\leq\beta\leq1,\\ I_{a^+}^{1-\gamma}y(a^+)=y_a,\quad \alpha\leq\gamma=\alpha+\beta-\alpha\beta,\end{cases}$$

if and only if y satisfies the following Volterra integral equation:

$$y(t)=\frac{y_a}{\Gamma(\gamma)}(t-a)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s,y(s))ds.$$

Next we obtain the integral solution of the problem (1.1) by using Lemma 2.7.

Lemma 2.8. *Let $f:J\times X\times X\to X$ be a function such that $f\in C_{1-\gamma}[J,X]$ for any $y\in C_{1-\gamma}[J,X]$. Then $y\in C_{1-\gamma}^{\gamma}[J,X]$ is a solution of the problem (1.1) if and only if y satisfies the following integral equation*

$$\begin{aligned} y(t)=&\frac{w}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}-\frac{v}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{\Gamma(1-\gamma+\alpha)}\\ &\times\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds \\ &+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s,y(s),(Sy)(s))ds.\end{aligned}\tag{2.1}$$

Proof. In view of Lemma 2.7, the solution of (1.1) can be written as

$$y(t)=\frac{I_{a^+}^{1-\gamma}y(a^+)}{\Gamma(\gamma)}(t-a)^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s,y(s),(Sy)(s))ds.\tag{2.2}$$

Applying $I_{a^+}^{1-\gamma}$ on both sides of (2.2) and taking the limit $t\to b^-$, we obtain

$$I_{a^+}^{1-\gamma}y(b^-)=I_{a^+}^{1-\gamma}y(a^+)+\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds.\tag{2.3}$$

In a similar manner, we find that

$$I_{a^+}^{1-\gamma}y(a^+)=\frac{1}{1+\frac{v}{u}}\left\{\frac{w}{u}-\frac{v}{u}\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds\right\}$$
$$=\frac{1}{u+v}\left\{w-v\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds\right\}. \tag{2.4}$$

Submitting (2.4) into (2.2), we obtain

$$y(t)=\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{u+v}\left\{w-v\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds\right\}$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s,y(s),(Sy)(s))ds,$$
$$=\frac{w}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}-\frac{v}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{\Gamma(1-\gamma+\alpha)}$$
$$\times\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds$$
$$+\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}f(s,y(s),(Sy)(s))ds.$$

Conversely, applying $I_{a^+}^{1-\gamma}$ on both sides of (2.1) and using Lemmas 2.1 and 2.2, we get

$$I_{a^+}^{1-\gamma}y(t)=\frac{w}{u+v}-\frac{v}{u+v}\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds$$
$$+I_{a^+}^{1-\beta(1-\alpha)}f(t,y(t),(Sy)(t)). \tag{2.5}$$

Next, taking the limit $t\to a^+$ of (2.5) and using Lemma 2.4, with $1-\gamma<1-\beta(1-\alpha)$, we obtain

$$I_{a^+}^{1-\gamma}y(a^+)=\frac{w}{u+v}-\frac{v}{u+v}\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds. \tag{2.6}$$

Now, taking the limit $t\to b^-$ of (2.5), we get

$$I_{a^+}^{1-\gamma}y(b^-)=\frac{w}{u+v}-\frac{v}{u+v}\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds$$
$$+I_{a^+}^{1-\beta(1-\alpha)}f(b,y(b),(Sy)(b)). \tag{2.7}$$

From (2.6) and (2.7), we find that

$$
\begin{aligned}
&uI_{a^+}^{1-\gamma}y(a^+)+vI_{a^+}^{1-\gamma}y(b^-)\\
&=\frac{uw}{u+v}-\frac{uv}{u+v}\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds\\
&\quad+\frac{vw}{u+v}-\frac{v^2}{u+v}\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b(b-s)^{\alpha-\gamma}f(s,y(s),(Sy)(s))ds.\\
&\quad+vI_{a^+}^{1-\beta(1-\alpha)}f(b,y(b),(Sy)(b))\\
&=\frac{w(u+v)}{u+v}-\frac{v(u+v)}{u+v}I_{a^+}^{1-\gamma+\alpha}f(b,y(b),(Sy)(b))\\
&\quad+vI_{a^+}^{1-\beta(1-\alpha)}f(b,y(b),(Sy)(b))\\
&=w,
\end{aligned}
$$

which shows that the boundary condition $I_{a^+}^{1-\gamma}[uy(a^+)+vy(b^-)]=w$ is satisfied.

Next, applying $D_{a^+}^{\gamma}$ on both sides of (2.1) and using Lemmas 2.1 and 2.5, we have

$$D_{a^+}^{\gamma}y(t)=D_{a^+}^{\beta(1-\alpha)}f(t,y(t),(Sy)(t)). \tag{2.8}$$

Since $y\in C_{1-\gamma}^{\gamma}[J,X]$ and by definition of $C_{1-\gamma}^{\gamma}[J,X]$, we have $D_{a^+}^{\gamma}y\in C_{1-\gamma}[J,X]$, therefore, $D_{a^+}^{\beta(1-\alpha)}f=DI_{a^+}^{1-\beta(1-\alpha)}f\in C_{1-\gamma}[J,X]$. For $f\in C_{1-\gamma}[J,X]$, it is clear that $I_{a^+}^{1-\beta(1-\alpha)}f\in C_{1-\gamma}[J,X]$. Hence f and $I_{a^+}^{1-\beta(1-\alpha)}f$ satisfy the hypothesis of Lemma 2.3.

Now, applying $I_{a^+}^{\beta(1-\alpha)}$ on both sides of (2.8), and using Lemma 2.3, we get

$$D_{a^+}^{\alpha,\beta}y(t)=f(t,y(t),(Sy)(t))-\frac{I_{a^+}^{1-\beta(1-\alpha)}f(a,y(a),(Sy)(a))}{\Gamma(\beta(1-\alpha))}(t-a)^{\beta(1-\alpha)-1}.$$

By Lemma 2.4, we have $I_{a^+}^{1-\beta(1-\alpha)}f(a,y(a),(Sy)(a))=0$. Therefore, we have $D_{a^+}^{\alpha,\beta}y(t)=f(t,y(t),(Sy)(t))$. This completes the proof. □

Next, we recall definition of noncompactness measure of Hausdorff $\Psi(\cdot)$ on each bounded subset Ω of Banach space X defined by

$$\Psi(\Omega)=\inf\{r>0,\Omega \text{ can be covered by finite number of balls with radii } r\}.$$

Lemma 2.9 ([3]). *For all nonempty subsets $A,B\subset X$, the Hausdorff measure of noncompactness $\Psi(\cdot)$ satisfies the following properties:*

(1) A is precompact if and only if $\Psi(A)=0$;

(2) $\Psi(A)=\Psi(\overline{A})=\Psi(conv\,A)$, where $\overline{A}$ and $conv\,A$ denote the closure and convex hull of A respectively;

(3) $\Psi(A) \leq \Psi(B)$ *when* $A \subseteq B$;

(4) $\Psi(A+B) \leq \Psi(A) + \Psi(B)$, *where* $A+B = \{a+b; a \in A, b \in B\}$;

(5) $\Psi(A \cup B) \leq \max\{\Psi(A), \Psi(B)\}$;

(6) $\Psi(\lambda A) = |\lambda| \Psi(A)$ *for any* $\lambda \in \mathbb{R}$;

(7) $\Psi(\{x\} \cup A) \leq \Psi(A)$ *for any* $x \in X$.

Lemma 2.10 ([3]). *If* $\mathbb{B} \subseteq C([a,b], X)$ *is bounded and equicontinuous, then* $\Psi(\mathbb{B}(t))$ *is continuous for* $t \in [a,b]$ *and* $\Psi(\mathbb{B}) = \sup\{\Psi(\mathbb{B}(t)), t \in [a,b]\}$, *where* $\mathbb{B}(t) = \{x(t); x \in \mathbb{B}\} \subseteq X$.

Lemma 2.11 ([16]). *If* $\{u_n\}_{n=1}^{\infty}$ *is a sequence of Bochner integrable functions from* J *into* X *with* $\|u_n(t)\| \leq \mu(t)$ *for almost all* $t \in J$ *and every* $n \geq 1$, *where* $\mu \in L^1(J, \mathbb{R})$, *then the function* $\Psi(t) = \Psi(\{u_n(t) : n \geq 1\})$ *belongs to* $L^1(J, \mathbb{R})$ *with*

$$\Psi\left(\left\{\int_0^t u_n(s)ds : n \geq 1\right\}\right) \leq 2\int_0^t \Psi(s)ds.$$

In order to prove the existence of solutions for our problem with lesser number of constraints, we will introduce another type of measure of noncompactness as follows.

Let Φ denote the measure of noncompactness in the Banach space $C[J, X]$ defined by

$$\Phi(\Omega) = \max_{E \in \Delta(\Omega)} (\delta(E), mod_c(E)), \tag{2.9}$$

for all bounded subsets Ω of $C[J, X]$, where $\Delta(\Omega)$ is the set of countable subsets of Ω, δ is the real measure of noncompactness given by

$$\delta(E) = \sup_{t \in [0,b]} e^{-Lt} \Psi(E(t)),$$

with $E(t) = \{x(t) : x \in E\}, t \in J$, L is a suitably chosen constant and $mod_c(E)$ is the modulus of equicontinuity of the function set E defined as

$$mod_c(E) = \lim_{\delta \to 0} \sup_{x \in E} \max_{|t_2 - t_1| \leq \delta} \|x(t_2) - x(t_1)\|.$$

Observe that Φ is well defined [9] (i.e., $E_0 \in \Delta(\Omega)$ which attends the maximum in (2.9)) and is nonsingular, monotone and regular measure of noncompactness.

Lemma 2.12 (*Mönch fixed point theorem*, [15]). *Let* D *be a closed convex subset of a Banach space* X *with* $0 \in D$. *Suppose that* $F : D \to X$ *is a continuous map satisfying the Mönch's condition (if* $M \subseteq D$ *is countable and* $M \subseteq conv(\{0\} \cup F(M))$, *then* $\overline{M}$ *is compact), then* F *has a fixed point in* D.

3. Existence of solutions

Let us begin this section by introducing the hypotheses needed to prove the existence of solutions for the problem at hand.

(H1) The function $f: J \times X \times X \to X$ satisfies (i) $f(\cdot, x, y): J \to X$ is measurable for all $x, y \in X$ and (ii) $f(t, \cdot, \cdot): X \times X \to X$ is continuous for a.e $t \in J$.

(H2) There exists a constant $N > 0$ such that

$$\|f(t, y, Sy)\| \le N(1 + \zeta\|y\|),$$

for each $t \in J$ and all $y \in X$.

(H3) There exist constants $m_1, m_2 > 0$ such that

$$\Psi(f(t, x, y)) \le m_1\Psi(x) + m_2\Psi(y),$$

for bounded sets $x, y \subset X$, a.e $t \in J$.

Now, we are ready to present the existence result for the problem (1.1), which is based on Mönch fixed point theorem.

Theorem 3.1. *Suppose that $f: J \times X \times X \to X$ is such that $f(\cdot, y(\cdot), Sy(\cdot)) \in C_{1-\gamma}^{\beta(1-\alpha)}[J, X]$ for any $y \in C_{1-\gamma}[J, X]$ and satisfies the hypotheses (H1)-(H3). Then the Hilfer problem (1.1) has at least one solution in $C_{1-\gamma}^{\gamma}[J, X] \subset C_{1-\gamma}^{\alpha,\beta}[J, X]$, provided that*

$$Q := \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{N\zeta}{\Gamma(1-\gamma+\alpha)}(b-a)^{\alpha} B(\gamma, \alpha-\gamma+1) + \frac{N\zeta}{\Gamma(\alpha)}(b-a)^{\alpha} B(\gamma, \alpha) < 1.$$

Proof. Introduce the operator $Q: C_{1-\gamma}[J, X] \to C_{1-\gamma}[J, X]$ defined by

$$\begin{aligned}(Qy)(t) = {} & \frac{w}{u+v} \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} - \frac{v}{u+v} \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} \frac{1}{\Gamma(1-\gamma+\alpha)} \\ & \times \int_a^b (b-s)^{\alpha-\gamma} f(s, y(s), (Sy)(s))ds \\ & + \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, y(s), (Sy)(s))ds.\end{aligned} \tag{3.1}$$

Notice that the solutions of problem (1.1) are the fixed points of the operator Q. Define a bounded closed convex set $B_r := \{y \in C_{1-\gamma}[J, X] : \|y\|^{C}{}_{1-\gamma} \le r, t \in J\}$ with $r \ge \frac{\omega}{1-\varrho}(\varrho < 1)$ and

$$\omega := \frac{1}{\Gamma(\gamma)} \frac{|w|}{|u+v|} + \frac{N(b-a)^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{N(b-a)^{\alpha-\gamma+1}}{\Gamma(2-\gamma+\alpha)}.$$

In order to satisfy the hypotheses of the Mönch fixed point theorem, we split the proof into four steps.

Step 1. The operator Q maps the set B_r into itself.

By the assumption (H2), we have

$$
\begin{aligned}
&\|(Qy)(t)(t-a)^{1-\gamma}\| \\
&= \left\| \frac{1}{\Gamma(\gamma)} \frac{w}{u+v} - \frac{1}{\Gamma(\gamma)} \frac{v}{u+v} \frac{1}{\Gamma(1-\gamma+\alpha)} \int_a^b (b-s)^{\alpha-\gamma} f(s, y(s), (Sy)(s)) ds \right. \\
&\quad \left. + \frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s, y(s), (Sy)(s)) ds \right\| \leq \frac{1}{\Gamma(\gamma)} \frac{|w|}{|u+v|} + \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{1}{\Gamma(1-\gamma+\alpha)} \\
&\quad \times \int_a^b (b-s)^{\alpha-\gamma} \|f(s, y(s), (Sy)(s))\| ds + \frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \|f(s, y(s), (Sy)(s))\| ds \\
&\leq \frac{1}{\Gamma(\gamma)} \frac{|w|}{|u+v|} + \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{1}{\Gamma(1-\gamma+\alpha)} \int_a^b (b-s)^{\alpha-\gamma} N(1+\zeta\|y(s)\|) ds \\
&\quad + \frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} N(1+\zeta\|y(s)\|) ds \leq \frac{1}{\Gamma(\gamma)} \frac{|w|}{|u+v|} \\
&\quad + \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{N}{\Gamma(1-\gamma+\alpha)} \int_a^b (b-s)^{\alpha-\gamma} ds + \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{N}{\Gamma(1-\gamma+\alpha)} \int_a^b (b-s)^{\alpha-\gamma} \zeta \|y(s)\| ds \\
&\quad + \frac{N(t-a)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} ds + \frac{N(t-a)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} \zeta \|y(s)\| ds \\
&\leq \frac{1}{\Gamma(\gamma)} \frac{|w|}{|u+v|} + \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{N}{\Gamma(1-\gamma+\alpha)} \frac{(b-a)^{\alpha-\gamma+1}}{(\alpha-\gamma+1)} \\
&\quad + \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{N\zeta}{\Gamma(1-\gamma+\alpha)} \int_a^b (b-s)^{\alpha-\gamma} (s-a)^{\gamma-1} \|y\|_{C_{1-\gamma}} ds + \frac{N(t-a)^{1-\gamma}}{\Gamma(\alpha)} \frac{(t-a)^{\alpha}}{\alpha} \\
&\quad + \frac{N\zeta(t-a)^{1-\gamma}}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} (s-a)^{\gamma-1} \|y\|_{C_{1-\gamma}} ds \leq \frac{1}{\Gamma(\gamma)} \frac{|w|}{|u+v|} + \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{N(b-a)^{\alpha-\gamma+1}}{\Gamma(2-\gamma+\alpha)} \\
&\quad + \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{N\zeta r}{\Gamma(1-\gamma+\alpha)} (b-a)^{\alpha} B(\gamma, \alpha-\gamma+1) + \frac{N(b-a)^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{N\zeta r}{\Gamma(\alpha)} (t-a)^{\alpha} B(\gamma, \alpha) \\
&\leq \frac{1}{\Gamma(\gamma)} \frac{|w|}{|u+v|} + \frac{N(b-a)^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{N(b-a)^{\alpha-\gamma+1}}{\Gamma(2-\gamma+\alpha)} \\
&\quad + \left[\frac{1}{\Gamma(\gamma)} \frac{|v|}{|u+v|} \frac{N\zeta}{\Gamma(1-\gamma+\alpha)} (b-a)^{\alpha} B(\gamma, \alpha-\gamma+1) + \frac{N\zeta}{\Gamma(\alpha)} (b-a)^{\alpha} B(\gamma, \alpha) \right] r,
\end{aligned}
$$

where we used the fact

$$
\begin{aligned}
\int_a^t (t-s)^{\alpha-1} \|y(s)\| ds &\leq \left(\int_a^t (t-s)^{\alpha-1} (s-a)^{\gamma-1} ds \right) \|y\|_{C_{1-\gamma}} \\
&= (t-a)^{\alpha+\gamma-1} B(\gamma, \alpha) \|y\|_{C_{1-\gamma}}
\end{aligned}
$$

In consequence, we get $\|Qy\|_{C_{1-\gamma}} \leq \omega + \varrho r \leq r$, that is, $QB_r \subset B_r$. Thus $Q: B_r \to B_r$.

Step 2. The operator Q is continuous.

Suppose that $\{y_n\}$ is a sequence such that $y_n \to y$ in B_r as $n \to \infty$. Since f satisfies (H1), for each $t \in J$, we get

$$
\begin{aligned}
&\left\|((Qy_n)(t)-(Qy)(t))(t-a)^{1-\gamma}\right\| \\
&\leq \frac{1}{\Gamma(\gamma)}\frac{|v|}{|u+v|}\frac{1}{\Gamma(1-\gamma+\alpha)}\times\int_a^b (b-s)^{\alpha-\gamma}\|f(s,y_n(s),(Sy_n)(s))-f(s,y(s),(Sy)(s))\|ds \\
&+\frac{(t-a)^{1-\gamma}}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\|f(s,y_n(s),(Sy_n)(s))-f(s,y(s),(Sy)(s))\|ds \\
&\leq \frac{1}{\Gamma(\gamma)}\frac{|v|}{|u+v|}\frac{(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)}{\Gamma(1-\gamma+\alpha)} \\
&\times\|f(\cdot,y_n(\cdot),(Sy_n)(\cdot))-f(\cdot,y(\cdot),(Sy)(\cdot))\|_{C_{1-\gamma}} \\
&+\frac{(t-a)^{\alpha}}{\Gamma(\alpha)}B(\gamma,\alpha)\|f(\cdot,y_n(\cdot),(Sy_n)(\cdot))-f(\cdot,y(\cdot),(Sy)(\cdot))\|_{C_{1-\gamma}}.
\end{aligned}
$$

By (H1) and using the Lebesgue dominated convergence theorem, we have

$$\|(Qy_n-Qy)\|_{C_{1-\gamma}}\to 0\, as\, n\to\infty,$$

which implies that the operator Q is continuous on B_r.

Step 3. The operator Q is equicontinuous.

For any $a<t_1<t_2<b$ and $y\in B_r$, we get

$$
\begin{aligned}
&\left\|(t_2-a)^{1-\gamma}(Qy)(t_2)-(t_1-a)^{1-\gamma}(Qy)(t_1)\right\| \\
&\leq\frac{1}{\Gamma(\alpha)}\left\|(t_2-a)^{1-\gamma}\int_a^{t_2}(t_2-s)^{\alpha-1}f(s,y(s),(Sy)(s))ds\right. \\
&\left.-(t_1-a)^{1-\gamma}\int_a^{t_1}(t_1-s)^{\alpha-1}f(s,y(s),(Sy)(s))ds\right\| \\
&\leq\frac{\|f\|_{C_{1-\gamma}}}{\Gamma(\alpha)}\left\|(t_2-a)^{1-\gamma}\int_a^{t_2}(t_2-s)^{\alpha-1}(s-a)^{\gamma-1}ds\right. \\
&\left.-(t_1-a)^{1-\gamma}\int_a^{t_1}(t_1-s)^{\alpha-1}(s-a)^{\gamma-1}ds\right\| \\
&\leq\frac{\|f\|_{C_{1-\gamma}}}{\Gamma(\alpha)}B(\gamma,\alpha)\left\|(t_2-a)^{1-\gamma}(t_2-a)^{\alpha+\gamma-1}-(t_1-a)^{1-\gamma}(t_1-a)^{\alpha+\gamma-1}\right\| \\
&\leq\frac{\|f\|_{C_{1-\gamma}}}{\Gamma(\alpha)}B(\gamma,\alpha)\|(t_2-a)^{\alpha}-(t_1-a)^{\alpha}\|,
\end{aligned}
$$

which tends to zero as $t_2\to t_1$, independent of $y\in B_r$. Thus we conclude that $Q(B_r)$ is equicontinuous, that is, $mod_c(Q(B_r))=0$.

Step 4. The Mönch condition is satisfied.

Suppose that $D \subset B_r$ is a countable set and $D \subseteq conv(\{0\} \cup Q(D))$. In order to show that D is precompact, it is enough to obtain that $\Phi(D) = (0,0)$. Since $\Phi(Q(D))$ is maximum, let $\{x_n\}_{n=1}^{\infty} \subseteq Q(D)$ be a countable set attaining its maximum. Then, there exists a set $\{y_n\}_{n=1}^{\infty} \subseteq D$ such that $x_n = (Qy_n)(t)$ for all $t \in J$, $n \geq 1$.

Now, using (H3) together with Lemmas 2.9–2.11, we obtain

$$
\begin{aligned}
&\Psi\left(\{x_n\}_{n=1}^{\infty}\right) = \Psi\left(\{(Qy_n)(t)\}_{n=1}^{\infty}\right)\\
&\leq \frac{2|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{\Gamma(1-\gamma+\alpha)} \times \int_a^b (b-s)^{\alpha-\gamma}\Psi\left(f(s,\{y_n(s)\}_{n=1}^{\infty},(S\{y_n(s)\}_{n=1}^{\infty}))\right)ds\\
&+\frac{2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Psi\left(f(s,\{y_n(s)\}_{n=1}^{\infty},(S\{y_n(s)\}_{n=1}^{\infty}))\right)ds\\
&\leq \frac{2|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{\Gamma(1-\gamma+\alpha)} \times \int_a^b (b-s)^{\alpha-\gamma}\left(m_1\Psi\left(\{y_n(s)\}_{n=1}^{\infty}\right)+m_2\Psi((S\{y_n(s)\}_{n=1}^{\infty}))\right)ds\\
&+\frac{2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\left(m_1\Psi(\{y_n(s)\}_{n=1}^{\infty})+m_2\Psi\left((S\{y_n(s)\}_{n=1}^{\infty})\right)\right)ds\\
&\leq \frac{2|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{\Gamma(1-\gamma+\alpha)} \times \int_a^b (b-s)^{\alpha-\gamma}\Big(m_1\sup_{t\in[a,b]}\Psi(\{y_n(t)\}_{n=1}^{\infty})\\
&+2m_2\zeta\sup_{t\in[a,b]}\Psi\left(\{y_n(t)\}_{n=1}^{\infty}\right)\Big)ds+\frac{2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}\Big(m_1\sup_{t\in[a,b]}\Psi(\{y_n(t)\}_{n=1}^{\infty})\\
&+2m_2\zeta\sup_{t\in[a,b]}\Psi\left(\{y_n(t)\}_{n=1}^{\infty}\right)\Big)ds \leq \frac{2|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b (b-s)^{\alpha-\gamma}\\
&\times e^{Ls}\left(m_1\sup_{t\in[a,b]}e^{-Lt}\Psi(\{y_n(t)\}_{n=1}^{\infty})+2m_2\zeta\sup_{t\in[a,b]}e^{-Lt}\Psi\left(\{y_n(t)\}_{n=1}^{\infty}\right)\right)ds\\
&+\frac{2}{\Gamma(\alpha)}\int_a^b (t-s)^{\alpha-1}\times e^{Ls}(m_1\sup_{t\in[a,b]}e^{-Lt}\Psi(\{y_n(t)\}_{n=1}^{\infty})+2m_2\zeta\sup_{t\in[a,b]}e^{-Lt}\Psi(\{y_n(t)\}_{n=1}^{\infty}))ds\\
&\leq \frac{2|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{\delta\left(\{y_n\}_{n=1}^{\infty}\right)}{\Gamma(1-\gamma+\alpha)}\int_a^b (b-s)^{\alpha-\gamma}e^{Ls}(m_1+2m_2\zeta)ds\\
&+\frac{2\delta\left(\{y_n\}_{n=1}^{\infty}\right)}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}e^{Ls}(m_1+2m_2\zeta)ds\\
&\leq \Bigg[\frac{2|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b (b-s)^{\alpha-\gamma}e^{Ls}(m_1+2m_2\zeta)ds\\
&+\frac{2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}e^{Ls}(m_1+2m_2\zeta)ds\Bigg]\delta\left(\{y_n\}_{n=1}^{\infty}\right).
\end{aligned}
$$

Hence

$$\delta(\{x_n\}_{n=1}^{\infty})$$

$$\leq \sup_{t\in[a,b]} e^{-Lt}\left[\frac{2|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{\Gamma(1-\gamma+\alpha)}\int_a^b (b-s)^{\alpha-\gamma}e^{Ls}(m_1+2m_2\zeta)ds\right.$$

$$\left.+\frac{2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}e^{Ls}(m_1+2m_2\zeta)ds\right]\delta(\{y_n\}_{n=1}^{\infty}).$$

Fixing a suitable constant $0<L^{'}<1$ given by

$$L^{'} = \sup_{t\in[a,b]} e^{-Lt}\left[\frac{2|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{\Gamma(1-\gamma+\alpha)}\right.$$

$$\times\int_a^b (b-s)^{\alpha-\gamma}e^{Ls}(m_1+2m_2\zeta)ds$$

$$\left.+\frac{2}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}e^{Ls}(m_1+2m_2\zeta)ds\right].,$$

we get $\delta(\{x_n\}_{n=1}^{\infty})\leq L^{'}\delta(\{y_n\}_{n=1}^{\infty})$. Thus

$$\delta(\{y_n\}_{n=1}^{\infty})\leq\delta(D)\leq\delta(conv(\{0\}\cup Q(D)))=\delta(\{x_n\}_{n=1}^{\infty})\leq L^{'}\delta(\{y_n\}_{n=1}^{\infty}),$$

which implies that $\delta(\{y_n\}_{n=1}^{\infty})=0$ and hence $\delta(\{x_n\}_{n=1}^{\infty})=0$.

Now, according to the Step 3, we have found an equicontinuous set $\{x_n\}_{n=1}^{\infty}$ on J. Hence $\Phi(D)\leq\Phi(conv(\{0\}\cup Q(D)))\leq\Phi(Q(D))$, where $\Phi(Q(D))=\Phi(\{x_n\}_{n=1}^{\infty})=(0,0)$. Therefore, D is precompact. Hence, by Lemma 2.12, there is a fixed point y of operator Q, which is a solution of the problem (1.1) in $C_{1-\gamma}[J,X]$.

Next, we show that such a solution is indeed in $C_{1-\gamma}^{\gamma}[J,X]$. By applying $D_{a^+}^{\gamma}$ on both sides of (2.1), we get

$$D_{a^+}^{\gamma}y(t)=D_{a^+}^{\beta(1-\alpha)}f(t,y(t),(Sy)(t)).$$

Since $f(t,y(t),(Sy)(t))\in C_{1-\gamma}^{\beta(1-\alpha)}[J,X]$, it follows by definition of the space $C_{1-\gamma}^{\beta(1-\alpha)}[J,X]$ that $D_{a^+}^{\gamma}y(t)\in C_{1-\gamma}[J,X]$, which implies that $y\in C_{1-\gamma}^{\gamma}[J,X]$. $\square$

4. $\epsilon-$Approximate solution

Definition 4.1. A function $z\in C_{1-\gamma}^{\gamma}[J,X]$ satisfying the Hilfer fractional integrodifferential inequality

$$\left\|D_{a^+}^{\alpha,\beta}z(t)-f(t,z(t),(Sz)(t))\right\|\leq\epsilon,\quad t\in J,$$

and

$$I_{a^+}^{1-\gamma}[uz(a^+)+vz(b^-)]=\overline{w},$$

is called an $\epsilon-$approximate solutions of Hilfer fractional integrodifferential equation (1.1).

Lemma 4.1 (*See [22]*). *For $\beta > 0$, let $v(t)$ be a nonnegative function locally integrable on $0 < t < T$ (some $T \leq +\infty$) and $g(t)$ be a nonnegative, nondecreasing continuous function defined on $0 < t < T$ with $g(t) \leq M$ (constant) and $u(t)$ be a nonnegative and locally integrable function on $0 < t < T$ such that*

$$u(t) \leq v(t) + g(t) \int_0^t (t-s)^{\beta-1} u(s) ds, 0 < t < T.$$

Then

$$u(t) \leq v(t) + \int_0^t \left[\sum_{n=1}^{\infty} \frac{(g(t)\Gamma(\beta))^n}{\Gamma(n\beta)} (t-s)^{n\beta-1} v(s) \right] ds, 0 < t < T.$$

Theorem 4.1. *Suppose that the function $f : J \times X \times X \to X$ satisfies the condition:*

$$\|f(t, y_1, x_1) - f(t, y_2, x_2)\| \leq n_1 \|y_1 - y_2\| + n_2 \|x_1 - x_2\|,$$

for each $t \in J$ and all $y_1, y_2, x_1, x_2 \in X$, where $n_1, n_2 > 0$ are constants. Let $z_i \in C^{\gamma}_{1-\gamma}[J, X]$, $i = 1, 2$, be an $\epsilon-$approximate solution of the following Hilfer fractional integrodifferential equation

$$\begin{cases} D^{\alpha,\beta}_{a^+} z_i(t) = f(t, z_i(t), (Sz_i)(t)), \; t \in J, 0 < \alpha < 1, 0 \leq \beta \leq 1, \\ I^{1-\gamma}_{a^+}[u z_i(a^+) + v z_i(b^-)] = \overline{w_i}, \alpha \leq \gamma = \alpha + \beta - \alpha\beta, i = 1, 2. \end{cases} \tag{4.1}$$

Then

$$\begin{aligned} \|z_1 - z_2\|_{C_{1-\gamma}} &\leq Z^{-1} \\ &\times \left[(\epsilon_1 + \epsilon_2) \left(\frac{(b-a)^{\alpha-\gamma+1}}{\Gamma(\alpha+1)} + \sum_{n=1}^{\infty} (n_1 + \zeta n_2)^n \frac{1}{\Gamma((n+1)\alpha+1)} (b-a)^{(n+1)\alpha-\gamma+1} \right) \right. \\ &\left. + \frac{|\overline{w_1} - \overline{w_2}|}{|u+v|} \left(\frac{1}{\Gamma(\gamma)} + \sum_{n=1}^{\infty} (n_1 + \zeta n_2)^n \frac{1}{\Gamma(n\alpha+\gamma)} (b-a)^{n\alpha} \right) \right], \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} Z = \Bigg(1 - \frac{|v|}{|u+v|} & \frac{(n_1 + \zeta n_2)}{\Gamma(\alpha-\gamma+1)} (b-a)^{\alpha} B(\gamma, \alpha-\gamma+1) \\ & \times \left\{ \frac{1}{\Gamma(\gamma)} + \sum_{n=1}^{\infty} (n_1 + \zeta n_2)^n \frac{1}{\Gamma(n\alpha+\gamma)} (b-a)^{n\alpha} \right\} \Bigg) \neq 0. \end{aligned} \tag{4.3}$$

Proof. Let $z_i \in C^{\gamma}_{1-\gamma}[J, X]$, $(i = 1, 2)$ be an $\epsilon-$approximate solution of problem (4.1). Then $I^{1-\gamma}_{a^+}[u z_i(a^+) + v z_i(b^-)] = \overline{w_i}$ and

$$\left\| D^{\alpha,\beta}_{a^+} z_i(t) - f(t, z_i(t), (Sz_i)(t)) \right\| \leq \epsilon_i, i = 1, 2, \quad t \in J. \tag{4.4}$$

Applying $I^{\alpha}_{a^+}$ on both sides of the above inequality and using Lemma 2.3, we get

$$I^{\alpha}_{a^+}\epsilon_i \geq I^{\alpha}_{a^+}\left\|D^{\alpha,\beta}_{a^+}z_i(t)-f(t,z_i(t),(Sz_i)(t))\right\| \geq \left\|z_i(t)-\frac{\overline{w_i}}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\right.$$
$$\left.+\frac{v}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}I^{\alpha-\gamma+1}_{a^+}f(b,z_i(b),(Sz_i)(b))-I^{\alpha}_{a^+}f(t,z_i(t),(Sz_i)(t))\right\|,$$

which implies that

$$\frac{\epsilon_i}{\Gamma(\alpha+1)}(t-a)^{\alpha} \geq \left\|z_i(t)-\frac{\overline{w_i}}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}+\frac{v}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}I^{\alpha-\gamma+1}_{a^+}f(b,z_i(b),(Sz_i)(b))\right.$$
$$\left.-I^{\alpha}_{a^+}f(t,z_i(t),(Sz_i)(t))\right\|,\ i=1,2.$$

Using $|x|-|y|\leq|x-y|\leq|x|+|y|$ in the above inequality yields

$$\frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}(t-a)^{\alpha}$$
$$\geq \left\|z_1(t)-\frac{\overline{w_1}}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}+\frac{v}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}I^{\alpha-\gamma+1}_{a^+}f(b,z_1(b),(Sz_1)(b))-I^{\alpha}_{a^+}f(t,z_1(t),(Sz_1)(t))\right\|$$
$$+\left\|z_2(t)-\frac{\overline{w_2}}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}+\frac{v}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}I^{\alpha-\gamma+1}_{a^+}f(b,z_2(b),(Sz_2)(b))-I^{\alpha}_{a^+}f(t,z_2(t),(Sz_2)(t))\right\|$$
$$\geq \left\|\left[z_1(t)-\frac{\overline{w_1}}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}+\frac{v}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}I^{\alpha-\gamma+1}_{a^+}f(b,z_1(b),(Sz_1)(b))\right.\right.$$
$$\left.-I^{\alpha}_{a^+}f(t,z_1(t),(Sz_1)(t))\right]-\left[z_2(t)-\frac{\overline{w_2}}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\right.$$
$$\left.\left.+\frac{v}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}I^{\alpha-\gamma+1}_{a^+}f(b,z_2(b),(Sz_2)(b))-I^{\alpha}_{a^+}f(t,z_2(t),(Sz_2)(t))\right]\right\|$$
$$\geq \left\|(z_1(t)-z_2(t))-\frac{(\overline{w_1}-\overline{w_2})}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\right.$$
$$+\frac{v}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}I^{\alpha-\gamma+1}_{a^+}[f(b,z_1(b),(Sz_1)(b))-f(b,z_2(b),(Sz_2)(b))]$$
$$\left.-I^{\alpha}_{a^+}[f(t,z_1(t),(Sz_1)(t))-f(t,z_2(t),(Sz_2)(t))]\right\|$$
$$\geq \|(z_1(t)-z_2(t))\|-\left|\frac{(\overline{w_1}-\overline{w_2})}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\right|$$
$$+\left\|\frac{v}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}I^{\alpha-\gamma+1}_{a^+}[f(b,z_1(b),(Sz_1)(b))-f(b,z_2(b),(Sz_2)(b))]\right\|$$
$$-\left\|I^{\alpha}_{a^+}[f(t,z_1(t),(Sz_1)(t))-f(t,z_2(t),(Sz_2)(t))]\right\|.$$

In consequence, we have

$$\|(z_1(t)-z_2(t))\|$$

$$\leq\frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}(t-a)^{\alpha}+\left|\frac{(\overline{w_1}-\overline{w_2})}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\right|$$

$$-\left\|\frac{|u|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}I_{a^+}^{\alpha-\gamma+1}[f(b,z_1(b),(Sz_1)(b))-f(b,z_2(b),(Sz_2)(b))]\right\|$$

$$+\|I_{a^+}^{\alpha}[f(t,z_1(t),(Sz_1)(t))-f(t,z_2(t),(Sz_2)(t))]\|$$

$$\leq\frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}(t-a)^{\alpha}+\frac{|\overline{w_1}-\overline{w_2}|}{u+v}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}$$

$$+\frac{|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\left\|I_{a^+}^{\alpha-\gamma+1}[f(b,z_1(b),(Sz_1)(b))-f(b,z_2(b),(Sz_2)(b))]\right\|$$

$$+\|I_{a^+}^{\alpha}[f(t,z_1(t),(Sz_1)(t))-f(t,z_2(t),(Sz_2)(t))]\|$$

$$\leq\frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}(t-a)^{\alpha}+\frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}$$

$$+\frac{|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}\int_a^b(b-s)^{\alpha-\gamma}\|z_1(s)-z_2(s)\|ds$$

$$+\frac{(n_1+\zeta n_2)}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-\gamma}\|z_1(s)-z_2(s)\|ds$$

$$\leq\frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}(t-a)^{\alpha}+\frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}$$

$$+\frac{|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)\|z_1-z_2\|_{C_{1-\gamma}}$$

$$+\frac{(n_1+\zeta n_2)}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}\|z_1(s)-z_2(s)\|ds.$$

Using Lemma 4.1 with $u(t)=\|(z_1(t)-z_2(t))\|$, $g(t)=\frac{(n_1+\zeta n_2)}{\Gamma(\alpha)}$ and $v(t)=\frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}(t-a)^{\alpha}+\frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}+\frac{|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)\|z_1-z_2\|_{C_{1-\gamma}}$, we get

$$
\begin{aligned}
\|(z_1(t)-z_2(t))\| \leq & \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}(t-a)^{\alpha} + \frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} \\
& + \frac{|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)\|z_1-z_2\|_{C_{1-\gamma}} \\
& + \int_a^t \sum_{n=1}^{\infty}\frac{(n_1+\zeta n_2)^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}\left(\frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}(s-a)^{\alpha} + \frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\frac{(s-a)^{\gamma-1}}{\Gamma(\gamma)}\right. \\
& \left. + \frac{|v|}{|u+v|}\frac{(s-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)\|z_1-z_2\|_{C_{1-\gamma}}\right)ds \\
\leq & \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}(t-a)^{\alpha} + \frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} + \frac{|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} \\
& \frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)\|z_1-z_2\|_{C_{1-\gamma}} + \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n I_{a^+}^{n\alpha}(t-a)^{\alpha} \\
& + \frac{|\overline{w_1}-\overline{w_2}|}{\Gamma(\gamma)|u+v|}\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n I_{a^+}^{n\alpha}(t-a)^{\gamma-1} \\
& + \frac{|v|\|z_1-z_2\|_{C_{1-\gamma}}}{\Gamma(\gamma)|u+v|}\frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1) \\
& \times \sum_{n=1}^{\infty}(n_1+\zeta n_2)^n I_{a^+}^{n\alpha}(t-a)^{\gamma-1} \leq \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}(t-a)^{\alpha} + \frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} \\
& + \frac{|v|}{|u+v|}\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)\|z_1-z_2\|_{C_{1-\gamma}} \\
& + \frac{(\epsilon_1+\epsilon_2)}{\Gamma(\alpha+1)}\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{\Gamma(\alpha+1)}{\Gamma((n+1)\alpha+1)}(t-a)^{(n+1)\alpha} \\
& + \frac{|\overline{w_1}-\overline{w_2}|}{\Gamma(\gamma)|u+v|}\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{\Gamma(\gamma)}{\Gamma(n\alpha+\gamma)}(t-a)^{n\alpha+\gamma-1} + \frac{|v|\|z_1-z_2\|_{C_{1-\gamma}}}{\Gamma(\gamma)|u+v|} \\
& \frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)\times\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{\Gamma(\gamma)}{\Gamma(n\alpha+\gamma)}(t-a)^{n\alpha+\gamma-1} \\
= & (\epsilon_1+\epsilon_2)\left(\frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} + \sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma((n+1)\alpha+1)}(t-a)^{(n+1)\alpha}\right) \\
& + \frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\left(\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} + \sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma(n\alpha+\gamma)}(t-a)^{n\alpha+\gamma-1}\right) \\
& + \frac{|v|}{|u+v|}\frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)\|z_1-z_2\|_{C_{1-\gamma}} \\
& \times\left(\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} + \sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma(n\alpha+\gamma)}(t-a)^{n\alpha+\gamma-1}\right).
\end{aligned}
$$

Hence, for each $t \in J$, we have

$$
\begin{aligned}
&(t-a)^{1-\gamma}\|(z_1(t)-z_2(t))\| \\
&\leq (\epsilon_1+\epsilon_2)\left(\frac{(t-a)^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma((n+1)\alpha+1)}(t-a)^{(n+1)\alpha-\gamma+1}\right) \\
&+\frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\left(\frac{1}{\Gamma(\gamma)}+\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma(n\alpha+\gamma)}(t-a)^{n\alpha}\right) \\
&+\frac{|v|}{|u+v|}\frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)\|z_1-z_2\|_{C_{1-\gamma}} \\
&\times\left(\frac{1}{\Gamma(\gamma)}+\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma(n\alpha+\gamma)}(t-a)^{n\alpha}\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\|z_1-z_2\|_{C_{1-\gamma}} \\
&\leq (\epsilon_1+\epsilon_2)\left(\frac{(b-a)^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma((n+1)\alpha+1)}(b-a)^{(n+1)\alpha-\gamma+1}\right) \\
&+\frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\left(\frac{1}{\Gamma(\gamma)}+\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma(n\alpha+\gamma)}(b-a)^{n\alpha}\right) \\
&+\frac{|v|}{|u+v|}\frac{(n_1+\zeta n_2)}{\Gamma(\alpha-\gamma+1)}(b-a)^{\alpha}B(\gamma,\alpha-\gamma+1)\|z_1-z_2\|_{C_{1-\gamma}} \\
&\times\left(\frac{1}{\Gamma(\gamma)}+\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma(n\alpha+\gamma)}(b-a)^{n\alpha}\right),
\end{aligned}
$$

which, together with (4.3), yields

$$
\begin{aligned}
&\|z_1-z_2\|_{C_{1-\gamma}} \leq Z^{-1}[(\epsilon_1+\epsilon_2) \\
&\times\left(\frac{(b-a)^{\alpha-\gamma+1}}{\Gamma(\alpha+1)}+\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma((n+1)\alpha+1)}(b-a)^{(n+1)\alpha-\gamma+1}\right) \\
&+\frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\left(\frac{1}{\Gamma(\gamma)}+\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma(n\alpha+\gamma)}(b-a)^{n\alpha}\right)\Bigg]. \qquad \square
\end{aligned}
\tag{4.5}
$$

Remark 4.1. If $\epsilon_1=\epsilon_2=0$ in the inequality (4.4), then z_1, z_2 are solutions of the problem (1.1) in the space $C^{\gamma}_{1-\gamma}[J,X]$ and the inequality (4.5) takes the form

$$
\|z_1-z_2\|_{C_{1-\gamma}} \leq Z^{-1}\frac{|\overline{w_1}-\overline{w_2}|}{|u+v|}\left(\frac{1}{\Gamma(\gamma)}+\sum_{n=1}^{\infty}(n_1+\zeta n_2)^n\frac{1}{\Gamma(n\alpha+\gamma)}(b-a)^{n\alpha}\right),
$$

which provides the information with respect to continuous dependence on the solution of the problem (1.1). In addition, if $\overline{w_1} = \overline{w_2}$ we get $\|z_1 - z_2\|_{C_{1-\gamma}} = 0$, which proves the uniqueness of solutions of the system (*1.1*).

Remark 4.2. One can note that our results for the Hilfer fractional integrodifferential equation (1.1) correspond to initial boundary value problem for $u = 1, v = 0$, terminal boundary value problem for $u = 0, v = 1$ and anti-periodic problem for $u = 1, v = 1, w = 0$.

Remark 4.3. If $\beta = 1$, then Eq. (1.1) reduces to the Caputo fractional integrodifferential equation with boundary conditions as in [12].

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