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The implicit midpoint rule for nonexpansive mappings in 2-uniformly convex hyperbolic spaces

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Abstract

The purpose of this paper is to introduce the implicit midpoint rule (IMR) of nonexpansive mappings in 2- uniformly convex hyperbolic spaces and study its convergence. Strong and $\triangle$-convergence theorems based on this algorithm are proved in this new setting. The results obtained hold concurrently in uniformly convex Banach spaces, CAT(0) spaces and Hilbert spaces as special cases.

Keywords Uniformly convex hyperbolic space, Nonexpansive mapping, Midpoint rule, Fixed point, Condition(A), Convergence

Paper type Original Article

1. Introduction

The iterative methods for approximating fixed points of nonexpansive mappings have received a great attention due to the fact that in many practical problems, the controlling operators are nonexpansive (cf. [16]). The iterative methods of Mann [17] and Halpern [9] are very popular (see also [20]). An implicit iterative method was proposed [25] and studied in [7,12]. The IMR is a powerful numerical method for solving ordinary differential equations and differential algebraic equations. For related works in this context, we refer the reader to [2,5,20,22].

For the ordinary differential equation

$$y'(t) = g(t), \quad y_0 = y(0), \tag{1.1}$$

JEL Classification — 47H09, 47H10

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IMR generates a sequence $\{y_n\}$ via the relation

$$\frac{1}{h}(y_{n+1} - y_n) = g\left(\frac{y_{n+1} + y_n}{2}\right)$$

where $h > 0$ is a step size. It is well known that if g: $\mathbb{R}^k \to \mathbb{R}^k$ is Lipschitzian continuous and sufficiently smooth, then the sequence $\{y_n\}$ converges to the exact solution of (1.1) as $h \to 0$ uniformly over $t \in [0, a]$ for any fixed $a > 0$.

Based on the above fact, Alghamdi et al. [1] presented the following IMR for nonexpansive mappings in the setting of a Hilbert space H:

$$y_{n+1} = (1 - t_n)y_n + t_n T\left(\frac{y_{n+1} + y_n}{2}\right) \tag{1.2}$$

where $t_n \in (0, 1)$ and $T : H \to H$ is a nonexpansive mapping and established weak convergence of (1.2) to the fixed point of T under some control conditions on $\{t_n\}$.

The extension of a linear version of a known result (usually in Banach spaces or Hilbert spaces) to metric spaces is very important. As an IMR for nonexpansive mappings involves general convex combinations, so we need some convex structure in a metric space to define an IMR on a nonlinear domain.

Let C be a nonempty subset of a metric space (M, d) and $T : C \to C$ a mapping. Set $F\ (T) = \{x \in M : Tx = x\}$. The mapping T is: (i) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in C$ (ii) quasi-nonexpansive if $d(Tx, y) \leq d(x, y)$ for all $x \in C$ and $y \in F(T)$ (iii) semi-compact if for any bounded sequence $\{x_n\}$ in C satisfying $d(x_n, Tx_n) \to 0$, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x \in C$ (iv) completely continuous if every bounded sequence $\{x_n\}$ in C implies that $\{Tx_n\}$ has a convergent subsequence. A sequence $\{x_n\}$ is Fejér monotone with respect to a subset C of M if $d(x_{n+1}, x) \leq d(x_n, x)$ for all $x \in C$.

For a bounded sequence $\{x_n\}$ in a metric space M, set

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n)$$

for all $x \in M$.

The asymptotic radius of $\{x_n\}$ with respect to $C \subseteq M$ is defined as

$$r(\{x_n\}) = \inf_{x \in C} r\ (x, \{x_n\}).$$

A point $y \in C$ is called the asymptotic centre of $\{x_n\}$ with respect to $C \subseteq M$ if

$$r(y, \{x_n\}) \leq r(x, \{x_n\}) \text{ for all } x \in C.$$

The set of all asymptotic centres of $\{x_n\}$ is denoted by $A(\{x_n\})$.

A sequence $\{x_n\}$ in M, is $\triangle$-convergent to $x \in M$ ($\triangle - \lim_n x_n = x$) if x is the unique asymptotic centre of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. It has been observed that $\triangle$-convergence in metric spaces constitutes an analogue of weak convergence in Hilbert spaces and both coincide in Hilbert spaces.

Let (M, d) be a metric space. Suppose that there exists a family F of metric segments such that any two points x, y in M are endpoints of a unique metric segment $[x, y] \in F$ ($[x, y]$ is an isometric image of the real line interval $[0, d(x, y)]$). We denote by z the unique point $\alpha x \oplus (1 - \alpha)y$ of $[x, y]$ which satisfies

$$d(x, z) = (1 - \alpha)d(x, y) \text{ and } d(z, y) = \alpha d(x, y) \text{ for } \alpha \in I = [0, 1].$$

Such metric spaces are usually called convex metric spaces [18]. A convex metric space M is hyperbolic if

$$d(\alpha x \oplus (1-\alpha)y, \alpha z \oplus (1-\alpha)w) \leq \alpha d(x,z) + (1-\alpha)\, d(y,w) \tag{1.3}$$

for all $x,y,z,w \in M$ and $\alpha \in I$.

For $z = w$, the hyperbolic inequality reduces to convex structure of Takahashi [23]

$$d(\alpha x \oplus (1-\alpha)y, z) \leq \alpha d(x,z) + (1-\alpha) d(y,z).$$

A nonempty subset C of a hyperbolic space M is convex if $\alpha x \oplus (1-\alpha)y \in C$ for all $x,y \in C$ and $\alpha \in I$. A few examples of nonlinear hyperbolic spaces are Hadamard manifolds [4], the Hilbert open unit ball equipped with the hyperbolic metric [8] and the $CAT(0)$ spaces [14,15] while normed spaces and their subsets are linear hyperbolic spaces. Throughout this paper, we denote $\frac{1}{2}x \oplus \frac{1}{2}y$ by $\frac{x \oplus y}{2}$.

A hyperbolic space M is uniformly convex if

$$\delta(r,\varepsilon) = \inf\left\{1 - \frac{1}{r} d\left(a, \frac{x \oplus y}{2}\right) : d(a,x) \leq r, d(a,y) \leq r, d(x,y) \geq r\varepsilon\right\} > 0,$$

for any $a \in M$, $r > 0$ and $\varepsilon > 0$.

Xu [24], extensively used the concept of p-uniform convexity; its nonlinear version in hyperbolic spaces for $p = 2$ has been introduced by Khamsi and Khan [13] as under:

For a fixed $a \in M, r > 0, \varepsilon > 0$, define

$$\psi(r,\varepsilon) = \inf\left\{\frac{1}{2} d(a,x)^2 + \frac{1}{2} d(a,y)^2 - d\left(a, \frac{x \oplus y}{2}\right)^2\right\}$$

where the infimum is taken over all $x,y \in M$ such that $d(a,x) \leq r$, $d(a,y) \leq r$ and $d(x,y) \geq r\varepsilon$.

We say that M is 2-uniformly convex if

$$c_M = \inf\left\{\frac{\psi(r,\varepsilon)}{r^2 \varepsilon^2} : r > 0, \varepsilon > 0\right\} > 0.$$

It has been shown in [13] that any $CAT(0)$ space is 2-uniformly convex hyperbolic space with $c_M = \frac{1}{4}$.

From now onwards we assume that M is a uniformly convex hyperbolic space with the property that for every $s \geq 0, \varepsilon > 0$, there exists $\eta(s,\varepsilon) > 0$ depending on s and ε such that $\delta(r,\varepsilon) > \eta(s,\varepsilon) > 0$ for any $r > s$.

Using the concept of metric segment $[x,y]$, we translate (1.2) for nonexpansive mappings in a hyperbolic space as follows:

$$\begin{aligned} x_0 &= x \in C, \\ x_{n+1} &= \alpha_n T\left(\frac{x_n \oplus x_{n+1}}{2}\right) \oplus (1-\alpha_n) x_n, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$ is the sequence in $(0,1)$ satisfying (C1): $\liminf_{n\to\infty} \alpha_n > 0$ and (C2): $\alpha_{n+1}^2 \leq \lambda \alpha_n^2$ for some $\lambda > 0$.

The following known results are needed in the sequel.

Lemma 1.1 ([3]). *Let C be a nonempty closed subset of a complete metric space (M,d) and $\{x_n\}$ be a Fejér monotone with respect to C. Then $\{x_n\}$ strongly converges to $x \in C$ if and only if* $\lim_{n\to\infty} d(x_n, C) = 0$.

Lemma 1.2 ([6]). *Let C be a nonempty closed and convex subset of a complete uniformly convex hyperbolic space M. Then every bounded sequence $\{y_n\}$ in M has a unique asymptotic centre with respect to C that lies in C.*

Lemma 1.3 ([10]). *Suppose that M is a 2-uniformly convex hyperbolic space. Then for any $\theta \in (0,1)$, we have that*

$$d(u, \theta x \oplus (1-\theta)y)^2 \leq \theta d(u,x)^2 + (1-\theta)d(u,y)^2 - 4c_M \min\{\theta^2, (1-\theta)^2\}d(x,y)^2,$$

for all $u, x, y \in M$ and c_M is the number as given above.

Our purpose in this paper is to approximate fixed point of nonexpansive mappings using iterative method (1.4) in a 2-uniformly convex hyperbolic spaces. This work provides a unified approach to convergence results in Hilbert spaces, uniformly convex Banach spaces and CAT(0) spaces.

2. Convergence in 2-uniformly convex hyperbolic spaces

Lemma 2.1. *Let C be a nonempty convex subset of a complete hyperbolic space M and $T: C \to C$ a nonexpansive mapping. Then the sequence $\{x_n\}$ in (1.4) is well defined.*

Proof. Define $S: C \to C$ by

$$Sx = \alpha_0 T\left(\frac{x_0 \oplus x}{2}\right) \oplus (1-\alpha_0)x_0.$$

With the help of (1.3), we have

$$\begin{aligned} d(Sx, Sy) &= d\left(\alpha_0 T\left(\frac{x_0 \oplus x}{2}\right) \oplus (1-\alpha_0)x_0, \alpha_0 T\left(\frac{x_0 \oplus y}{2}\right) \oplus (1-\alpha_0)x_0\right) \\ &\leq \alpha_0 d\left(T\left(\frac{x_0 \oplus x}{2}\right), T\left(\frac{x_0 \oplus y}{2}\right)\right) \\ &\leq \alpha_0 d\left(\frac{x_0 \oplus x}{2}, \frac{x_0 \oplus y}{2}\right) \\ &\leq \frac{\alpha_0}{2} d(x,y). \end{aligned}$$

This gives that S is a contraction with contraction constant $\frac{\alpha_0}{2} \in (0,1)$. Therefore by Banach contraction principle, there is a unique element $x_1 \in C$ such that $x_1 = Sx_1 = \alpha_0 T\left(\frac{x_0 \oplus x_1}{2}\right) \oplus (1-\alpha_0)x_0$. Hence x_1 is achieved. Similarly, we can find x_2 and so on. So in general,

$$x_{n+1} = \alpha_n T\left(\frac{x_n \oplus x_{n+1}}{2}\right) \oplus (1-\alpha_n)x_n. \ \square$$

Lemma 2.2. *Let C be a nonempty convex subset of a complete 2-uniformly convex hyperbolic space M and $T: C \to C$ a nonexpansive mapping such that $F(T) \neq \phi$. Then for the sequence $\{x_n\}$ in (1.4), we have the following: (i)* $\lim_{n\to\infty} d(x_n, p)$ *exists for all $p \in F(T)$*

(ii) $\sum_{n=1}^{\infty} \alpha_n d(x_n, x_{n+1}) < \infty$

(iii) $\sum_{n=1}^{\infty} \alpha_n^2 (1-\alpha_n)^2 d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right)^2 < \infty.$

Proof. Let $p \in F(T)$. Applying Lemma 1.3 to (1.4), we have that

$$
\begin{aligned}
d(x_{n+1},p)^2 &= d\left(\alpha_n T\left(\frac{x_n \oplus x_{n+1}}{2}\right) \oplus (1-\alpha_n)x_n, p\right)^2\\
&\leq \alpha_n d\left(T\left(\frac{x_n \oplus x_{n+1}}{2}\right), p\right)^2 + (1-\alpha_n)d(x_n,p)^2\\
&\quad - 4c_M \min\{\alpha_n^2, (1-\alpha_n)^2\} d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right)^2\\
&\leq \alpha_n d\left(\frac{x_n \oplus x_{n+1}}{2}, p\right)^2 + (1-\alpha_n)d(x_n,p)^2\\
&\quad - 4c_M \alpha_n^2(1-\alpha_n)^2 d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right)^2\\
&\leq \alpha_n d\left(\frac{1}{2}d(x_n,p)^2 + \frac{1}{2}d(x_{n+1},p)^2 - \frac{C_M}{4}d(x_n,x_{n+1})^2\right)\\
&\quad + (1-\alpha_n)d(x_n,p)^2 - 4c_M \alpha_n^2(1-\alpha_n)^2 d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right)^2.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\left(1-\frac{\alpha_n}{2}\right)d(x_{n+1},p) \leq \left(1-\frac{\alpha_n}{2}\right)d(x_n,p) - \frac{\alpha_n C_M}{4}d(x_n,x_{n+1})^2\\
- 4c_M \alpha_n^2(1-\alpha_n)^2 d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right)^2
\end{aligned}
$$

which further implies that

$$
\begin{aligned}
d(x_{n+1},p) \leq d(x_n,p) - \frac{\alpha_n C_M}{2(2-\alpha_n)}d(x_n,x_{n+1}).\\
- \frac{8c_M \alpha_n^2(1-\alpha_n)^2}{2(2-\alpha_n)} d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right)^2.
\end{aligned}
$$

The above inequality provides the following three inequalities:

$$
d(x_{n+1},p) \leq d(x_n,p), \tag{2.1}
$$

$$
\frac{\alpha_n C_M}{2(2-\alpha_n)} d(x_n,x_{n+1}) \leq d(x_n,p) - d(x_{n+1},p) \tag{2.2}
$$

and

$$
\frac{8c_M \alpha_n^2(1-\alpha_n)^2}{2(2-\alpha_n)} d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right)^2 \leq d(x_n,p) - d(x_{n+1},p). \tag{2.3}
$$

From (2.1), it follows that $\lim_{n\to\infty} d(x_n,p)$ exists, that is, (i) holds.

Since $\alpha_n \in (0,1)$, therefore $\alpha_n \leq \frac{\alpha_n}{2(2-\alpha_n)}$. Hence (2.2) becomes

$$
\alpha_n d(x_n,x_{n+1}) \leq \frac{1}{C_M}[d(x_n,p) - d(x_{n+1},p)]. \tag{2.4}
$$

Let $m \geq 1$ be any positive integer. Then from (2.4), we have that

$$
\sum_{n=1}^{m} \alpha_n d(x_n,x_{n+1}) \ \leq \ \frac{1}{C_M}[d(x_1,p) - d(x_{m+1},p)] \leq \frac{d(x_1,p)}{C_M}.
$$

Let $m \to \infty$. Then

$$\sum_{n=1}^{\infty} \alpha_n d(x_n, x_{n+1}) \le \frac{d(x_1, p)}{C_M} < \infty.$$

That is,

$$\sum_{n=1}^{\infty} \alpha_n d(x_n, x_{n+1}) < \infty,$$

proving (ii). Similarly, from (2.3), we have

$$\sum_{n=1}^{\infty} \alpha_n^2 (1-\alpha_n)^2 d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right)^2 < \infty. \quad \square$$

Lemma 2.3. *Let C be a nonempty convex subset of a complete 2-uniformly convex hyperbolic space M and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \phi$. Then for the sequence $\{x_n\}$ in (1.4), we have that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.*

Proof. Consider

$$\begin{aligned}
d(x_{n+1}, x_{n+2}) &= d\left(\alpha_{n+1} T\left(\frac{x_{n+1} \oplus x_{n+2}}{2}\right) \oplus (1-\alpha_{n+1})x_{n+1}, x_{n+1}\right) \\
&\le \alpha_{n+1} d\left(x_{n+1}, T\left(\frac{x_{n+1} \oplus x_{n+2}}{2}\right)\right) \\
&\le \alpha_{n+1} d\left(x_{n+1}, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right) \\
&\quad + \alpha_{n+1} d\left(T\left(\frac{x_n \oplus x_{n+1}}{2}\right), T\left(\frac{x_{n+1} \oplus x_{n+2}}{2}\right)\right) \\
&\le \alpha_{n+1} d\left(x_{n+1}, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right) \\
&\quad + \alpha_{n+1} d\left(\frac{x_n \oplus x_{n+1}}{2}, \frac{x_{n+1} \oplus x_{n+2}}{2}\right) \\
&\le \alpha_{n+1}(1-\alpha_n) d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right) \\
&\quad + \alpha_{n+1} d\left(\frac{x_n \oplus x_{n+1}}{2}, \frac{x_{n+1} \oplus x_{n+2}}{2}\right) \\
&\le \alpha_{n+1}(1-\alpha_n)\, d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right) \\
&\quad + \frac{\alpha_{n+1}}{2}\left(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\right).
\end{aligned}$$

Therefore

$$\left(1 - \frac{\alpha_{n+1}}{2}\right) d(x_{n+2}, x_{n+1}) \le \alpha_{n+1}(1-\alpha_n) d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right) + \frac{\alpha_{n+1}}{2} d(x_n, x_{n+1})$$

which further implies that

$$
\begin{aligned}
d(x_{n+1}, x_{n+2}) \leq & \frac{2\alpha_{n+1}(1-\alpha_n)}{2-\alpha_{n+1}} d\Big(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\Big) \\
& + \frac{\alpha_{n+1}}{2-\alpha_{n+1}} d(x_n, x_{n+1}) \\
\leq & 2\alpha_{n+1}(1-\alpha_n) d\Big(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\Big) \\
& + \alpha_{n+1} d\,(x_n, x_{n+1}).
\end{aligned}
$$

For some $A > 0, B > 0$ and using the assumption $\alpha_{n+1}^2 \leq \lambda \alpha_n^2$, we further derive that

$$
\begin{aligned}
d(x_{n+1}, x_{n+2})^2 \leq & 4A\alpha_{n+1}^2(1-\alpha_n)^2 d\Big(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\Big)^2 \\
& + B\alpha_{n+1}^2 d(x_n, x_{n+1})^2 \\
\leq & 4A\lambda\alpha_{n+1}^2(1-\alpha_n)^2 d\Big(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\Big)^2 \\
& + B\alpha_{n+1}^2 d(x_n, x_{n+1})^2 \\
\leq & 4A\lambda\alpha_n^2(1-\alpha_n)^2 d\Big(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\Big)^2 \\
& + B\lambda\alpha_n^2 d(x_n, x_{n+1})^2 \\
\leq & 4A\lambda\alpha_n^2(1-\alpha_n)^2 d\Big(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\Big)^2 \\
& + B\lambda\alpha_n d(x_n, x_{n+1})^2.
\end{aligned}
$$

Hence by Lemma 2.2(ii)–(iii), we have that

$$
\sum_{n=1}^{\infty} d(x_{n+1}, x_{n+2})^2 < \infty.
$$

This in turn implies that

$$
\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \; \square \tag{2.5}
$$

Lemma 2.4. *Let C be a nonempty closed and convex subset of a complete 2-uniformly convex hyperbolic space M and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \phi$. Then for the sequence $\{x_n\}$ in (1.4), we have that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.*

Proof. The condition $\liminf_{n\to\infty} \alpha_n > 0$ implies that $0 < \frac{1}{\alpha_n} \leq \frac{1}{\alpha}$ for sufficiently large n. The inequality

$$
\begin{aligned}
d\Big(x_n, T\Big(\frac{x_n \oplus x_{n+1}}{2}\Big)\Big) \leq & d(x_n, x_{n+1}) + d\Big(x_{n+1}, T\Big(\frac{x_n \oplus x_{n+1}}{2}\Big)\Big) \\
\leq & d(x_n, x_{n+1}) + (1-\alpha_n) d\Big(x_n, T\Big(\frac{x_n \oplus x_{n+1}}{2}\Big)\Big)
\end{aligned}
$$

implies that

$$d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right) \le \frac{1}{\alpha_n} d(x_n, x_{n+1}) \le \frac{1}{\alpha} d(x_n, x_{n+1}).$$

By taking $\limsup_{n\to\infty}$ on both sides in the above inequality and then appealing to Lemma 2.3, we get that

$$\lim_{n\to\infty} d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right) = 0. \tag{2.6}$$

Finally, the inequality

$$\begin{aligned} d(x_n, Tx_n) &\le d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right) + d\left(T\left(\frac{x_n \oplus x_{n+1}}{2}\right), Tx_n\right) \\ &\le d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right) + d\left(\frac{x_n \oplus x_{n+1}}{2}, x_n\right) \\ &\le d\left(x_n, T\left(\frac{x_n \oplus x_{n+1}}{2}\right)\right) + \frac{1}{2} d(x_{n+1}, x_n) \end{aligned}$$

together with (2.5) and (2.6) provides that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \ \square \tag{2.7}$$

The following concept is needed to establish strong convergence of (1.4).

Let f be a nondecreasing function on $[0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$. Then the mapping $T : C \to C$ with $F(T) \neq \phi$, satisfies condition (*A*) [21] if

$$d(x, Tx) \ge f(d(x, F(T))) \text{ for } x \in C,$$

where $d(x, F(T)) = \inf\{d(x, y) : y \in F(T)\}$.

Using condition(A) and Lemma 2.4, we obtain the following strong convergence result.

Theorem 2.5. *Let C be a nonempty closed and convex subset of a complete 2-uniformly convex hyperbolic space M and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \phi$. If the mapping $T : C \to C$ satisfies condition(A), then the sequence $\{x_n\}$ in (1.4), strongly converges to a fixed point of T.*

Proof. By Lemma 2.4, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Now condition(A) implies that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Finally, by Lemma 1.1, $\{x_n\}$ strongly converges to a fixed point of T. $\square$

Here are our other strong convergence results.

Theorem 2.6. *Let C be a nonempty closed and convex subset of a complete 2-uniformly convex hyperbolic space M and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \phi$. If T is semi-compact, then the sequence $\{x_n\}$ in (1.4) strongly converges to a fixed point of T.*

Proof. By Lemma 2.4, we have that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$, $\{x_n\}$ is bounded. As $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ and T is semi-compact, so there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to q \in C$ and hence $Tx_{n_i} \to Tq$. Therefore, $\lim_{i\to\infty} d(x_{n_i}, Tx_{n_i}) = 0$ implies that $d(Tq, q) = 0$. That is, $q \in F(T)$. Since $\lim_{n\to\infty} d(x_n, p)$ exists and $x_{n_i} \to q$, $x_n \to q$. $\square$

Theorem 2.7. *Let C be a nonempty closed and convex subset of a complete 2-uniformly convex hyperbolic space M and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \phi$. If T is completely continuous, then the sequence $\{x_n\}$ in (1.4), strongly converges to a fixed point of T.*

Proof. Since $\{x_n\}$ is bounded and T is completely continuous, $\{Tx_n\}$ has a convergent subsequence say $\{Tx_{n_i}\}$. Therefore by (2.7), $\{x_{n_i}\}$ converges. Let $\lim_{i\to\infty} x_{n_i} = v$. By continuity of T and (2.7), we have that $Tv = v$. By Lemma 2.2, $\lim_{n\to\infty} d(x_n, v)$ exists and so $\{x_n\}$ strongly converges to v. □

We now present our $\triangle$-convergence result.

Theorem 2.8. *Let C be a nonempty closed and convex subset of a complete 2-uniformly convex hyperbolic space M and $T : C \to C$ a nonexpansive mapping such that $F(T) \neq \phi$. Then the sequence $\{x_n\}$ in (1.4), $\triangle$-converges to a fixed point of T.*

Proof. It follows from Lemma 2.1 that $\{x_n\}$ is bounded in C. By Lemma 1.2, $\{x_n\}$ has a unique asymptotic centre, that is, $A_C(\{x_n\}) = \{y\}$. Let $\{w_n\}$ be any subsequence of $\{x_n\}$ such that $A_C(\{w_n\}) = \{w\}$. We claim that $w \in F(T)$. By Lemma 2.4, we have that

$$\lim_{n\to\infty} d(w_n, Tw_n) = 0.$$

The nonexpansive mapping T satisfies the following inequality:

$$d(w_n, Tw) \leq d(w_n, Tw_n) + d(w_n, w)$$

which further implies that

$$\limsup_{n\to\infty} d(w_n, Tw) \leq \limsup_{n\to\infty} d(w_n, Tw_n) + \limsup_{n\to\infty} d(w_n, w) = \limsup_{n\to\infty} d(w_n, w).$$

By the uniqueness of asymptotic centre, we have $Tw = w$. Therefore $F(T) \neq \phi$. If $y \neq w$, then by the uniqueness of asymptotic centre and the fact that $\lim_{n\to\infty} d(x_n, x)$ exists for each $x \in F(T)$, we have that

$$\begin{aligned} \limsup_{n\to\infty} d(w_n, w) &< \limsup_{n\to\infty} d(w_n, y) \\ &\leq \limsup_{n\to\infty} d(x_n, y) \\ &< \limsup_{n\to\infty} d(x_n, w) \\ &= \limsup_{n\to\infty} d(w_n, w). \end{aligned}$$

This is a contradiction and therefore $y = w$. This proves that $\{x_n\}$, $\triangle$-converges to $x \in F(T)$. □

Remark 2.9. (1) All the results of this paper instantly hold in Hilbert spaces, uniformly convex Banach spaces satisfying Opial property and CAT(0) spaces; (2) The results of Alghamdi et al. [1] are corollaries of our corresponding results; (3) The interested reader is referred to [11] for another notion of p-uniformly convex metric spaces; (4) The two control conditions: (C1)and (C2) in our algorithm (1.4) are satisfied by the sequence $\alpha_n = 1 - \frac{1}{n+1}$.

3. Application

We know that $L^2[0, 1]$ is a Hilbert space and hence it is a 2-uniformly convex hyperbolic space. Suppose that h: $[0, 1] \to [0, 1]$ and $F : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and F satisfies the Lipschitz continuity condition, i.e.,

$$|F(t, \lambda, x) - F(t, s, y)| \leq |x - y| \text{ for } t, s \in [0, 1] \text{ and } x, y \in \mathbb{R}.$$

Consider a Fredholm integral equation of the form

$$x(t) = h(t) + \int_0^1 F(t, s, x(s))ds \quad \text{for } t \in [0, 1]. \tag{3.1}$$

It has been shown in [19] that the solution of Eq. (3.1) exists in $L^2[0, 1]$. To find an approximate solution of this equation, we define $S : L^2[0, 1] \to L^2[0, 1]$ by

$$Sx(t) = h(t) + \int_0^1 F(t, s, x(s))ds \quad \text{for } t \in [0, 1].$$

For $x, y \in L^2$ $[0, 1]$, we calculate

$$\begin{aligned}||Sx - Sy||^2 &= \int_0^1 |Sx(t) - Sy(t)|^2 dt \\ &= \int_0^1 \left| \int_0^1 (F(t, s, x(s)) - F(t, s, y(s)))ds \right|^2 dt \\ &\leq \int_0^1 \left| \int_0^1 |x(s) - y(s)|ds \right|^2 dt \\ &\leq \int_0^1 |x(s) - y(s)|^2 ds = ||x - y||^2.\end{aligned}$$

So S is nonexpansive. For any function $x_0 \in L^2[0, 1]$, we define a sequence of functions $\{x_n\}$ in $L^2[0, 1]$ by

$$x_{n+1} = \alpha_n S\left(\frac{x_n + x_{n+1}}{2}\right) + (1 - \alpha_n)x_n$$

where $\alpha_n \in (0, 1)$ such that $\liminf_{n\to\infty}\alpha_n > 0$ and $\alpha_{n+1}^2 \leq \lambda\alpha_n^2$ for some $\lambda > 0$. Now by Theorem 2.8, $\{x_n\}$ weakly converges to the fixed point of S which is a solution of Eq. (3.1).

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