Approximation of fixed point of multivalued ρ-quasi-contractive mappings in modular function spaces

Received 11 November 2018
Accepted 3 February 2019

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Abstract
The purpose of this paper is to extend the recent results of Okeke et al. (2018) to the class of multivalued ρ-quasi-contractive mappings in modular function spaces. We approximate fixed points of this class of nonlinear multivalued mappings in modular function spaces. Moreover, we extend the concepts of T-stability, almost T-stability and summably almost T-stability to modular function spaces and give some results.

Keywords Multivalued ρ-quasi-contractive mappings, Multivalued mappings, Approximation of fixed point, Modular function spaces, S-iterative process, ρ-T-stable, ρ-almost T-stable, ρ-summably almost T-stable
Paper type Original Article

1. Introduction
It is known that there is a close relationship between the problem of solving a nonlinear equation and that of approximating fixed points of a corresponding contractive type operator (see, e.g. [4,17]). Hence, there is a practical and theoretical interest in approximating fixed points of several contractive type operators. For over a century now, the study of fixed point theory of multivalued nonlinear mappings has attracted many well-known mathematicians and mathematical scientists (see, e.g. Khan et al. [13]). The motivation for such studies stems mainly from the usefulness of fixed point theory results in real-world applications, as in *Game*

JEL Classification — 47H09, 47H10, 49M05, 54H25

Conflicts of interest: The authors declare that they do not have any conflicts of interest.
Authors' contributions: All authors contributed equally in writing this research paper. Each author read and approved the final manuscript.
The publisher wishes to inform readers that the article "Approximation of fixed point of multivalued ρ-quasi-contractive mappings in modular function spaces" was originally published by the previous publisher of the *Arab Journal of Mathematical Sciences* and the pagination of this article has been subsequently changed. There has been no change to the content of the article. This change was necessary for the journal to transition from the previous publisher to the new one. The publisher sincerely apologises for any inconvenience caused. To access and cite this article, please use Amechi Okeke, G., Hussain Khan, S. (2019), "Approximation of fixed point of multivalued ρ-quasi-contractive mappings in modular function spaces" *Arab Journal of Mathematical Sciences*, Vol. 26 No. 1/2, pp. 75-93. The original publication date for this paper was 08/02/2019.

Arab Journal of Mathematical Sciences
Vol. 26 No. 1/2, 2020
pp. 75-93
Emerald Publishing Limited
e-ISSN: 2588-9214
p-ISSN: 1319-5166
DOI 10.1016/j.ajmsc.2019.02.001

Theory and Market Economy and in other areas of mathematical sciences such as in *Nonsmooth Differential Equations.*

Modular function spaces are natural generalizations of both function and sequence variants of several important, from application perspective, spaces like Musielak–Orlicz, Orlicz, Lorentz, Orlicz–Lorentz, Kothe, Lebesgue, Calderon–Lozanovskii spaces and several others. Interest in quasi-nonexpansive mappings in modular function spaces stems mainly in the richness of structure of modular function spaces, that – besides being Banach spaces (or F-spaces in a more general settings) – are equipped with modular equivalents of norm or metric notions and also equipped with almost everywhere convergence and convergence in submeasure. It is known that modular type conditions are much more natural as modular type assumptions can be more easily verified than their metric or norm counterparts, particularly in applications to integral operators, approximation and fixed point results. Moreover, there are certain fixed point results that can be proved only using the apparatus of modular function spaces. Hence, fixed point theory results in modular function spaces, in this perspective, should be considered as complementary to the fixed point theory in normed and metric spaces (see, e.g. [10]). Several authors have proved very interesting fixed points results in the framework of modular function spaces, (see, e.g. [10,11,15,18]).

It is our purpose in the present paper to extend the recent results of Okeke et al. [17] to the class of multivalued ρ-quasi-contractive mappings, which is known to be wider than the class of Zamfirescu operators (see, e.g. [5]) in modular function spaces. We approximate the fixed point of these classes of nonlinear multivalued mappings in modular function spaces. Moreover, we extend the concepts of T-stability, almost T-stability and summably almost T-stability to modular function spaces. Consequently, we define the concepts of ρ-T-stable, ρ-almost T-stable and ρ-summably almost T-stable in modular function spaces. We prove that some fixed point iterative processes are ρ-summably almost T-stable with respect to T, where T is a multivalued ρ-quasi-contractive mapping in modular function spaces.

2. Preliminaries

In this study, we let Ω denote a nonempty set and Σ a nontrivial σ-algebra of subsets of Ω. Let $\mathcal{P}$ be a δ-ring of subsets of Ω, such that $E \cap A \in \mathcal{P}$ for any $E \in \mathcal{P}$ and $A \in \Sigma$. Let us assume that there exists an increasing sequence of sets $K_n \in \mathcal{P}$ such that $\Omega = \cup K_n$ (for instance, $\mathcal{P}$ can be the class of sets of finite measure in a σ-finite measure space). By 1_A, we denote the characteristic function of the set A in Ω By ε we denote the linear space of all simple functions with supports from $\mathcal{P}$. By $\mathcal{M}_\infty$ we denote the space of all extended measurable functions, i.e., all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \varepsilon$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for each $\omega \in \Omega$.

Definition 2.1. Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex and even function. We say that ρ is a regular convex function pseudomodular if

(1) $\rho(0) = 0$;

(2) ρ is monotone, i.e.,$|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;

(3) ρ is orthogonally subadditive, i.e., $\rho(f1_{A\cup B}) \leq \rho(f1_A) + \rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cap B \neq \emptyset$, $f \in \mathcal{M}_\infty$;

(4) ρ has Fatou property, i.e.,$|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;

(5) ρ is order continuous in ε, i.e., $g_n \in \varepsilon$ and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

A set $A \in \Sigma$ is said to be ρ-null if $\rho(g 1_A) = 0$ for every $g \in \varepsilon$. A property $p(\omega)$ is said to hold ρ-almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega : p(\omega)$ does not hold$\}$ is ρ-null. As usual, we identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define

$$\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho) = \{f \in \mathcal{M}_\infty : |f(\omega)| < \infty \, \rho\text{-}a.e.\},$$

where $f \in \mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$ is actually an equivalence class of functions equal ρ-a.e. rather than an individual function. Where no confusion exists, we shall write $\mathcal{M}$ instead of $\mathcal{M}(\Omega, \Sigma, \mathcal{P}, \rho)$.

The following definitions were given in [12].

Definition 2.2. Let ρ be a regular function pseudomodular;

(a) we say that ρ is a regular convex function modular if $\rho(f) = 0$ implies $f = 0$ ρ-a.e.

(b) we say that ρ is a regular convex function semimodular if $\rho(\alpha f) = 0$ for every $\alpha > 0$ implies $f = 0$ ρ-a.e.

It is known (see, e.g. [10]) that ρ satisfies the following properties:

(1) $\rho(0) = 0$ iff $f = 0$ ρ-a.e.

(2) $\rho(\alpha f) = \rho(f)$ for every scalar α with $|\alpha| = 1$ and $f \in \mathcal{M}$.

(3) $\rho(\alpha f + \beta g) \le \rho(f) + \rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \ge 0$ and $f, g \in \mathcal{M}$.

ρ is called a convex modular if, in addition, the following property is satisfied:
$(3')$ $\rho(\alpha f + \beta g) \le \alpha\rho(f) + \beta\rho(g)$ if $\alpha + \beta = 1$, $\alpha, \beta \ge 0$ and $f, g \in \mathcal{M}$.

The class of all nonzero regular convex function modulars on Ω is denoted by $\mathfrak{R}$.

Definition 2.3. The convex function modular ρ defines the modular function space L_ρ as

$$L_\rho = \{f \in \mathcal{M}; \rho(\lambda f) \to 0 \ \ as \ \ \lambda \to 0\}.$$

Generally, the modular ρ is not subadditive and therefore does not behave as a norm or a distance. However, the modular space L_ρ can be equipped with an F-norm defined by

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le \alpha\right\}.$$

In the case ρ is convex modular,

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le 1\right\}.$$

defines a norm on the modular space L_ρ, and it is called the Luxemburg norm.

Lemma 2.1 ([10]). *Let $\rho \in \mathfrak{R}$. Defining $L_\rho^0 = \{f \in L_\rho; \rho(f, .)$ is order continuous$\}$ and $E_\rho = \{f \in L_\rho; \lambda f \in L_\rho^0$ for every $\lambda > 0\}$, we have*

(i) $L_\rho \supset L_\rho^0 \supset E_\rho$;

(ii) E_ρ has the Lebesgue property, i.e., $\rho(\alpha f, D_k) \to 0$, for $\alpha > 0$, $f \in E_\rho$ and $D_k \downarrow \varnothing$;

(iii) E_ρ is the closure of ε (in the sense of $\|.\|_\rho$).

Definition 2.4. A nonzero regular convex function ρ is said to satisfy the Δ_2-condition, if $\sup_{n\geq 1}\rho(2f_n, D_k) \to 0$ as $k \to \infty$ whenever $\{D_k\}$ decreases to $\emptyset$ and $\sup_{n\geq 1}\rho(f_n, D_k) \to 0$ as $k \to \infty$.

If ρ is convex and satisfies Δ_2-condition, then $L_\rho = E_\rho$.

The following uniform convexity type properties of ρ can be found in [6].

Definition 2.5. Let ρ be a nonzero regular convex function modular defined on Ω

(i) Let $r > 0$, $\epsilon > 0$. Define

$$D_1(r,\epsilon) = \{(f,g) : f,g \in L_\rho, \rho(f) \leq r, \rho(g) \leq r, \rho(f-g) \geq \epsilon r\}.$$

Let

$$\delta_1(r,\epsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f,g) \in D_1(r,\epsilon)\right\} \quad \text{if } D_1(r,\epsilon) \neq \emptyset,$$

and $\delta_1(r,\epsilon) = 1$ if $D_1(r,\epsilon) = \emptyset$. We say that ρ satisfies $(UC1)$ if for every $r > 0$, $\epsilon > 0$, $\delta_1(r,\epsilon) > 0$. Observe that for every $r > 0$, $D_1(r,\epsilon) \neq \emptyset$, for $\epsilon > 0$ small enough.

(ii) We say that ρ satisfies $(UUC1)$ if for every $s \geq 0$, $\epsilon > 0$, there exists $\eta_1(s,\epsilon) > 0$ depending only on s and ϵ such that $\delta_1(r,\epsilon) > \eta_1(s,\epsilon) > 0$ for any $r > s$.

(iii) Let $r > 0$, $\epsilon > 0$. Define

$$D_2(r,\epsilon) = \left\{(f,g) : f,g \in L_\rho, \rho(f) \leq r, \rho(g) \leq r, \rho\left(\frac{f-g}{2}\right) \geq \epsilon r\right\}.$$

Let

$$\delta_2(r,\epsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+g}{2}\right) : (f,g) \in D_2(r,\epsilon)\right\}, \quad \text{if } D_2(r,\epsilon) \neq \emptyset,$$

and $\delta_2(r,\epsilon) = 1$ if $D_2(r,\epsilon) = \emptyset$. We say that ρ satisfies $(UC2)$ if for every $r > 0$, $\epsilon > 0$, $\delta_2(r,\epsilon) > 0$. Observe that for every $r > 0$, $D_2(r,\epsilon) \neq \emptyset$, for $\epsilon > 0$ small enough.

(iv) We say that ρ satisfies $(UUC2)$ if for every $s \geq 0$, $\epsilon > 0$, there exists $\eta_2(s,\epsilon) > 0$ depending only on s and ϵ such that $\delta_2(r,\epsilon) > \eta_2(s,\epsilon) > 0$ for any $r > s$.

(v) We say that ρ is strictly convex (SC), if for every $f, g \in L_\rho$ such that $\rho(f) = \rho(g)$ and $\rho\left(\frac{f+g}{2}\right) = \frac{\rho(f)+\rho(g)}{2}$, there holds $f = g$.

Proposition 2.1. ([10]). *The following conditions characterize relationship between the above defined notions:*

(i) $(UUCi) \Rightarrow (UCi)$ for $i = 1, 2$.

(ii) $\delta_1(r,\epsilon) \leq \delta_2(r,\epsilon)$.

(iii) $(UC1) \Rightarrow (UC2)$.

(iv) $(UUC1) \Rightarrow (UUC2)$.

(v) *If ρ is homogeneous (e.g. it is a norm), then all the conditions $(UC1), (UC2), (UUC1), (UUC2)$ are equivalent and $\delta_1(r, 2\epsilon) = \delta_1(1, 2\epsilon) = \delta_2(1,\epsilon) = \delta_2(r,\epsilon)$.*

Definition 2.6. Let L_ρ be a modular space. The sequence $\{f_n\} \subset L_\rho$ is called:

(1) ρ-convergent to $f \in L_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$;

(2) ρ-Cauchy, if $\rho(f_n - f_m) \to 0$ as n and $m \to \infty$.

Observe that ρ-convergence does not imply ρ-Cauchy since ρ does not satisfy the triangle inequality. In fact, one can easily show that this will happen if and only if ρ satisfies the Δ_2-condition.

Kilmer et al. [14] defined ρ-distance from an $f \in L_\rho$ to a set $D \subset L_\rho$ as follows:

$$\mathrm{dist}_\rho(f, D) = \inf\{\rho(f - h) : h \in D\}.$$

Definition 2.7. A subset $D \subset L_\rho$ is called:

(1) ρ-closed if the ρ-limit of a ρ-convergent sequence of D always belongs to D;

(2) ρ-a.e. closed if the ρ-a.e. limit of a ρ-a.e. convergent sequence of D always belongs to D;

(3) ρ-compact if every sequence in D has a ρ-convergent subsequence in D;

(4) ρ-a.e. compact if every sequence in D has a ρ-a.e. convergent subsequence in D;

(5) ρ-bounded if

$$\mathrm{diam}_\rho(D) = \sup\{\rho(f - g) : f, g \in D\} < \infty.$$

The following famous result was proved by Zamfirescu [19]

Theorem 2.1. ([19]). *Let (X, d) be a complete metric space, and let $T : X \to X$ be a mapping for which there exist real numbers a, b and c satisfying $0 < a < 1$, $0 < b, c < \frac{1}{2}$ such that for each pair $x, y \in X$ at least one of the following is true:*

(z1) $d(Tx, Ty) \leq ad(x, y)$,

(z2) $d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$,

(z3) $d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Then T has a unique fixed point p and the Picard iteration process $\{x_n\}$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots$$

converges to p for any $x_0 \in X$.

Remark 2.1. Any operator T which satisfies the contractive conditions (z1)–(z3) of Theorem 2.1 is called a Zamfirescu operator (see e.g. [5]) and is denoted by Z.

The following class of quasi-contractive operators was introduced on a normed space E by Berinde [5]:

$$\|Tx - Ty\| \leq \delta\|x - y\| + L\|Tx - x\|,$$

for any $x, y \in E$, $0 \leq \delta < 1$ and $L \geq 0$. He proved that this class is wider than the class of Zamfirescu operators.

A set $D \subset L_\rho$ is called ρ-proximinal if for each $f \in L_\rho$ there exists an element $g \in D$ such that $\rho(f - g) = dist_\rho(f, D)$. We shall denote the family of nonempty ρ-bounded ρ-proximinal subsets of D by $P_\rho(D)$, the family of nonempty ρ-closed ρ-bounded subsets of D by $C_\rho(D)$ and the family of ρ-compact subsets of D by $K_\rho(D)$. Let $H_\rho(.,.)$be the ρ-Hausdorff distance on $C_\rho(L_\rho)$, that is,

$$H_\rho(A, B) = \max\left\{\sup_{f\in A}\mathrm{dist}_\rho(f, B), \sup_{g\in B}\mathrm{dist}_\rho(g, A)\right\}, A, B \in C_\rho(L_\rho).$$

A multivalued map $T : D \to C_\rho(L_\rho)$ is said to be:

(a) ρ-contraction mapping if there exists a constant $k \in [0, 1)$ such that

$$H_\rho(Tf, Tg) \leq k\rho(f - g), \quad \text{for all} f, g \in D. \tag{2.1}$$

(b) ρ-nonexpansive (see, e.g. Khan and Abbas [12]) if

$$H_\rho(Tf, Tg) \leq \rho(f - g), \quad \text{for all } f, g \in D. \tag{2.2}$$

(c) ρ-quasi-nonexpansive mapping if

$$H_\rho(Tf, p) \leq \rho(f - p) \quad \text{for all } f \in D \text{ and } p \in F_\rho(T). \tag{2.3}$$

(d) ρ-quasi-contractive mapping if

$$H_\rho(Tf, Tg) \leq \delta\rho(f - g) + L\rho(Tf - f), \quad \text{for all } f, g \in D, 0 \leq \delta < 1 \text{ and } L \geq 0. \tag{2.4}$$

A sequence $\{t_n\} \subset (0, 1)$ is called bounded away from 0 if there exists $a > 0$ such that $t_n \geq a$ for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0, 1)$ is called bounded away from 1 if there exists $b < 1$ such that $t_n \leq b$ for every $n \in \mathbb{N}$.

Recently, Okeke et al. [17] approximated the fixed point of multivalued ρ-quasi-nonexpansive mappings using the Picard–Krasnoselskii hybrid iterative process. It is known that this iteration process converges faster than all of Picard, Mann, Krasnoselskii and Ishikawa iterative processes when applied to contraction mappings (see, Okeke and Abbas [16]). The following is the analogue of the Picard–Krasnoselskii hybrid iterative process in modular function spaces: Let $T : D \to P_\rho(D)$ be a multivalued mapping and $\{f_n\} \subset D$ be defined by the following iteration process:

$$\begin{cases} f_{n+1} \in P_\rho^T(g_n) \\ g_n = (1 - \lambda)f_n + \lambda P_\rho^T(v_n), \quad n \in \mathbb{N}, \end{cases} \tag{2.5}$$

where $v_n \in P_\rho^T(f_n)$ and $0 < \lambda < 1$. It is our purpose in the present paper to prove some new fixed point theorems using this iteration process in the framework of modular function spaces.

The following is the analogue of the S-iteration, introduced by Agarwal et al. [1] in modular function spaces.

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1 - \alpha_n)u_n + \alpha_n v_n \\ g_n = (1 - \beta_n) f_n + \beta_n u_n, \end{cases} \tag{2.6}$$

where $u_n \in P_\rho^T(f_n)$, $v_n \in P_\rho^T(g_n)$, the sequences $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ are bounded away from both 0 and 1. It is known (see, e.g. [9]) that the S-iteration converges faster than the Mann iteration process and the Ishikawa iteration process for Zamfirescu operators.

Definition 2.8. A sequence $\{f_n\} \subset D$ is said to be Fejér monotone with respect to subset $P_\rho(D)$ of D if $\rho(f_{n+1} - p) \leq \rho(f_n - p)$, for all $p \in P_\rho^T(D)$ of D, $n \in \mathbb{N}$.

Definition 2.9. (*[12]*). A multivalued mapping $T : D \to C_\rho(D)$ is said to satisfy condition (I) if there exists a nondecreasing function $l : [0, \infty) \to [0, \infty)$ with $l(0) = 0$, $l(r) > 0$ for all $r \in (0, \infty)$ such that $\text{dist}_\rho(f, Tf) \geq l(\text{dist}_\rho(f, F_\rho(T)))$ for all $f \in D$.

The following Lemma will be needed in this study.

Lemma 2.2. (*[2]*). *Let $\rho \in \Re$ satisfy the Δ_2-condition. Let $\{f_n\}$ and $\{g_n\}$ be two sequences in L_ρ. Then*

$$\lim_{n\to\infty} \rho(g_n) = 0 \Rightarrow \limsup_{n\to\infty} \rho(f_n + g_n) = \limsup_{n\to\infty} \rho(f_n)$$

and

$$\lim_{n\to\infty}\rho(g_n) = 0 \Rightarrow \liminf_{n\to\infty}\rho(f_n + g_n) = \liminf_{n\to\infty}\rho(f_n).$$

Lemma 2.3. ([6]). *Let ρ satisfy $(UUC1)$ and let $\{t_k\}\subset(0,1)$ be bounded away from 0 and 1. If there exists $R > 0$ such that*

$$\limsup_{n\to\infty}\rho(f_n)\le R, \quad \limsup_{n\to\infty}\rho(g_n)\le R$$

and

$$\lim_{n\to\infty}\rho(t_nf_n + (1-t_n)g_n) = R,$$

then $\lim_{n\to\infty}\rho(f_n - g_n) = 0$.

A function $f\in L_\rho$ is called a fixed point of $T : L_\rho \to P_\rho(D)$ if $f\in Tf$. The set of all fixed points of T will be denoted by $F_\rho(T)$.

Lemma 2.4. ([12]). *Let $T : D\to P_\rho(D)$ be a multivalued mapping and*

$$P_\rho^T(f) = \{g\in Tf : \rho(f-g) = \mathrm{dist}_\rho\,(f, Tf)\}.$$

Then the following are equivalent:

(1) $f\in F_\rho(T)$, that is, $f\in Tf$.

(2) $P_\rho^T(f) = \{f\}$, that is, $f = g$ for each $g\in P_\rho^T(f)$.

(3) $f\in F(P_\rho^T(f))$, that is, $f\in P_\rho^T(f)$. Further $F_\rho(T) = F(P_\rho^T(f))$ where $F(P_\rho^T(f))$ denotes the set of fixed points of $P_\rho^T(f)$.

Lemma 2.5. ([3]). *Let $\{a_n\}_{n=0}^\infty, \{b_n\}_{n=0}^\infty$ be sequences of nonnegative numbers and $0\le q < 1$, such that*

$$a_{n+1}\le qa_n + b_n, \quad \textit{for all} \quad n\ge 0.$$

(i) If $\lim_{n\to\infty}b_n = 0$, then $\lim_{n\to\infty}a_n = 0$.

(ii) If $\sum_{n=0}^\infty b_n < \infty$, then $\sum_{n=0}^\infty a_n < \infty$.

3. Approximation of fixed points in modular function spaces

We begin this section with the following proposition

Proposition 3.1. *Let ρ satisfy $(UUC1)$ and Δ_2-condition. Let D be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D\to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-contractive mapping, satisfying contractive condition (2.4) and $F_\rho(T)\ne\emptyset$. Let $\{f_n\}\subset D$ be defined by the two step S-iterative process (2.6), such that the sequences $\{\alpha_n\}\subset(0,1)$ and $\{\beta_n\}\subset(0,1)$ are bounded away from both 0 and 1. Then the S-iterative process (2.6) is Fejér monotone with respect to $F_\rho(T)$.*

Proof. Let $p\in F_\rho(T)$. By Lemma 2.4, $P_\rho^T(p) = \{p\}$ and $F_\rho(T) = F(P_\rho^T)$. Using relation (2.4) and (2.6), we obtain the following estimate:

$$\begin{aligned}\rho(f_{n+1} - p) &= \rho[(1-\alpha_n)u_n + \alpha_n v_n - p] \\ &= \rho[(1-\alpha_n)(u_n - p) + \alpha_n(v_n - p)].\end{aligned} \tag{3.1}$$

The convexity of ρ implies

$$\begin{aligned}\rho(f_{n+1}-p) &\leq (1-\alpha_n)\rho(u_n-p)+\alpha_n\rho(v_n-p)\\ &\leq (1-\alpha_n)H_\rho(P_\rho^T(f_n),P_\rho^T(p))+\alpha_n H_\rho(P_\rho^T(g_n),P_\rho^T(p)).\end{aligned} \tag{3.2}$$

From relation (2.4), with $f=p, g=f_n$ and also $f=p, g=g_n$, then we obtain the following estimates from relation (3.2):

$$H_\rho(P_\rho^T(f_n),P_\rho^T(p)) \leq \delta\rho(f_n-p). \tag{3.3}$$

$$H_\rho(P_\rho^T(g_n),P_\rho^T(p)) \leq \delta\rho(g_n-p). \tag{3.4}$$

Using (3.3), (3.4) and the fact that $0\leq\delta<1$ in (3.2), we have

$$\begin{aligned}\rho(f_{n+1}-p) &\leq (1-\alpha_n)\delta\rho(f_n-p)+\alpha_n\delta\rho(g_n-p)\\ &\leq (1-\alpha_n)\rho(f_n-p)+\alpha_n\rho(g_n-p).\end{aligned} \tag{3.5}$$

Next, we have

$$\begin{aligned}\rho(g_n-p) &= \rho[(1-\beta_n)f_n+\beta_n u_n-p]\\ &= \rho[(1-\beta_n)(f_n-p)+\beta_n(u_n-p)].\end{aligned} \tag{3.6}$$

By convexity of ρ, we have

$$\rho(g_n-p) \leq (1-\beta_n)\rho(f_n-p)+\beta_n H_\rho(P_\rho^T(f_n),P_\rho^T(p)). \tag{3.7}$$

Using (2.4) with $f=p$ and $g=f_n$ and the fact that $0\leq\delta<1$, relation (3.7) yields:

$$\begin{aligned}\rho(g_n-p) &\leq (1-\beta_n)\rho(f_n-p)+\beta_n\delta\rho(f_n-p)\\ &\leq (1-\beta_n)\rho(f_n-p)+\beta_n\rho(f_n-p)\\ &= \rho(f_n-p).\end{aligned} \tag{3.8}$$

Using (3.8) in (3.5), we obtain: (3.9)

$$\begin{aligned}\rho(f_{n+1}-p) &\leq (1-\alpha_n)\rho(f_n-p)+\alpha_n\rho(f_n-p)\\ &= \rho(f_n-p).\end{aligned} \tag{3.9}$$

Hence, the S-iteration (2.6) is Fejér monotone with respect to $F_\rho(T)$. The proof of Proposition 3.1 is completed. □

Next, we prove the following proposition.

Proposition 3.2. *Let ρ satisfy the $(UUC1)$ and Δ_2-condition. Suppose that D is a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T: D\to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-contractive mapping, satisfying contractive condition (2.4) and $F_\rho(T)\neq\emptyset$. Let $\{f_n\}\subset D$ be defined by the two step S-iterative process (2.6), such that the sequences $\{\alpha_n\}\subset(0,1)$ and $\{\beta_n\}\subset(0,1)$ are bounded away from both 0 and 1. Then*

(i) the sequence $\{f_n\}$ is bounded.

(ii) for each $f\in D, \{\rho(f_n-f)\}$ converges.

Proof. Since $\{f_n\}$ is Fejér monotone as shown in Proposition 3.1. Using the fact that ρ satisfies the Δ_2-condition, we can easily show (i) and (ii). This completes the proof of Proposition 3.2. □

Theorem 3.1. *Let ρ satisfy $(UUC1)$ and Δ_2-condition. Let D be a ρ-closed, ρ-bounded and convex subset of a ρ-complete modular space L_ρ and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-contractive mapping, satisfying contractive condition (2.4) and $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset D$ be defined by the two step S-iterative process (2.6) and $f_0 \in D$, where the sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ are bounded away from both 0 and 1, satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then $\{f_n\}$ converges strongly to the fixed point of T.*

Proof. Let $p \in F_\rho(T)$. By Lemma 2.4, $P_\rho^T(p) = \{p\}$ and $F_\rho(T) = F(P_\rho^T)$. Using relation (2.4) and (2.6), we obtain the following estimate:

$$\begin{aligned} \rho(f_{n+1} - p) &= \rho[(1-\alpha_n)u_n + \alpha_n v_n - p] \\ &= \rho[(1-\alpha_n)(u_n - p) + \alpha_n(v_n - p)]. \end{aligned} \tag{3.10}$$

The convexity of ρ implies (3.11)

$$\begin{aligned} \rho(f_{n+1} - p) &\leq (1-\alpha_n)\rho(u_n - p) + \alpha_n \rho(v_n - p) \\ &\leq (1-\alpha_n)H_\rho(P_\rho^T(f_n), P_\rho^T(p)) + \alpha_n H_\rho(P_\rho^T(g_n), P_\rho^T(p)). \end{aligned} \tag{3.11}$$

From relation (2.4), with $f = p, g = f_n$ and also $f = p, g = g_n$, then we obtain the following estimates from relation (3.11):

$$H_\rho(P_\rho^T(f_n), P_\rho^T(p)) \leq \delta\rho(f_n - p). \tag{3.12}$$

$$H_\rho(P_\rho^T(g_n), P_\rho^T(p)) \leq \delta\rho(g_n - p). \tag{3.13}$$

Using (3.12) and (3.13) in (3.11), we have

$$\rho(f_{n+1} - p) \leq (1-\alpha_n)\delta\rho(f_n - p) + \alpha_n \delta\rho(g_n - p). \tag{3.14}$$

Next, we have

$$\begin{aligned} \rho(g_n - p) &= \rho[(1-\beta_n)f_n + \beta_n u_n - p] \\ &= \rho[(1-\beta_n)(f_n - p) + \beta_n(u_n - p)]. \end{aligned} \tag{3.15}$$

By convexity of ρ, we have

$$\rho(g_n - p) \leq (1-\beta_n)\rho(f_n - p) + \beta_n H_\rho(P_\rho^T(f_n), P_\rho^T(p)). \tag{3.16}$$

Using (2.4) with $f = p$ and $g = f_n$, then relation (3.16) yields:

$$\rho(g_n - p) \leq (1-\beta_n)\rho(f_n - p) + \beta_n \delta\rho(f_n - p). \tag{3.17}$$

Using (3.17) in (3.14), we have

$$\begin{aligned} \rho(f_{n+1} - p) &\leq (1-\alpha_n)\delta\rho(f_n - p) + \alpha_n \delta(1 - \beta_n(1-\delta))\rho(f_n - p) \\ &\leq [1 - \alpha_n(1 - \delta(1 - \beta_n(1-\delta)))]\rho(f_n - p). \end{aligned} \tag{3.18}$$

Using (3.18), we inductively obtain

$$\begin{aligned} \rho(f_{n+1} - p) \leq \prod_{k=0}^{n} [1 - \alpha_k(1 - \delta(1 - \beta_k(1-\delta)))]\rho(f_0 - p), \\ n = 0, 1, 2, 3, \ldots \end{aligned} \tag{3.19}$$

Using the fact that $0 \leq \delta < 1$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ are bounded away from both 0 and 1, satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$, relation (3.19) yields

$$\lim_{n\to\infty}\prod_{k=0}^{n}[1-\alpha_k(1-\delta(1-\beta_k(1-\delta)))]=0, \tag{3.20}$$

which implies that (3.19) becomes:

$$\lim_{n\to\infty}\rho(f_{n+1}-p)=0. \tag{3.21}$$

Consequently, $f_n\to p\in F_\rho(T)$. The proof of Theorem 3.1 is completed. □

Theorem 3.2. *Let ρ satisfy $(UUC1)$ and Δ_2-condition. Let D be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T:D\to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-contractive mapping, satisfying contractive condition (2.4) and $F_\rho(T)\neq\emptyset$. Let $\{f_n\}\subset D$ be defined by the two step S-iterative process (2.6) and $f_0\in D$, where the sequences $\{\alpha_n\},\{\beta_n\}\subset(0,1)$ are bounded away from both 0 and 1. Then $\lim_{n\to\infty}\rho(f_n-p)$ exists for all $p\in F_\rho(T)$ and $\lim_{n\to\infty}dist_\rho(f_n,P_\rho^T(f_n))=0$.*

Proof. Let $p\in F_\rho(T)$. By Lemma 2.4, $P_\rho^T(p)=\{p\}$ and $F_\rho(T)=F(P_\rho^T)$. Using relation (2.4) and (2.6), we obtain the following estimate:

$$\begin{aligned}\rho(f_{n+1}-p)&=\rho[(1-\alpha_n)u_n+\alpha_n v_n-p]\\&=\rho[(1-\alpha_n)(u_n-p)+\alpha_n(v_n-p)].\end{aligned} \tag{3.22}$$

The convexity of ρ implies

$$\begin{aligned}\rho(f_{n+1}-p)&\le(1-\alpha_n)\rho(u_n-p)+\alpha_n\rho(v_n-p)\\&\le(1-\alpha_n)H_\rho(P_\rho^T(f_n),P_\rho^T(p))+\alpha_n H_\rho(P_\rho^T(g_n),P_\rho^T(p)).\end{aligned} \tag{3.23}$$

From relation (2.4), with $f=p$, $g=f_n$ and also $f=p$, $g=g_n$, then we obtain the following estimates from relation (3.23):

$$H_\rho(P_\rho^T(f_n),P_\rho^T(p))\le\delta\rho(f_n-p). \tag{3.24}$$

$$H_\rho(P_\rho^T(g_n),P_\rho^T(p))\le\delta\rho(g_n-p). \tag{3.25}$$

Using (3.24), (3.25) and the fact that $0\le\delta<1$ in (3.23), we have

$$\begin{aligned}\rho(f_{n+1}-p)&\le(1-\alpha_n)\delta\rho(f_n-p)+\alpha_n\delta\rho(g_n-p)\\&\le(1-\alpha_n)\rho(f_n-p)+\alpha_n\rho(g_n-p).\end{aligned} \tag{3.26}$$

Next, we have

$$\begin{aligned}\rho(g_n-p)&=\rho[(1-\beta_n)f_n+\beta_n u_n-p]\\&=\rho[(1-\beta_n)(f_n-p)+\beta_n(u_n-p)].\end{aligned} \tag{3.27}$$

By convexity of ρ, we have

$$\rho(g_n-p)\le(1-\beta_n)\rho(f_n-p)+\beta_n H_\rho\left(P_\rho^T(f_n),P_\rho^T(p)\right). \tag{3.28}$$

Using (3.25) with $f=p$ and $g=f_n$ and the fact that $0\le\delta<1$, relation (3.28) yields:

$$\begin{aligned}\rho(g_n-p)&\le(1-\beta_n)\rho(f_n-p)+\beta_n\delta\rho(f_n-p)\\&\le(1-\beta_n)\rho(f_n-p)+\beta_n\rho(f_n-p)\\&=\rho(f_n-p).\end{aligned} \tag{3.29}$$

Using (3.29) in (3.26), we obtain:

$$\begin{aligned}\rho(f_{n+1}-p) &\leq (1-\alpha_n)\rho(f_n-p)+\alpha_n\rho(f_n-p)\\ &= \rho(f_n-p).\end{aligned} \tag{3.30}$$

This implies that $\lim_{n\to\infty}\rho(f_n-p)$ exists for all $p\in F_\rho(T)$.

Let

$$\lim_{n\to\infty}\rho(f_n-p)=K, \quad \text{where } K\geq 0. \tag{3.31}$$

Now, we show that

$$\lim_{n\to\infty}\operatorname{dist}_\rho\left(f_n, P_\rho^T(f_n)\right)=0. \tag{3.32}$$

Since $\operatorname{dist}_\rho(f_n, P_\rho^T(f_n))\leq\rho(f_n-u_n)$, it suffices to show that

$$\lim_{n\to\infty}\rho(f_n-u_n)=0. \tag{3.33}$$

Now,

$$\rho(u_n-p)\leq H_\rho\left(P_\rho^T(f_n),P_\rho^T(p)\right)\leq\rho(f_n-p). \tag{3.34}$$

This implies that

$$\limsup_{n\to\infty}\rho(u_n-p)\leq\limsup_{n\to\infty}\rho(f_n-p). \tag{3.35}$$

By (3.31), we have

$$\limsup_{n\to\infty}\rho(u_n-p)\leq K. \tag{3.36}$$

Also from (3.29), we have

$$\limsup_{n\to\infty}\rho(g_n-p)\leq\limsup_{n\to\infty}\rho(f_n-p), \tag{3.37}$$

so that

$$\limsup_{n\to\infty}\rho(g_n-p)\leq K. \tag{3.38}$$

Moreover, the inequality

$$\rho(v_n-p)\leq H_\rho(P_\rho^T(g_n),P_\rho^T(p))\leq\rho(g_n-p)\leq\rho(f_n-p), \tag{3.39}$$

this implies that

$$\limsup_{n\to\infty}\rho(v_n-p)\leq\limsup_{n\to\infty}\rho(f_n-p), \tag{3.40}$$

hence,

$$\limsup_{n\to\infty}\rho(v_n-p)\leq K. \tag{3.41}$$

Now,

$$\begin{aligned}\lim_{n\to\infty}\rho(f_{n+1}+p) &= \lim_{n\to\infty}\rho[(1-\alpha_n)u_n+\alpha_n v_n-p]\\ &= \lim_{n\to\infty}\rho[(1-\alpha_n)(u_n-p)+\alpha_n(v_n-p)]\\ &= K.\end{aligned} \tag{3.42}$$

Using (3.35), (3.41), (3.42) and Lemma 2.3, we have

$$\lim_{n\to\infty} \rho(v_n - u_n) = 0. \tag{3.43}$$

Now,

$$\begin{aligned}\rho(f_{n+1} - p) &= \rho[(1-\alpha_n)u_n + \alpha_n v_n - p] \\ &= \rho[(u_n - p) + \alpha_n(v_n - u_n)].\end{aligned} \tag{3.44}$$

Using Lemma 2.2 and (3.44), we have

$$\begin{aligned}K &= \liminf_{n\to\infty} \rho(f_{n+1} - p) = \liminf_{n\to\infty} \rho[(u_n - p) + \alpha_n(v_n - u_n)] \\ &= \liminf_{n\to\infty} \rho(u_n - p).\end{aligned} \tag{3.45}$$

This means that

$$K = \liminf_{n\to\infty} \rho(u_n - p). \tag{3.46}$$

Using (3.35) and (3.46), we have

$$\lim_{n\to\infty} \rho(u_n - p) = K. \tag{3.47}$$

Using (3.43), we have

$$\liminf_{n\to\infty} \rho(u_n - p) = \liminf_{n\to\infty} \rho[(u_n - v_n) + (v_n - p)] = \liminf_{n\to\infty} \rho(v_n - p). \tag{3.48}$$

But

$$\rho(v_n - p) \le H_\rho(P_\rho^T(g_n), P_\rho^T(p)) \le \rho(g_n - p). \tag{3.49}$$

Hence,

$$\liminf_{n\to\infty} \rho(v_n - p) \le \liminf_{n\to\infty} \rho(g_n - p). \tag{3.50}$$

By (3.41), we have

$$K \le \liminf_{n\to\infty} \rho(g_n - p). \tag{3.51}$$

From (3.41) and (3.51), we have

$$\lim_{n\to\infty} \rho(g_n - p) = K. \tag{3.52}$$

Since

$$\begin{aligned}\lim_{n\to\infty} \rho(g_n - p) &= \lim_{n\to\infty} \rho[(1-\beta_n)f_n + \beta_n u_n - p] \\ &= \lim_{n\to\infty} \rho[(1-\beta_n)(f_n - p) + \beta_n(u_n - p)] = K.\end{aligned} \tag{3.53}$$

Using (3.31), (3.35) and Lemma 2.3, we have

$$\lim_{n\to\infty} \rho(f_n - u_n) = 0. \tag{3.54}$$

Hence,

$$\lim_{n\to\infty} \operatorname{dist}_\rho\left(f_n, P_\rho^T(f_n)\right) = 0. \tag{3.55}$$

The proof of Theorem 3.2 is completed. □

Theorem 3.3. *Let ρ satisfy $(UUC1)$ and Δ_2-condition. Let D be a nonempty ρ-compact, ρ-bounded and convex subset of L_ρ. Let $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-contractive mapping, satisfying contractive condition (2.4) and $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset D$ be defined by the two step S-iterative process (2.6) and $f_0 \in D$, where the sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ are bounded away from both 0 and 1. Then $\{f_n\}$ ρ-converges to a fixed point of T.*

Proof. Using relation (2.4) with $f = q$, $g = f_{n_k}$ and the fact that $0 \leq \delta < 1$. Since D is ρ-compact, there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\lim_{n\to\infty}(f_{n_k} - q) = 0$ for some $q \in D$. Next, we show that q is a fixed point of T. Suppose t is an arbitrary point in $P_\rho^T(q)$ and $f \in P_\rho^T(f_{n_k})$. Observe that

$$\begin{aligned}
\rho\left(\frac{q-t}{3}\right) &= \rho\left(q - \frac{f_{n_k}}{3} + \frac{f_{n_k} - f}{3} + \frac{f-t}{3}\right) \\
&\leq \frac{1}{3}\rho(q - f_{n_k}) + \frac{1}{3}\rho(f_{n_k} - f) + \frac{1}{3}\rho(f-t) \\
&\leq \rho(q - f_{n_k}) + \mathrm{dist}_\rho\left(f_{n_k}, P_\rho^T(f_{n_k})\right) + \mathrm{dist}_\rho\left(P_\rho^T(f_{n_k}), t\right) \\
&\leq \rho(q - f_{n_k}) + \mathrm{dist}_\rho\left(f_{n_k}, P_\rho^T(f_{n_k})\right) + H_\rho\left(P_\rho^T(f_{n_k}), P_\rho^T(q)\right) \\
&\leq \rho(q - f_{n_k}) + \mathrm{dist}_\rho\left(f_{n_k}, P_\rho^T(f_{n_k})\right) + \delta\rho(q - f_{n_k}) \\
&\leq \rho(q - f_{n_k}) + \mathrm{dist}_\rho\left(f_{n_k}, P_\rho^T(f_{n_k})\right) + \rho(q - f_{n_k}).
\end{aligned} \tag{3.56}$$

By Theorem 3.2, we obtain $\lim_{n\to\infty}\mathrm{dist}_\rho(f_n, P_\rho^T(f_n)) = 0$. So that $\rho\left(\frac{q-t}{3}\right) = 0$. Therefore, q is a fixed point of P_ρ^T. By Lemma 2.4, we see that the set of fixed points of P_ρ^T is the same as that of T, hence, we have that $\{f_n\}$ ρ-converges to a fixed point of T. The proof of Theorem 3.3 is completed. □

Theorem 3.4. *Let ρ satisfy $(UUC1)$ and Δ_2-condition. Let D be a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to P_\rho(D)$ be a multivalued mapping satisfying condition (I) such that P_ρ^T is a ρ-quasi-contractive mapping, satisfying contractive condition (2.4) and $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset D$ be defined by the two step S-iterative process (2.6) and $f_0 \in D$, where the sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ are bounded away from both 0 and 1. Then $\{f_n\}$ ρ-converges to a fixed point of T.*

Proof. The proof of Theorem 3.4 is similar to the proof of Theorem 3 of Khan and Abbas [12]. □

4. ρ-Stability of fixed point iterations in modular function spaces

In this section, we define the concepts of ρ-T-stable, ρ-almost T-stable and ρ-summably almost T-stable in modular function spaces. We prove that some fixed point iterative processes are ρ-summably almost T-stable with respect to T, where T is a multivalued ρ-quasi-contractive mapping in modular function spaces.

Let ρ satisfy $(UUC1)$ and D a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to P_\rho(D)$ be a mapping with $F_\rho(T) \neq \emptyset$. Suppose that $\{f_n\}_{n=0}^\infty$ is a fixed point iterative process, i.e. a sequence $\{f_n\}_{n=0}^\infty$ defined by $f_0 \in D$ and (4.1)

$$f_{n+1} = F(T, f_n), \quad n = 0, 1, 2, 3, \ldots, \tag{4.1}$$

where F is a given function.

Several fixed point iterations exist in literature. For instance, Mann iteration, with $F(T, f_n) = (1-\alpha_n)f_n + \alpha_n Tf_n$, where $\{\alpha_n\} \subset [0,1]$ such that $\{\alpha_n\}$ is bounded away from both 0 and 1. The Ishikawa iteration, with $F(T, f_n) = (1-\alpha_n)f_n + \alpha_n T[(1-\beta_n)f_n + \beta_n Tf_n]$, such that $\{\alpha_n\}_{n=0}^{\infty}, \{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ are both bounded away from both 0 and 1.

Let $\{f_n\}_{n=0}^{\infty}$ converge strongly to some $p \in F_\rho(T)$. In practice, we compute $\{f_n\}_{n=0}^{\infty}$ as follows:

(i) Choose the initial guess (approximation) $f_0 \in D$;

(ii) Compute $f_1 = F(T, f_0)$. However, as a result of various errors that occur during computations (numerical approximations of functions, rounding errors, derivatives, integration, etc.), we do not obtain the exact value of f_1, but a different one, say , which is close enough to f_1, this means that $h_1 \approx f_1$;

(iii) Therefore, during the computation of $f_2 = F(T, f_1)$ we have

$$f_2 = F(T, h_1). \tag{4.2}$$

This means that instead of the theoretical value of f_2, we expect another value h_2 will be obtained, and h_2 being close enough to f_2, i.e. $h_2 \approx f_2$, and so on.

Continuing this process, we see that instead of the theoretical sequence $\{f_n\}_{n=0}^{\infty}$ defined by the fixed point iteration (4.1), we obtain practically an approximate sequence $\{h_n\}_{n=0}^{\infty}$.

The fixed point iteration (4.1) is considered to be numerically stable if and only if for h_n close enough to f_n at each stage, we have that the approximate $\{h_n\}_{n=0}^{\infty}$ still converges to the fixed point p of $F_\rho(T)$.

Next, we give the following definition, which is the analogue of the concept of T-stability introduced by Harder and Hicks (see, [7,8]) in modular function spaces.

Definition 4.1. Let ρ satisfy $(UUC1)$ and D a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to P_\rho(D)$ be a mapping with $F_\rho(T) \neq \emptyset$. Suppose that the fixed point iterative process (4.1) converges to a fixed point p of T. Let $\{h_n\}_{n=0}^{\infty}$ be an arbitrary sequence in D and set

$$\varepsilon_n = \rho(h_{n+1} - F(T, h_n)), \quad n = 0, 1, 2, 3, \ldots \tag{4.3}$$

The fixed point iterative process (4.1) is said to be ρ-T-stable, or *ρ-stable* or *ρ-stable with respect to T* if and only if

$$\lim_{n\to\infty} \varepsilon_n = 0 \Rightarrow \lim_{n\to\infty} h_n = p. \tag{4.4}$$

Definition 4.2. Let ρ satisfy $(UUC1)$ and D a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T : D \to P_\rho(D)$ be a mapping with $F_\rho(T) \neq \emptyset$. Suppose that the fixed point iterative process (4.1) converges to a fixed point p of T. Let $\{h_n\}_{n=0}^{\infty}$ be an arbitrary sequence in D and let $\{\varepsilon_n\}_{n=0}^{\infty}$ be defined by (4.3). The fixed point iterative process (4.1) is said to be *ρ-almost T-stable* or *ρ-almost stable with respect to T* if and only if

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty \Rightarrow \lim_{x\to\infty} h_n = p. \tag{4.5}$$

Remark 4.1. It is clear from the definitions that any ρ-stable fixed point iteration $\{f_n\}$ is also ρ-almost stable.

A sharper concept of almost stability was introduced by Berinde [4]. He showed some almost stable fixed point iterations which are also summably almost stable with respect to some classes of contractive operators. We next define the analogue of this concept in modular function spaces.

Definition 4.3. Let ρ satisfy $(UUC1)$ and D a nonempty ρ-closed, ρ-bounded and convex subset of L_ρ. Let $T: D \to P_\rho(D)$ be a mapping with $F_\rho(T) \neq \emptyset$. Suppose that the fixed point iterative process (4.1) converges to a fixed point p of T. Let $\{h_n\}_{n=0}^{\infty}$ be an arbitrary sequence in D and let $\{\varepsilon_n\}_{n=0}^{\infty}$ be defined by (4.3). The fixed point iterative process (4.1) is said to be *ρ-summably almost T-stable* or *ρ-summably almost stable with respect to T* if and only if

$$\sum_{n=0}^{\infty} \varepsilon_n < \infty \Rightarrow \sum_{n=0}^{\infty} \rho(h_n - p) < \infty. \tag{4.6}$$

Remark 4.2. Clearly, any fixed point iteration $\{f_n\}$ that is ρ-almost stable is also ρ-summably almost stable, since

$$\sum_{n=0}^{\infty} \rho(h_n - p) < \infty \Rightarrow \lim_{n\to\infty} h_n = p.$$

However, we show that the converse is generally not true (see Example 4.1 below).

Example 4.1. Let the real number system $\mathbb{R}$ be the space modulared as follows:

$$\rho(f) = |f|^k, \quad k \geq 1.$$

Let $D = \{f \in L_\rho : 0 \leq f(x) \leq 1\}$. Let $T: D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is ρ-nonexpansive satisfying $Tf = f$. Let $\{f_n\}$ be the Picard iteration. Then $\{f_n\}$ is not ρ-summably almost T-stable.

Clearly, D is a nonempty ρ-compact, ρ-bounded and convex subset of $L_\rho = \mathbb{R}$ which satisfies $UC1$ condition. Moreover, $\rho(f) = |f|^k$, $k \geq 1$ is homogeneous and it is of degree k, hence by Proposition 2.1 $(UUC1)$ hold. Clearly, $F_\rho(T) = [0, 1]$. Suppose $p = 0$. Take $h_n = \frac{1}{n}$, for each $n \geq 1$. Hence, $\lim_{n\to\infty} h_n = 0$, we see that

$$\begin{aligned}\varepsilon_n = \rho(h_{n+1} - F(T, h_n)) &= dist_\rho\left(\frac{1}{n+1}, \frac{1}{n}\right)\\ &= \left|\frac{1}{n+1} - \frac{1}{n}\right|^k = \left|\frac{1}{n(n+1)}\right| = \frac{1}{n(n+1)}.\end{aligned}$$

Hence, $\sum_{n=0}^{\infty} \varepsilon_n < \infty$.

However, we have

$$\sum_{n=0}^{\infty} \rho(h_n - p) = \sum_{n=0}^{\infty} dist_\rho\left(\frac{1}{n}, 0\right) = \sum_{n=0}^{\infty} \left|\frac{1}{n} - 0\right|^k = \sum_{n=0}^{\infty} \left|\frac{1}{n}\right| = \sum_{n=0}^{\infty} \frac{1}{n} = \infty.$$

This means that the Picard iteration $\{f_n\}$ is not ρ-summably almost T-stable.

It is known that the Picard iteration is not T-stable and hence not almost T-stable (see, e.g. [4]).

Next, we prove the following results.

Theorem 4.1. *Let ρ satisfy $(UUC1)$ and Δ_2-condition. Let D be a ρ-closed, ρ-bounded and convex subset of a ρ-complete modular space L_ρ and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-contractive mapping, satisfying contractive condition (2.4) and $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset D$ be defined by the two step S-iterative process as follows*

$$\begin{cases} f_0 \in D \\ f_{n+1} = (1-\alpha_n)u_n + \alpha_n v_n \\ g_n = (1-\beta_n)f_n + \beta_n u_n, \end{cases} \tag{4.7}$$

where $u_n \in P_\rho^T(f_n)$, $v_n \in P_\rho^T(g_n)$, the sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ are bounded away from both 0 and 1. Then $\{f_n\}$ is ρ-summably almost stable with respect to T.

Proof. Suppose $p \in F_\rho(T)$ and $\{h_n\}$ is an arbitrary sequence. Define

$$\begin{cases} s_n = (1-\beta_n)h_n + \beta_n w_n, \\ \varepsilon_n = \rho(h_{n+1} - (1-\alpha_n)w_n - \alpha_n z_n), \end{cases} \tag{4.8}$$

where $w_n \in P_\rho^T(h_n)$, $z_n \in P_\rho^T(s_n)$, the sequences $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ are bounded away from both 0 and 1.

Using the convexity of ρ, we have the following estimates:

$$\begin{aligned} \rho(h_{n+1} - p) &= \rho(h_{n+1} - (1-\alpha_n)w_n - \alpha_n z_n + (1-\alpha_n)(w_n - p) + \alpha_n(z_n - p)) \\ &\leq \varepsilon_n + (1-\alpha_n)\rho(w_n - p) + \alpha_n\rho(z_n - p) \\ &\leq \varepsilon_n + (1-\alpha_n)H_\rho(P_\rho^T(h_n), P_\rho^T(p)) + \alpha_n H_\rho(P_\rho^T(s_n), P_\rho^T(p)). \end{aligned} \tag{4.9}$$

Using (4.9), relation (2.4) with $f = p, g = h_n$ and also $f = p, g = s_n$, we have

$$\rho(h_{n+1} - p) \leq \varepsilon_n + (1-\alpha_n)\delta\rho(h_n - p) + \alpha_n\delta\rho(s_n - p). \tag{4.10}$$

Next, by convexity of ρ we have

$$\begin{aligned} \rho(s_n - p) &= \rho((1-\beta_n)h_n + \beta_n w_n - p) \\ &\leq (1-\beta_n)\rho(h_n - p) + \beta_n H_\rho(P_\rho^T(h_n), P_\rho^T(p)) \\ &\leq (1-\beta_n)\rho(h_n - p) + \beta_n\delta\rho(h_n - p) \\ &\leq (1-\beta_n)\rho(h_n - p) + \beta_n\rho(h_n - p) \\ &= \rho(h_n - p). \end{aligned} \tag{4.11}$$

Using (4.11) in (4.10), we obtain

$$\begin{aligned} \rho(h_{n+1} - p) &\leq \varepsilon_n + (1-\alpha_n)\delta\rho(h_n - p) + \alpha_n\delta\rho(h_n - p) \\ &= \varepsilon_n + \delta\rho(h_n - p). \end{aligned} \tag{4.12}$$

By Lemma 2.5, we have that the two step S-iteration (4.7) is ρ-summably almost stable with respect to T. The proof of Theorem 4.1 is completed. □

Theorem 4.2. *Let ρ satisfy $(UUC1)$ and Δ_2-condition. Let D be a ρ-closed, ρ-bounded and convex subset of a ρ-complete modular space L_ρ and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-contractive mapping, satisfying contractive condition (2.4) and $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset D$ be defined by the following iterative process*

$$\begin{cases} f_0 \in D \\ f_{n+1} \in P_\rho^T(u_n) \end{cases} \tag{4.13}$$

where $u_n \in P_\rho^T(f_n)$. Then $\{f_n\}$ is ρ -summably almost stable with respect to T.

Proof. Let $p \in F_\rho(T)$ and $\{h_n\}$ be an arbitrary sequence. Define

$$\varepsilon_n = \rho(h_{n+1} - m_n), \tag{4.14}$$

where $m_n \in P_\rho^T(h_n)$. Using (4.13), (4.14), relation (2.4) with $f = p, g = h_n$ and the convexity of ρ, we have the following estimate:

$$\begin{aligned} \rho(h_{n+1} - p) &= \rho(h_{n+1} - m_n + m_n - p) \\ &\leq \rho(h_{n+1} - m_n) + \rho(m_n - p) \\ &\leq \varepsilon_n + H_\rho(P_\rho^T(h_n), P_\rho^T(p)) \\ &\leq \varepsilon_n + \delta\rho(h_n - p). \end{aligned} \tag{4.15}$$

By Lemma 2.5, it follows that the fixed point iteration (4.13) is ρ-summably almost stable with respect to T. The proof of Theorem 4.2 is completed. □

Theorem 4.3. *Let ρ satisfy $(UUC1)$ and Δ_2-condition. Let D be a ρ-closed, ρ-bounded and convex subset of a ρ-complete modular space L_ρ and $T : D \to P_\rho(D)$ be a multivalued mapping such that P_ρ^T is a ρ-quasi-contractive mapping, satisfying contractive condition (2.4) and $F_\rho(T) \neq \emptyset$. Let $\{f_n\} \subset D$ be defined by the two step S-iterative process as follows*

$$\begin{cases} f_0 \in D \\ f_{n+1} \in \sum_{i=0}^{k} \alpha_i u_n^i, n \geq 0, \alpha_i \geq 0, \alpha_1 > 0, \sum_{i=0}^{k} \alpha_i = 1. \end{cases} \tag{4.16}$$

where $u_n^i \in P_\rho^{T^i}(f_n)$. Then $\{f_n\}$ is ρ-summably almost stable with respect to T.

Proof. Let $p \in F_\rho(T)$ and $\{h_n\}$ be any given sequence in D and define

$$\varepsilon_n = \rho\left(h_{n+1} - \sum_{i=0}^{k} \alpha_i z_n^i\right), \tag{4.17}$$

where $z_n^i \in P_\rho^{T^i}(h_n)$. Using (4.16), (4.17), relation (2.4) with $f = p, g = h_n$ and the convexity of ρ, we have the following estimate:

$$
\begin{aligned}
\rho(h_{n+1}-p) &= \rho\left(h_{n+1}-\sum_{i=0}^{k}\alpha_i z_n^i+\sum_{i=0}^{k}\alpha_i z_n^i-p\right)\\
&\le \rho\left(h_{n+1}-\sum_{i=0}^{k}\alpha_i z_n^i\right)+\rho\left(\sum_{i=0}^{k}\alpha_i z_n^i-p\right)\\
&\le \varepsilon_n+\rho\left(\sum_{i=0}^{k}\alpha_i z_n^i-p\right)\\
&\le \varepsilon_n+H_\rho\left(\sum_{i=0}^{k}\alpha_i P_\rho^{T^i}(h_n),P_\rho^{T}(p)\right)\\
&\le \varepsilon_n+\sum_{i=0}^{k}\alpha_i H_\rho\big(P_\rho^{T^i}(h_n),P_\rho^{T}(p)\big)\\
&\le \varepsilon_n+\left(\sum_{i=0}^{k}\alpha_i\delta^i\right)\rho(h_n-p)\\
&= \varepsilon_n+q\rho(h_n-p),
\end{aligned}
\tag{4.18}
$$

where $q=\sum_{i=0}^{k}\alpha_i\delta^i<1$. Hence, by Lemma 2.5 it follows that the fixed point iteration (4.16) is ρ-summably almost stable with respect to T. The proof of Theorem 4.3 is completed. □

References

[1] R.P. Agarwal, D. O'Regan, D.R. Sahu, Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J. Nonlinear Convex Anal. 8 (1) (2007) 61–79.

[2] T.D. Benavides, M.A. Khamsi, S. Samadi, Asymptotically non-expansive mappings in modular function spaces, J. Math. Anal. Appl. 265 (2002) 249–263.

[3] V. Berinde, Iterative Approximation of Fixed Points, Editura Efemeride, Baia Mare, 2002.

[4] V. Berinde, Summable almost stability of fixed point iteration procedures, Carpathian J. Math. 19 (2) (2003) 81–88.

[5] V. Berinde, A convergence theorem for some mean value fixed point iteration procedures, Demonstr. Math. 38 (1) (2005) 177–184.

[6] B.A.B. Dehaish, W.M. Kozlowski, Fixed point iteration for asymptotic pointwise nonexpansive mappings in modular function spaces, Fixed Point Theory Appl. 2012 (2012) 118.

[7] A.M. Harder, T.L. Hicks, Stability results for fixed point iteration procedures, Math. Japon. 33 (5) (1988) 693–706.

[8] A.M. Harder, T.L. Hicks, A stable iteration procedure for nonexpansive mappings, Math. Japon. 33 (5) (1988) 687–692.

[9] N. Hussain, A. Rafiq, B. Damjanović, R. Lazović, On rate of convergence of various iterative schemes, Fixed Point Theory Appl. 2011 (45) (2011) 6.

[10] M.A. Khamsi, W.M. Kozlowski, Fixed Point Theory in Modular Function Spaces, Springer International Publishing, Switzerland, 2015.

[11] S.H. Khan, Approximating fixed points of (λ,ρ)-firmly nonexpansive mappings in modular function spaces, Arab. J. Math. (2018) 7, http://dx.doi.org/10.1007/s40065-018-0204-x.

[12] S.H. Khan, M. Abbas, Approximating fixed points of multivalued ρ-nonexpansive mappings in modular function spaces, Fixed Point Theory Appl. 2014 (2014) 34.

[13] S.H. Khan, M. Abbas, S. Ali, Fixed point approximation of multivalued ρ-quasi-nonexpansive mappings in modular function spaces, J. Nonlinear Sci. Appl. 10 (2017) 3168–3179.

[14] S.J. Kilmer, W.M. Kozlowski, G. Lewicki, Sigma order continuity and best approximation in L_ρ-spaces, Comment. Math. Univ. Carolin. 3 (1991) 2241–2250.

[15] M.A. Kutbi, A. Latif, Fixed points of multivalued mappings in modular function spaces, Fixed Point Theory Appl. (2009) 786357, 12 pages.

[16] G.A. Okeke, M. Abbas, A solution of delay differential equations via Picard-Krasnoselskii hybrid iterative process, Arab. J. Math. 6 (2017) 21–29.

[17] G.A. Okeke, S.A. Bishop, S.H. Khan, Iterative approximation of fixed point of multivalued ρ-quasi-nonexpansive mappings in modular function spaces with applications, J. Funct. Spaces 2018 (2018) Article ID 1785702, 9 pages.

[18] M. Öztürk, M. Abbas, E. Girgin, Fixed points of mappings satisfying contractive condition of integral type in modular spaces endowed with a graph, Fixed Point Theory Appl. 2014 (2014) 220.

[19] T. Zamfirescu, Fixed point theorems in metric spaces, Arch. Math. 23 (1992) 292–298.

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