Some new fractional integral inequalities for generalized relative semi-$m$-$\{r; h_1, h_2\}$-preinvex mappings via generalized Mittag-Leffler function

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Abstract

The authors discover a new identity concerning differentiable mappings defined on $m$-invex set via fractional integrals. By using the obtained identity as an auxiliary result, some fractional integral inequalities for generalized relative semi-$m$-$\{r; h_1, h_2\}$-preinvex mappings by involving generalized Mittag-Leffler function are presented. It is pointed out that some new special cases can be deduced from main results of the paper. Also these inequalities have some connections with known integral inequalities. At the end, some applications to special means for different positive real numbers are provided as well.

Keywords: Hermite–Hadamard inequality, Hölder’s inequality, Minkowski inequality, Power mean inequality, Generalized Mittag-Leffler function, Fractional integrals, $m$-invex

1. Introduction

The following double inequality is known as Hermite–Hadamard inequality.

**Theorem 1.1.** Let $f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping on an interval $I$ of real numbers and $a, b \in I$ with $a < b$. Then the subsequent double inequality holds:
\[ f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x)dx \leq \frac{f(a) + f(b)}{2}. \] (1.1)

For recent results concerning Hermite–Hadamard type inequalities through various classes of convex functions the readers are referred to [3–5,7,8,12–20,21,23–25,29,32] and the references mentioned in these papers.

Let us recall some special functions and evoke some basic definitions as follows.

**Definition 1.2** ([20]). Let \( f \in L[a,b] \). The Riemann–Liouville integrals \( J_{a}^{\alpha}f \) and \( J_{b}^{\alpha}f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[ J_{a}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1}f(t)dt, \quad x > a \]

and

\[ J_{b}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} (t-x)^{\alpha-1}f(t)dt, \quad b > x, \]

where \( \Gamma(\alpha) = \int_{0}^{\infty} e^{-u}u^{\alpha-1}du \). Here \( f_{a}^{0} = J_{a}^{0}f = f(x) \).

Note that \( \alpha = 1 \), the fractional integral reduces to the classical integral.

**Definition 1.3** ([27]). Let \( \mu, \nu, k, l, \gamma \) be positive real numbers and \( \omega \in \mathbb{R} \). Then the generalized fractional integral operators containing Mittag-Leffler function \( E_{\mu,\nu,\omega,a}^{\gamma,\delta,k} \) and \( E_{\mu,\nu,\omega,b}^{\gamma,\delta,k} \) for a real valued continuous function \( f \) are defined by:

\[ \left( E_{\mu,\nu,\omega,a}^{\gamma,\delta,k} f \right)(x) = \int_{a}^{x} (x-t)^{\gamma+1}E_{\mu,\nu,\omega}^{\gamma+1}(\omega(x-t))t f(t)dt \] (1.2)

and

\[ \left( E_{\mu,\nu,\omega,b}^{\gamma,\delta,k} f \right)(x) = \int_{x}^{b} (t-x)^{\gamma+1}E_{\mu,\nu,\omega}^{\gamma+1}(\omega(t-x))t f(t)dt, \]

where the function \( E_{\mu,\nu,\omega}^{\gamma,\delta,k} \) is the generalized Mittag-Leffler function defined as

\[ E_{\mu,\nu,\omega}^{\gamma,\delta,k}(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n} t^{n}}{\Gamma(\mu+n)(\delta)_{n}}, \] (1.3)

and \((a)_{n}\) is the Pochhammer symbol, it defined as

\[(a)_{n} = a(a+1)(a+2)\ldots(a+n-1), \quad (a)_{0} = 1. \]

For \( \omega = 0 \) in (1.2), integral operator \( E_{\mu,\nu,\omega,a}^{\gamma,\delta,k} \) reduces to the Riemann–Liouville fractional integral operator.

In [27,30] properties of generalized integral operators and generalized Mittag-Leffler functions are studied in detail. In [27] it is proved that \( E_{\mu,\nu,\omega}^{\gamma,\delta,k}(t) \) is absolutely convergent for \( k < l + \mu \). Let \( S \) be the sum of series of absolute terms of \( E_{\mu,\nu,\omega}^{\gamma,\delta,k}(t) \). We will use this property of Mittag-Leffler function in sequel.

**Definition 1.4** ([11]). A set \( K \subseteq \mathbb{R}^{n} \) is said to be invex with respect to the mapping \( \Lambda : K \times K \rightarrow \mathbb{R}^{n} \), if \( x + t\Lambda(y,x) \in K \) for every \( x, y \in K \) and \( t \in [0, 1] \).
Definition 1.5 ([7]). A non-negative function $f : I \subseteq \mathbb{R} \rightarrow [0, +\infty)$ is said to be $P$-function, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \quad t \in [0, 1].$$

Definition 1.6 ([22]). Let $h : [0, 1] \rightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function $f$ on the invex set $K$ is said to be $h$-preinvex with respect to $\Lambda$, if

$$f(x + t\Lambda(y, x)) \leq h(1-t)f(x) + h(t)f(y)$$

(1.4)

for each $x, y \in K$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

Definition 1.7 ([31]). Let $f : K \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative function. A function $f : K \rightarrow \mathbb{R}$ is said to be a tgs-convex on $K$ if the inequality

$$f((1-t)x + ty) \leq t(1-t)[f(x) + f(y)]$$

(1.5)

holds for all $x, y \in K$ and $t \in (0, 1)$.

Definition 1.8 ([19]). A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be MT-convex, if it is non-negative and $\forall x, y \in I$ and $t \in (0, 1)$ satisfies the subsequent inequality:

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y).$$

(1.6)

Definition 1.9 ([25]). A function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be $m$-MT-convex, if $f$ is positive and for $\forall x, y \in I$, and $t \in (0, 1)$, among $m \in (0, 1]$, satisfies the following inequality

$$f(tx + m(1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{m\sqrt{1-t}}{2\sqrt{t}}f(y).$$

(1.7)

Definition 1.10 ([8]). A set $K \subseteq \mathbb{R}^n$ is named as $m$-invex with respect to the mapping $\Lambda : K \times K \rightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\Lambda(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 1.11. In Definition 1.10, under certain conditions, the mapping $\Lambda(y, mx)$ could be reduced to $\Lambda(y, x)$. For example when $m = 1$, then the $m$-invex set degenerates an invex set on $K$.

Definition 1.12 ([26]). Let $K \subseteq \mathbb{R}$ be an open $m$-invex set with respect to the mapping $\Lambda : K \times K \rightarrow \mathbb{R}$ and $h_1, h_2 : [0, 1] \rightarrow [0, +\infty)$. A function $f : K \rightarrow \mathbb{R}$ is said to be generalized $(m, h_1, h_2)$-preinvex, if

$$f(mx + t\Lambda(y, mx)) \leq mh_1(t)f(x) + h_2(t)f(y)$$

(1.8)

is valid for all $x, y \in K$ and $t \in [0, 1]$, for some fixed $m \in (0, 1]$.

Motivated by the above literatures, the main objective of this paper is to establish in Section 2, some new fractional integral inequalities for generalized relative semi-$m$-$(r; h_1, h_2)$-preinvex mappings by involving generalized Mittag-Leffler function. It is pointed out that some new special cases will be deduced from main results of the paper. Also we will see that these inequalities have some connections with known integral inequalities. In Section 3, some applications to special means for different positive real numbers will be given.
2. Main results

The following definitions will be used in this section.

Definition 2.1. Let \( m: [0, 1] \to (0, 1] \) be a function. A set \( K \subseteq \mathbb{R}^n \) is named as \( m \)-invex with respect to the mapping \( \Lambda: K \times K \to \mathbb{R}^n \), if \( m(t)x + \xi \Lambda(y, m(t)x) \in K \) holds for each \( x, y \in K \) and any \( t, \xi \in [0, 1] \).

Remark 2.2. In Definition 2.1, under certain conditions, the mapping \( \Lambda(y, m(t)x) \) for any \( t, \xi \in [0, 1] \) could be reduced to \( \Lambda(y, mx) \). For example when \( m(t) = m \) for all \( t \in [0, 1] \), then the \( m \)-invex set degenerates to an \( m \)-invex set on \( K \).

We next introduce the notion of generalized relative semi-\( m \)-(\( r; h_1, h_2 \))-preinvex mappings.

Definition 2.3. Let \( K \subseteq \mathbb{R} \) be an open \( m \)-invex set with respect to the mapping \( \Lambda: K \times K \to \mathbb{R} \). Suppose \( h_1, h_2: [0, 1] \to [0, +\infty), \psi: I \to K \) are continuous functions and \( m: [0, 1] \to (0, 1] \). A mapping \( f: K \to (0, +\infty) \) is said to be generalized relative semi-\( m \)-(\( r; h_1, h_2 \))-preinvex mappings, if

\[
f(m(t)\psi(x) + \xi \Lambda(\psi(y), m(t)\psi(x))) \leq \left[ m(\xi)h_1(\xi)f''(x) + h_2(\xi)f''(y) \right]^\frac{1}{r}
\]

holds for all \( x, y \in I \) and \( t, \xi \in [0, 1] \), where \( r \neq 0 \).

Remark 2.4. In Definition 2.3, if we choose \( m = m = r = 1 \), this definition reduces to the definition considered by Noor in [23] and Preda et al. in [11].

Remark 2.5. In Definition 2.3, if we choose \( m = m = r = 1 \) and \( \psi(x) = x \), then we get Definition 1.12.

Remark 2.6. Let us discuss some special cases in Definition 2.3 as follows.

(I) Taking \( h_1(t) = h_2(t) = 1 \), then we get the generalized relative semi-(\( m, P \))-preinvex mappings.

(II) Taking \( h_1(t) = (1 - t)^s \) and \( h_2(t) = t^s \) for \( s \in (0, 1] \), then we get the generalized relative semi-(\( m, s \))-Brecker-preinvex mappings.

(III) Taking \( h_1(t) = (1 - t)^{-s} \) and \( h_2(t) = t^{-s} \) for \( s \in (0, 1] \), then we get the generalized relative semi-(\( m, s \))-Godunova–Levin–Dragomir-preinvex mappings.

(IV) Taking \( h_1(t) = h(1 - t) \) and \( h_2(t) = h(t) \), then we get the generalized relative semi-(\( m, h \))-preinvex mappings.

(V) Taking \( h_1(t) = h_2(t) = t(1 - t) \), then we get the generalized relative semi-(\( m, tgs \))-preinvex mappings.

(VI) Taking \( h_1(t) = \frac{\sqrt[2n]{1}}{2n} \) and \( h_2(t) = \frac{\sqrt[2n]{1}}{2n\sqrt{1 - t}} \), then we get the generalized relative semi-(\( m, MT \))-preinvex mappings.

It is worth mentioning here that to the best of our knowledge all the special cases discussed above are new in the literature.

For establishing our main results we need to prove the following lemma.

Lemma 2.7. Let \( \psi: I \to K \) and \( g: K \to \mathbb{R} \) are continuous functions and \( m: [0, 1] \to (0, 1] \). Suppose \( K = \{ m(t)\psi(a), m(t)\psi(a) + \Lambda(\psi(b), m(t)\psi(a)) \} \subseteq \mathbb{R} \) be an open \( m \)-invex subset with respect to \( \Lambda: K \times K \to \mathbb{R} \) for \( \Lambda(\psi(b), m(t)\psi(a)) > 0 \) and \( \forall t \in [0, 1] \). Assume that \( f: K \to \mathbb{R} \) be a differentiable mapping on \( K^* \). If \( f \circ g \in L(K) \), then the following equality for \( \nu > 0 \) holds:
We denote \(E_{\mu,\nu}^{\gamma,k}(\omega \xi^\mu)\). This completes the proof of the lemma.

\[
\left( \int_{m(t)\psi(a)}^{m(t)\psi(a)+A(\psi(b), m(t)\psi(a))} g(s)E_{\mu,\nu}^{\gamma,k}(\omega \xi^\mu)ds \right)^{\nu} \times \left[ f(m(t)\psi(a)) + f(m(t)\psi(a) + A(\psi(b), m(t)\psi(a))) \right]
\]

\[
- \nu \int_{m(t)\psi(a)}^{m(t)\psi(a) + A(\psi(b), m(t)\psi(a))} \left( \int_{m(t)\psi(a)}^{\xi} g(s)E_{\mu,\nu}^{\gamma,k}(\omega \xi^\mu)ds \right) f(\xi) d\xi
\]

\[
= \int_{m(t)\psi(a)}^{m(t)\psi(a) + A(\psi(b), m(t)\psi(a))} \left( \int_{m(t)\psi(a)}^{\xi} g(s)E_{\mu,\nu}^{\gamma,k}(\omega \xi^\mu)ds \right)^{\nu} \times f'(\xi) d\xi.
\]

We denote

\[
I_{\xi, t, A, \psi, m}(\nu, a, b) := \int_{m(t)\psi(a)}^{m(t)\psi(a) + A(\psi(b), m(t)\psi(a))} \left( \int_{m(t)\psi(a)}^{\xi} g(s)E_{\mu,\nu}^{\gamma,k}(\omega \xi^\mu)ds \right)^{\nu} \times f(\xi) d\xi
\]

\[
= \int_{m(t)\psi(a)}^{m(t)\psi(a) + A(\psi(b), m(t)\psi(a))} \left( \int_{m(t)\psi(a)}^{\xi} g(s)E_{\mu,\nu}^{\gamma,k}(\omega \xi^\mu)ds \right)^{\nu} \times f'(\xi) d\xi.
\]

**Proof.** Integrating by parts, we get

\[
I_{\xi, t, A, \psi, m}(\nu, a, b) = \left( \int_{m(t)\psi(a)}^{\xi} g(s)E_{\mu,\nu}^{\gamma,k}(\omega \xi^\mu)ds \right)^{\nu} \times \left[ f(m(t)\psi(a)) + f(m(t)\psi(a) + A(\psi(b), m(t)\psi(a))) \right]
\]

\[
- \nu \int_{m(t)\psi(a)}^{m(t)\psi(a) + A(\psi(b), m(t)\psi(a))} \left( \int_{m(t)\psi(a)}^{\xi} g(s)E_{\mu,\nu}^{\gamma,k}(\omega \xi^\mu)ds \right) f(\xi) d\xi
\]

\[
= \int_{m(t)\psi(a)}^{m(t)\psi(a) + A(\psi(b), m(t)\psi(a))} \left( \int_{m(t)\psi(a)}^{\xi} g(s)E_{\mu,\nu}^{\gamma,k}(\omega \xi^\mu)ds \right)^{\nu} \times f'(\xi) d\xi.
\]

This completes the proof of the lemma.
Using Lemma 2.7, we now state the following theorems for the corresponding version for power of first derivative.

**Theorem 2.8.** Let \( h_1, h_2 : [0, 1] \rightarrow [0, +\infty), \gamma : I \rightarrow K \) and \( g : K \rightarrow \mathbb{R} \) are continuous functions and \( m : [0, 1] \rightarrow (0, 1] \). Suppose \( K = \{ m(t)\gamma(a), m(t)\gamma(b) + \Lambda(\gamma(b), \gamma(a)) \} \subseteq \mathbb{R} \) be an open \( m \)-invex subset with respect to \( \Lambda : K \times K \rightarrow \mathbb{R} \) for \( \Lambda(\gamma(b), \gamma(a)) > 0 \) and \( \forall t \in [0, 1] \). Assume that \( f : K \rightarrow (0, +\infty) \) be a differentiable mapping on \( K^* \) such that \( f \), \( g \in L(K) \). If \((f' (x))^q \) is generalized relative semi-\( m \)-(r; \( h_1, h_2 \))-preinvex function, \( 0 < r < l < l + \mu \), \( q > 1, p^1 + q^{-1} = 1 \) and \( ||g||_{\infty} = \sup_{s \in K} |g(s)| \), then the following inequality for \( \nu > 0 \) holds:

\[
|I_{f,g,E,\Lambda,m}(\nu, a, b)| \leq \frac{2||g||_{\infty}^{1/\nu}S^{\nu+1}(\gamma(b), m(\gamma(a)))}{\sqrt{\nu + 1}}
\]

\[
\times \sqrt[3]{(f' (\xi))^q I_1(h_1(\xi); m(\xi), r) + (f' (\xi))^q I_2(h_2(\xi); r)},
\]

where

\[
I_1(h_1(\xi); m(\xi), r) := \int_0^1 m^q(\xi)h_1^q(\xi) d\xi, \quad I_2(h_2(\xi); r) := \int_1^1 h_2^q(\xi) d\xi.
\]

**Proof.** From Lemma 2.7, the generalized relative semi-\( m \)-(r; \( h_1, h_2 \))-preinvexity of \((f' (x))^q \), Hölder inequality, Minkowski inequality, absolute convergence of Mittag-Leffler function, properties of the modulus, the fact \( g(s) \leq ||g||_{\infty} \forall s \in K \) and changing the variable \( u = m(\gamma(a)) + \xi\Lambda(\gamma(b), m(\gamma(a))) \), \( \forall t \in [0, 1] \), we have

\[
|I_{f,g,E,\Lambda,m}(\nu, a, b)| \leq \int_0^1 \left( \int_{m(\gamma(a))}^{m(\gamma(a)) + \Lambda(\gamma(b), m(\gamma(a)))} g(s)E_{\mu,k}^{\nu}(\omega^s) ds \right)^{1/\nu} \times |f' (\xi)| d\xi
\]

\[
+ \int_0^1 \left( \int_{m(\gamma(a))}^{m(\gamma(a)) + \Lambda(\gamma(b), m(\gamma(a)))} g(s)E_{\mu,k}^{\nu}(\omega^s) ds \right)^{1/\nu} \times |f' (\xi)| d\xi
\]

\[
\leq \left( \int_{m(\gamma(a))}^{m(\gamma(a)) + \Lambda(\gamma(b), m(\gamma(a)))} g(s)E_{\mu,k}^{\nu}(\omega^s) ds \right)^{1/\nu} \times \left( \int_{m(\gamma(a))}^{m(\gamma(a)) + \Lambda(\gamma(b), m(\gamma(a)))} (f' (\xi))^q d\xi \right)^{1/3}
\]

\[
+ \left( \int_{m(\gamma(a))}^{m(\gamma(a)) + \Lambda(\gamma(b), m(\gamma(a)))} g(s)E_{\mu,k}^{\nu}(\omega^s) ds \right)^{1/\nu} \times \left( \int_{m(\gamma(a))}^{m(\gamma(a)) + \Lambda(\gamma(b), m(\gamma(a)))} (f' (\xi))^q d\xi \right)^{1/3}
\]

\[
+ \left( \int_{m(\gamma(a))}^{m(\gamma(a)) + \Lambda(\gamma(b), m(\gamma(a)))} g(s)E_{\mu,k}^{\nu}(\omega^s) ds \right)^{1/\nu} \times \left( \int_{m(\gamma(a))}^{m(\gamma(a)) + \Lambda(\gamma(b), m(\gamma(a)))} (f' (\xi))^q d\xi \right)^{1/3}
\]
Remark 2.9. In Theorem 2.8, for \( h_1(t) = t, h_2(t) = 1 - t, r = 1 \), if we choose \( \Lambda(\psi(b), m(t)\psi(a)) = \psi(b) - m(t)\psi(a) \), where \( m(t) \equiv 1, \forall t \in [0, 1] \) and \( \psi(x) = x, \forall x \in I \), then

1. If we put \( \omega = 0 \), we get [[28], Theorem 7].
2. If we put \( \omega = 0 \) along with \( \nu = \frac{q}{r} \), we get [[10], Theorem 2.5].
3. If we put \( g(s) = 1 \) and \( \omega = 0 \), we get [[6], Theorem 2.3].
4. If we put \( \omega = 0 \) and \( \nu = 1 \), we get [[6], Corollary 3].
Remark 2.10. In Theorem 2.8, for $h_1(t) = t$, $h_2(t) = 1 - t$, $r = 1$, if we choose $\Lambda(\psi(b), m(t)\psi(a)) = \psi(b) - m(t)\psi(a)$, where $m(t) \equiv 1$, $\forall t \in [0, 1]$ and $\psi(x) = x$, $\forall x \in I$, we get [9], Corollary 3.8.

We point out some special cases of Theorem 2.8.

Corollary 2.11. In Theorem 2.8 for $p = q = 2$, we get the following inequality:

\[
\left| I_{f,g,E,\Lambda,\psi,m}(\nu, a, b) \right| \leq \frac{2\|g\|_{\infty}^\nu S^r \Lambda^{r+1}(\psi(b), m(t)\psi(a))}{\sqrt{2\nu + 1}} \left( 2 \int (f'(a))^2 I_1^r(h_1(\xi); m(\xi), r) + (f'(b))^2 I_2^r(h_2(\xi); r) \right).
\]  

(2.5)

Corollary 2.12. In Theorem 2.8 for $g(s) = 1$, we get the following inequality:

\[
\left| I_{f,g,E,\Lambda,\psi,m}(\nu, a, b) \right| = \left[ \int_{\nu}^{m(t)\psi(a) + \Lambda(\psi(b), m(t)\psi(a))} E_{\mu,\nu}^{\delta,h}(\omega s^a) ds \right]^\nu \left[ f(m(t)\psi(a)) + f(m(t)\psi(a)) + \Lambda(\psi(b), m(t)\psi(a)) \right]
- \nu \int_{\nu}^{m(t)\psi(a) + \Lambda(\psi(b), m(t)\psi(a))} \left( \int_{\nu}^{m(t)\psi(a)} E_{\mu,\nu}^{\delta,h}(\omega s^a) ds \right)^{\nu-1} E_{\mu,\nu}^{\delta,h}(\omega s^a) f(\psi) d\xi
\]

(2.6)

\[
\times E_{\mu,\nu}^{\delta,h}(\omega s^a) f(\psi) d\xi \leq \frac{2S^r \Lambda^{r+1}(\psi(b), m(t)\psi(a))}{\sqrt{2\nu + 1}} \sqrt{(f'(a))^2 I_1^r(h_1(\xi); m(\xi), r) + (f'(b))^2 I_2^r(h_2(\xi); r)}.
\]

Corollary 2.13. In Theorem 2.8 for $h_1(t) = h_2(t) = 1$ and $m(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we get the following inequality for the generalized relative semi-$(m, P)$-preinvex mappings (2.7):

\[
\left| I_{f,g,E,\Lambda,\psi,m}(\nu, a, b) \right| \leq \frac{2\|g\|_{\infty}^\nu S^r \Lambda^{r+1}(\psi(b), m\psi(a))}{\sqrt{2\nu + 1}} \sqrt{m(f'(a))^2 + (f'(b))^2}.
\]  

(2.7)

Corollary 2.14. In Theorem 2.8 for $h_1(t) = h(1 - t)$, $h_2(t) = h(t)$ and $m(t) = m \in (0, 1]$ for all $t \in [0, 1]$, we get the following inequality for the generalized relative semi-$(m, h)$-preinvex mappings:

\[
\left| I_{f,g,E,\Lambda,\psi,m}(\nu, a, b) \right| \leq \frac{2\|g\|_{\infty}^\nu S^r \Lambda^{r+1}(\psi(b), m\psi(a))}{\sqrt{2\nu + 1}} \times \sqrt{I_2^r(h(\xi); r) \sqrt{m(f'(a))^2 + (f'(b))^2}}.
\]  

(2.8)
Corollary 2.15. In Corollary 2.14 for \( h_1(t) = (1-t)^s \) and \( h_2(t) = t^r \), we get the following inequality for the generalized relative semi-\((m,s)\)-Brecker-preinvex mappings:

\[
|I_{f,g,E,A,p,m}(\nu,a,b)| \leq \frac{2\|g\|_\infty^s S^{r+1}(\psi(b),m\psi(a))}{\sqrt[p]{1+1}} \times \sqrt[n]{\frac{r}{r+s} m(f'(a))^{rq} + (f'(b))^{rq}}.
\]

(2.9)

Corollary 2.16. In Corollary 2.14 for \( h_1(t) = (1-t)^s \), \( h_2(t) = t^r \) and \( 0 < s < r \), we get the following inequality for the generalized relative semi-\((m,s)\)-Godunova–Levin–Dragomir-preinvex mappings:

\[
|I_{f,g,E,A,p,m}(\nu,a,b)| \leq \frac{2\|g\|_\infty^s S^{r+1}(\psi(b),m\psi(a))}{\sqrt[p]{1+1}} \times \sqrt[n]{\frac{r}{r-s} m(f'(a))^{rq} + (f'(b))^{rq}}.
\]

(2.10)

Corollary 2.17. In Theorem 2.8 for \( h_1(t) = h_2(t) = t(1-t) \) and \( m(t) = m \in (0,1) \) for all \( t \in [0,1] \), we get the following inequality for the generalized relative semi-\((m,tgs)\)-preinvex mappings:

\[
|I_{f,g,E,A,p,m}(\nu,a,b)| \leq \frac{2\|g\|_\infty^s S^{r+1}(\psi(b),m\psi(a))}{\sqrt[p]{1+1}} \times \sqrt[n]{\frac{r}{r-s} m(f'(a))^{rq} + (f'(b))^{rq}}.
\]

(2.11)

Corollary 2.18. In Corollary 2.14 for \( h_1(t) = \frac{1-t}{2\sqrt{t}}, h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}} \), and \( r \in (\frac{1}{2},1] \) we get the following inequality for the generalized relative semi-\((m,MT)\)-preinvex mappings:

\[
|I_{f,g,E,A,p,m}(\nu,a,b)| \leq \frac{2\|g\|_\infty^s S^{r+1}(\psi(b),m\psi(a))}{\sqrt[p]{1+1}} \times \sqrt[n]{\frac{r}{r-s} m(f'(a))^{rq} + (f'(b))^{rq}}.
\]

(2.12)

Theorem 2.19. Let \( h_1, h_2 : [0,1] \rightarrow [0, +\infty), \psi : I \rightarrow K \) and \( g : K \rightarrow \mathbb{R} \) are continuous functions and \( m : [0,1] \rightarrow (0,1] \). Suppose \( K = [m(t)\psi(a), m(t)\psi(a) + \Lambda(\psi(b), m(t)\psi(a))] \subseteq \mathbb{R} \) be an open \( m \)-invex subset with respect to \( \Lambda : K \times K \rightarrow \mathbb{R} \) for \( \Lambda(\psi(b), m(t)\psi(a)) \) and \( \forall t \in [0,1] \). Assume that \( f : K \rightarrow (0, +\infty) \) be a differentiable mapping on \( K \) such that \( f, g \in L(K) \). If \((f(x))^{-q} \) is the generalized relative semi-\( m \)-\((r_1, h_1, h_2)\)-preinvex mapping, \( 0 < r_1 \leq 1, k < l + \mu, q \geq 1 \) and \( \|g\|_\infty = \sup_{s \in K} |g(s)| \), then the following inequality for \( \nu > 0 \) holds:

\[
|I_{f,g,E,A,p,m}(\nu,a,b)| \leq \frac{\|g\|_\infty S^{r+1}(\psi(b),m(t)\psi(a))}{(\nu + 1)^{\frac{1}{q}}} \times \left\{ \sqrt[n]{(f'(a))^{rq}I_1^r(h_1(\xi); m(\xi), \nu, r) + (f'(b))^{rq}I_2^r(h_1(\xi); \nu, r)} + \sqrt[n]{(f'(a))^{rq}I_1^r(h_1(\xi); m(\xi), \nu, r) + (f'(b))^{rq}I_2^r(h_1(\xi); \nu, r)} \right\}.
\]

(2.13)
where

\[ I_1(h_1(\xi); m(\xi), \nu, r) := \int_0^1 m^r(\xi)\xi^{r^2} - h_1^2(\xi)d\xi; \quad I_2(h_2(\xi); \nu, r) := \int_0^1 \xi^{r^2} h_2^2(\xi)d\xi \]

and

\[ T_1(h_1(\xi); m(\xi), \nu, r) := \int_0^1 m^r(\xi)(1 - \xi)^{r^2} - h_1^2(\xi)d\xi; \quad T_2(h_2(\xi); \nu, r) := \int_0^1 (1 - \xi)^{r^2} h_2^2(\xi)d\xi. \]

**Proof.** From Lemma 2.7, the generalized relative semi-m(r; h_1, h_2)-preinvexity of \((f'(x)))^q, the well-known power mean inequality, Minkowski inequality, absolute convergence of Mittag-Leffler function, properties of the modulus, the fact \( g(s) \leq \|g\|_\infty, \forall s \in K \) and changing the variable \( u = m(t)\psi(a) + \xi\Lambda(\psi(b), m(t)\psi(a)), \forall t \in [0, 1], \) we have

\[
\begin{align*}
&|I_{f,g,E,A,\psi}(m(\nu, a, b)| \leq \int_{m(\nu)}^{m(a)} |g(s)E^{\delta,k}_{\mu/\nu}(\omega s^\nu)|^{\nu} \times |f'(\xi)|d\xi \\
&+ \int_{m(\nu)}^{m(a)} \left| \int_{\xi}^{\nu} g(s)E^{\delta,k}_{\mu/\nu}(\omega s^\nu)ds \right|^{\nu} \times |f'(\xi)|d\xi \\
&\leq \|g\|_\infty S^{\frac{\Lambda + 1}{\nu}}(\psi(b), m(t)\psi(a)) \quad (v + 1)^{1 - \frac{1}{\nu}} \\
&\times \left\{ \left[ \int_0^1 \xi^{\nu}(f'(m(t)\psi(a) + \xi\Lambda(\psi(b), m(t)\psi(a)))^\nu d\xi \right]^{\frac{1}{\nu}} \\
&+ \left[ \int_0^1 (1 - \xi)^{\nu}(f'(m(t)\psi(a) + \xi\Lambda(\psi(b), m(t)\psi(a)))^\nu d\xi \right]^{\frac{1}{\nu}} \right\} \\
&\leq \|g\|_\infty S^{\frac{\Lambda + 1}{\nu}}(\psi(b), m(t)\psi(a)) \quad (v + 1)^{1 - \frac{1}{\nu}}
\end{align*}
\]
Corollary 2.21. In Theorem 2.19 for \( g(s) = 1 \), we get the following inequality:

\[
|I_{f;E,A,y,m}(\nu, a, b)| \leq \frac{S_A^{\lambda + 1}(\psi(b), m(t)\psi(a))}{(\nu + 1)^{1-\frac{1}{\lambda}}}
\times \left\{ \sqrt[\nu]{(f' (a))^\nu I_1^\psi (h_1(\xi); \mathbf{m}(\xi), \nu, r) + (f' (b))^\nu I_2^\psi (h_2(\xi); \nu, r)} + \sqrt[\nu]{(f'' (a))^\nu T_1^\psi (h_1(\xi); \mathbf{m}(\xi), \nu, r) + (f'' (b))^\nu T_2^\psi (h_2(\xi); \nu, r)} \right\}.
\]

So, the proof of this theorem is completed.

We point out some special cases of Theorem 2.19.

Corollary 2.20. In Theorem 2.19 for \( q = 1 \), we get the following inequality:

\[
|I_{f;E,A,y,m}(\nu, a, b)| \leq \frac{S_A^{\lambda + 1}(\psi(b), m(t)\psi(a))}{(\nu + 1)^{1-\frac{1}{\lambda}}}
\times \left\{ \sqrt[\nu]{(f' (a))^\nu I_1^\psi (h_1(\xi); \mathbf{m}(\xi), \nu, r) + (f' (b))^\nu I_2^\psi (h_2(\xi); \nu, r)} + \sqrt[\nu]{(f'' (a))^\nu T_1^\psi (h_1(\xi); \mathbf{m}(\xi), \nu, r) + (f'' (b))^\nu T_2^\psi (h_2(\xi); \nu, r)} \right\}.
\]
Corollary 2.22. In Theorem 2.19 for \( h_1(t) = h_2(t) = 1 \) and \( m(t) = m \in (0, 1] \) for all \( t \in [0, 1] \), we get the following inequality for the generalized relative semi-(\( m, P \))-preinvex mappings:

\[
|I_{f,g,E,A,y,m}(\nu, a, b)| \leq \frac{2\|g\|_\infty S^\nu A_{\nu+1}(\psi(b), m\psi(a))}{(\nu + 1)^{1-\frac{\nu}{2}}} \sqrt[\nu]{m(f'(a))^\nu + (f'(b))^\nu}. \tag{2.16}
\]

Corollary 2.23. In Theorem 2.19 for \( h_1(t) = h(t) \), \( h_2(t) = h(t) \) and \( m(t) = m \in (0, 1] \) for all \( t \in [0, 1] \), we get the following inequality for the generalized relative semi-(\( m, h \))-preinvex mappings:

\[
|I_{f,g,E,A,y,m}(\nu, a, b)| \leq \frac{\|g\|_\infty S^\nu A_{\nu+1}(\psi(b), m\psi(a))}{(\nu + 1)^{1-\frac{\nu}{2}}} \times \left\{ \frac{\nu}{\nu} \sqrt{m(f'(a))^\nu \int_{h(1-\xi);\nu,r} + (f'(b))^\nu \int_{h(\xi);\nu,r}} \right\} + \frac{\nu}{\nu} \sqrt{m(f'(a))^\nu \int_{h(1-\xi);\nu,r} + (f'(b))^\nu \int_{h(\xi);\nu,r}} \right\}. \tag{2.17}
\]

Corollary 2.24. In Corollary 2.23 for \( h_1(t) = (1-t)^s \) and \( h_2(t) = t^s \), we get the following inequality for the generalized relative semi-(\( m, s \))-Brechner-preinvex mappings:

\[
|I_{f,g,E,A,y,m}(\nu, a, b)| \leq \frac{\|g\|_\infty S^\nu A_{\nu+1}(\psi(b), m\psi(a))}{(\nu + 1)^{1-\frac{\nu}{2}}} \times \left\{ \frac{\nu}{\nu} \sqrt{m(f'(a))^\nu \beta^\nu \left( \frac{s}{\nu+1} + 1, v+1 \right) + (f'(b))^\nu \beta^\nu \left( \frac{1}{\nu+1} \right)} \right\}. \tag{2.18}
\]

Corollary 2.25. In Corollary 2.23 for \( h_1(t) = (1-t)^{-s} \), \( h_2(t) = t^{-s} \) and \( 0 < s < r \), we get the following inequality for the generalized relative semi-(\( m, s \))-Godunova–Levin–Dragomir-preinvex mappings:

\[
|I_{f,g,E,A,y,m}(\nu, a, b)| \leq \frac{\|g\|_\infty S^\nu A_{\nu+1}(\psi(b), m\psi(a))}{(\nu + 1)^{1-\frac{\nu}{2}}} \times \left\{ \frac{\nu}{\nu} \sqrt{m(f'(a))^\nu \beta^\nu \left( 1 - \frac{s}{\nu+1}, v+1 \right) + (f'(b))^\nu \beta^\nu \left( \frac{1}{\nu+1} \right)} \right\}. \tag{2.19}
\]
Corollary 2.26. In Theorem 2.19 for \(h_1(t) = h_2(t) = t(1-t)\) and \(m(t) = m \in (0, 1)\) for all \(t \in [0, 1]\), we get the following inequality for the generalized relative semi-(m, tgs)-preinvex mappings:

\[
|I_{f,g,E,A,\psi,m}(\nu, a, b)| = \frac{2\|g\|^{\nu}_{\infty}S^{\nu}A^{\nu+1}(\psi(b), m\psi(a))}{(\nu + 1)^{\frac{1}{\nu}}} \frac{\sqrt{\beta(1 + \frac{1}{r}, \nu + \frac{1}{r} + 1)}}{m(f^\nu(a))^\nu + (f^\nu(b))^\nu}. \tag{2.20}
\]

Corollary 2.27. In Corollary 2.23 for \(h_1(t) = \sqrt[\nu]{1-t}, h_2(t) = \sqrt[\nu]{t}\) and \(r \in (\frac{1}{2}, 1]\), we get the following inequality for the generalized relative semi-m- MT-preinvex mappings:

\[
|I_{f,g,E,A,\psi,m}(\nu, a, b)| = \frac{\|g\|^{\nu}_{\infty}S^{\nu}A^{\nu+1}(\psi(b), m\psi(a))}{(\nu + 1)^{\frac{1}{\nu}}} \frac{\sqrt{\beta(1 + \frac{1}{r}, \nu + \frac{1}{r} + 1)}}{m(f^\nu(a))^\nu + (f^\nu(b))^\nu} \tag{2.21}
\]

Remark 2.28. By taking particular values of parameters used in Mittag-Leffler function in Theorems 2.8 and 2.19, several fractional integral inequalities can be obtained.

Remark 2.29. Also, applying our Theorems 2.8 and 2.19, for \(f'(x) \leq K\), for all \(x \in I\), we can get some new fractional integral inequalities.

3. Applications to special means

Definition 3.1. ([2]) A function \(M : \mathbb{R}^2_+ \to \mathbb{R}_+\), is called a Mean function if it has the following properties:

1. Homogeneity: \(M(ax, ay) = aM(x, y)\), for all \(a > 0\),
2. Symmetry: \(M(x, y) = M(y, x)\),
3. Reflexivity: \(M(x, x) = x\),
4. Monotonicity: If \(x \leq x'\) and \(y \leq y'\), then \(M(x, y) \leq M(x', y')\),
5. Internality: \(m\{x, y\} \leq M(x, y) \leq M\{x, y\}\).

Let us consider some special means for arbitrary positive real numbers \(a \neq b\) as follows: The arithmetic mean \(A := A(\alpha, \beta)\); The geometric mean \(G := G(\alpha, \beta)\); The harmonic mean \(H := H(\alpha, \beta)\); The power mean \(P_r := P_r(\alpha, \beta)\); The identric mean \(I := I(\alpha, \beta)\); The logarithmic mean \(L := L(\alpha, \beta)\); The generalized log-mean \(L_p := L_p(\alpha, \beta)\); The weighted \(p\)-power mean \(M_p := M_p\). Now, let \(a\) and \(b\) be positive real numbers such that \(a < b\). Consider the function \(M := M(\psi(a), \psi(b)) : [\psi(a), \psi(\alpha) + \Lambda(\psi(b), \psi(a))] \times [\psi(a), \psi(\alpha) + \Lambda(\psi(b), \psi(a))] \to \mathbb{R}_+\), which is one of the above mentioned means, therefore one can obtain various inequalities using the results of Section 2 for these means as follows: Replace
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\( M_\Psi(\psi(y), \psi(t)\psi(x)) \) with \( M_\Psi(\psi(y), \psi(x)) \) where \( \mu(t) \equiv 1 \), for all \( t \in [0, 1] \) and setting \( M_\Psi(\psi(y), \psi(x)) = M(\psi(x), \psi(y)) \) for all \( x, y \in I \), in (2.4) and (2.13), one can obtain the following interesting inequalities involving means:

\[
|I_{f,g,M_\Psi}(\nu,a,b)| \leq \frac{2\|g\|_\infty S^M}{\sqrt{pv+1}} \times \left( \sqrt{(f'(a))^q I_2'(h_1(\xi);\nu,r) + (f'(b))^q I_2'(h_2(\xi);\nu,r)} \right),
\]

(3.1)

\[
|I_{f,g,M_\Psi}(\nu,a,b)| \leq \frac{\|g\|_\infty S^M}{(v+1)^{1-\frac{1}{q}}}
\]

\[
\times \left\{ \sqrt{(f'(a))^q I_2'(h_1(\xi);\nu,r) + (f'(b))^q I_2'(h_2(\xi);\nu,r)} \right\}.
\]

(3.2)

Letting \( M := A, G, H, P_r, I, L, L_p, M_p \) in (3.1) and (3.2), we get the inequalities involving means for particular choices of \((f'(x))^q\) that are the generalized relative semi-1-(\(r; h_1, h_2\))-preinvex mappings.

**Remark 3.2.** Also, applying our Theorems 2.8 and 2.19 for appropriate choices of functions \( h_1 \) and \( h_2 \) (see Remark 2.6) such that \((f'(x))^q\) to be the generalized relative semi-1-(\(r; h_1, h_2\))-preinvex mappings (see examples: \( f(x) = x^a \), where \( a > 1, \forall x > 0; f(x) = \frac{1}{x}, \forall x > 0; f(x) = e^x, \forall x \in \mathbb{R}; f(x) = -\ln x, \forall x > 0 \), etc.), we can deduce some new inequalities using above special means. The details are left to the interested reader.

**References**


New fractional integral inequalities


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